Fractionalization, Quantum Circuits, and Exactly Solvable Models

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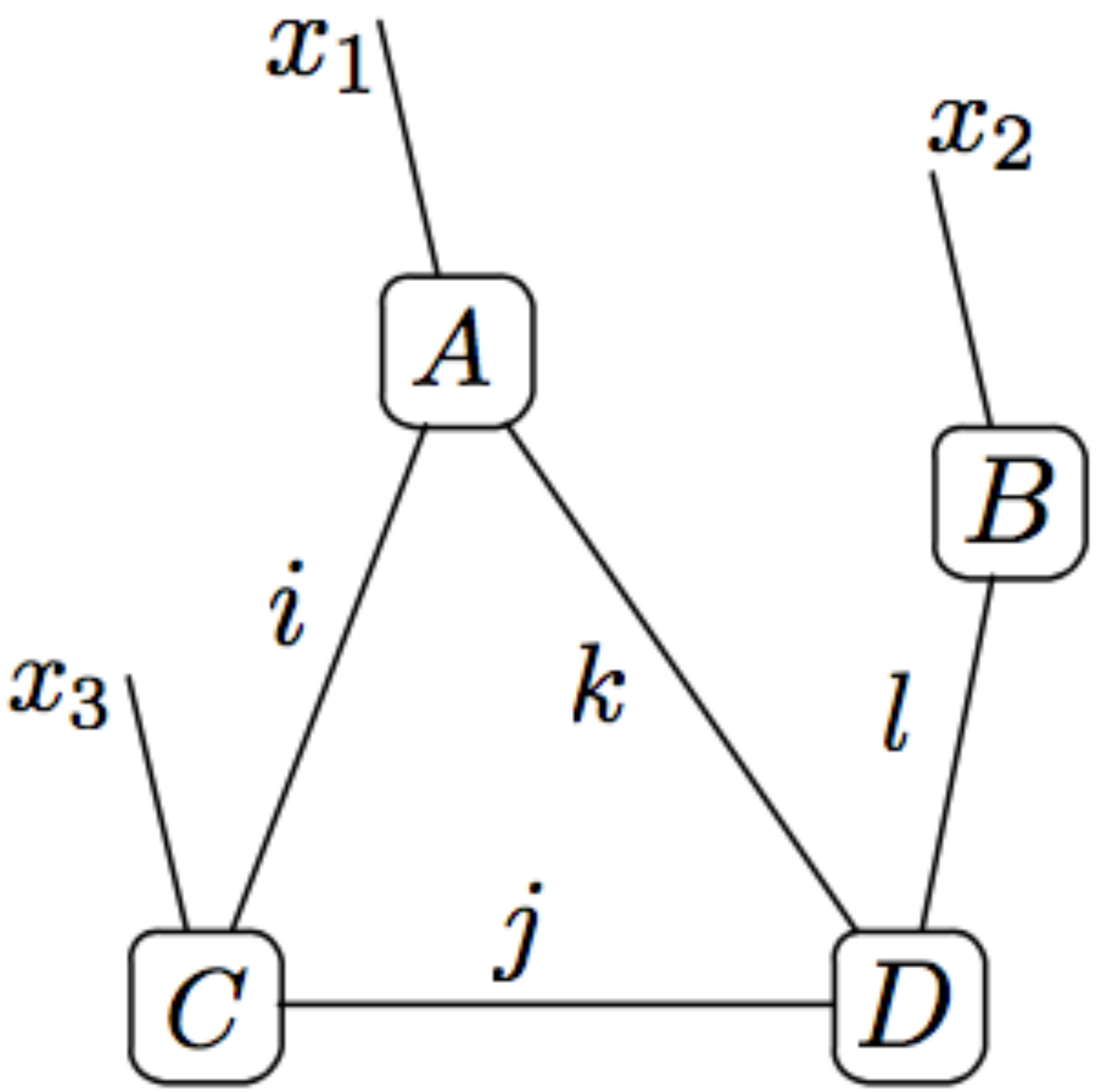
Harvard, MPHQ

with Yunxiang Ren, Shengtao Wang, Zhengwei Liu, Arthur Jaffe

Motivation

Open the black box of tensor networks (TN)



$$T_{x_1x_2x_3} = \sum_{ijkl} A_{x_1ik} B_{x_2l} C_{x_3ij} D_{kjl}$$

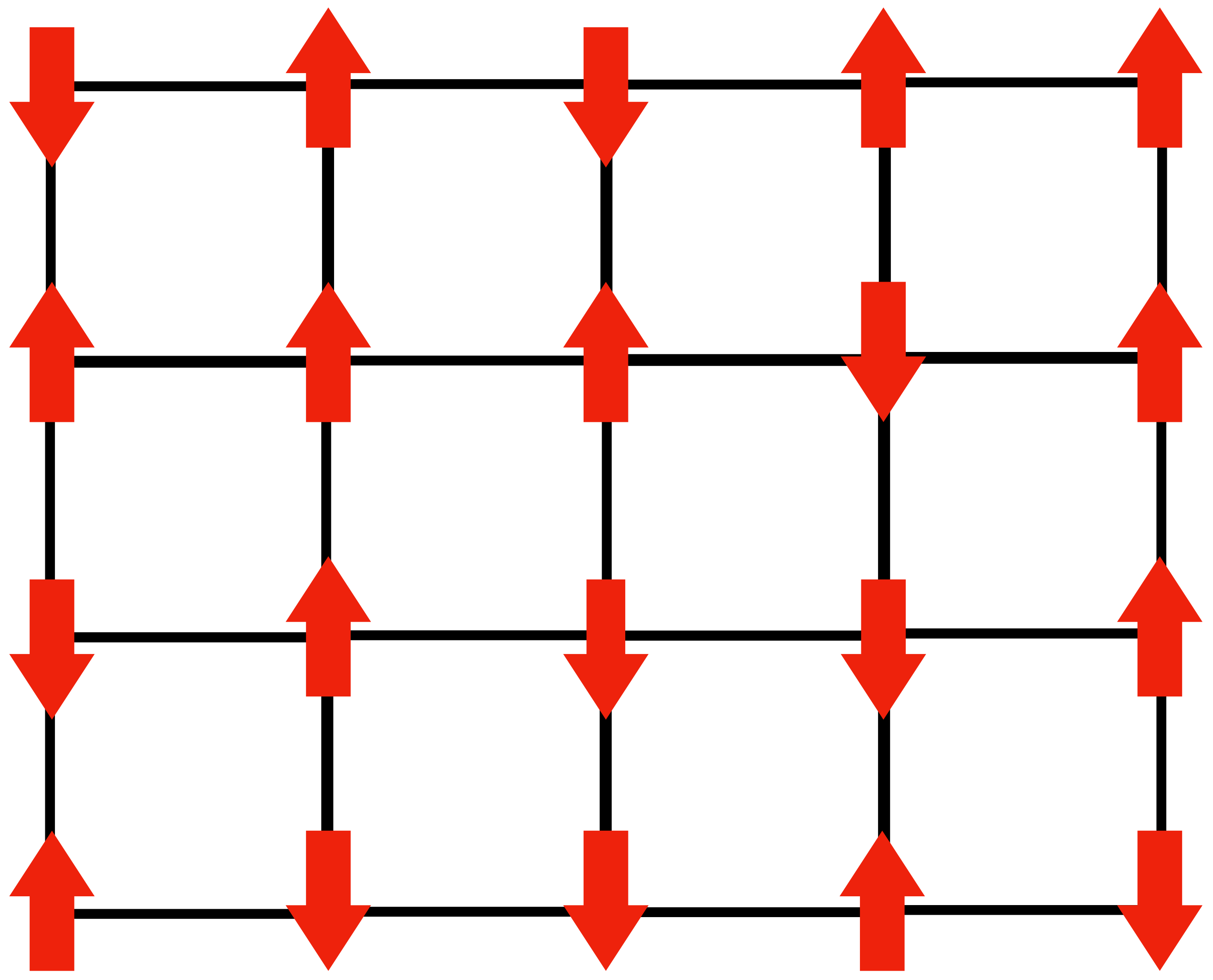
wide range of applications:
quantum many-body physics,
counting complexity,
machine learning, etc.

Quantum circuits & Partition function as special cases

The information of A,B,C,D are blackboxes in the diagram, hidden in the tables of A_xik, …

Motivation

Open the black box of tensor networks (TN)

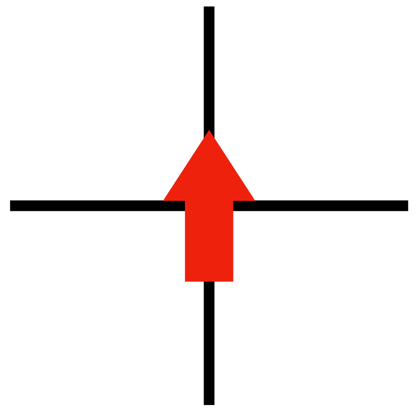


$$Z = \sum_{\sigma} e^{-\beta H(\sigma)}$$

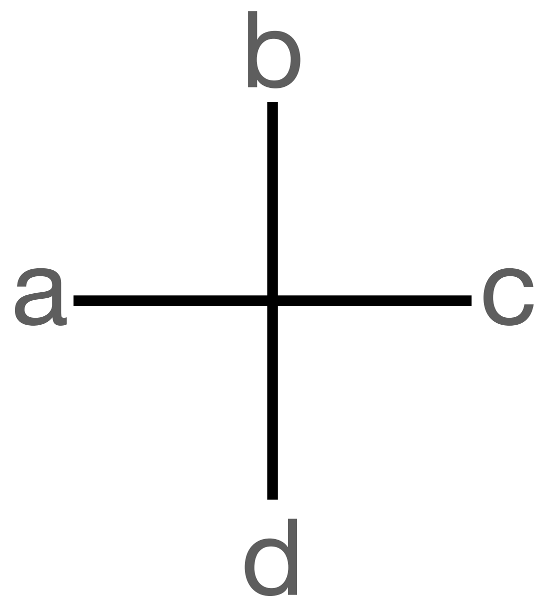
$$H(\sigma) = \sum_{(i,j)} J_{ij}\sigma_i\sigma_j$$

=> σ_i —●— $\sigma_j = e^{-\beta J_{ij}\sigma_i\sigma_j}$

define two tensors:



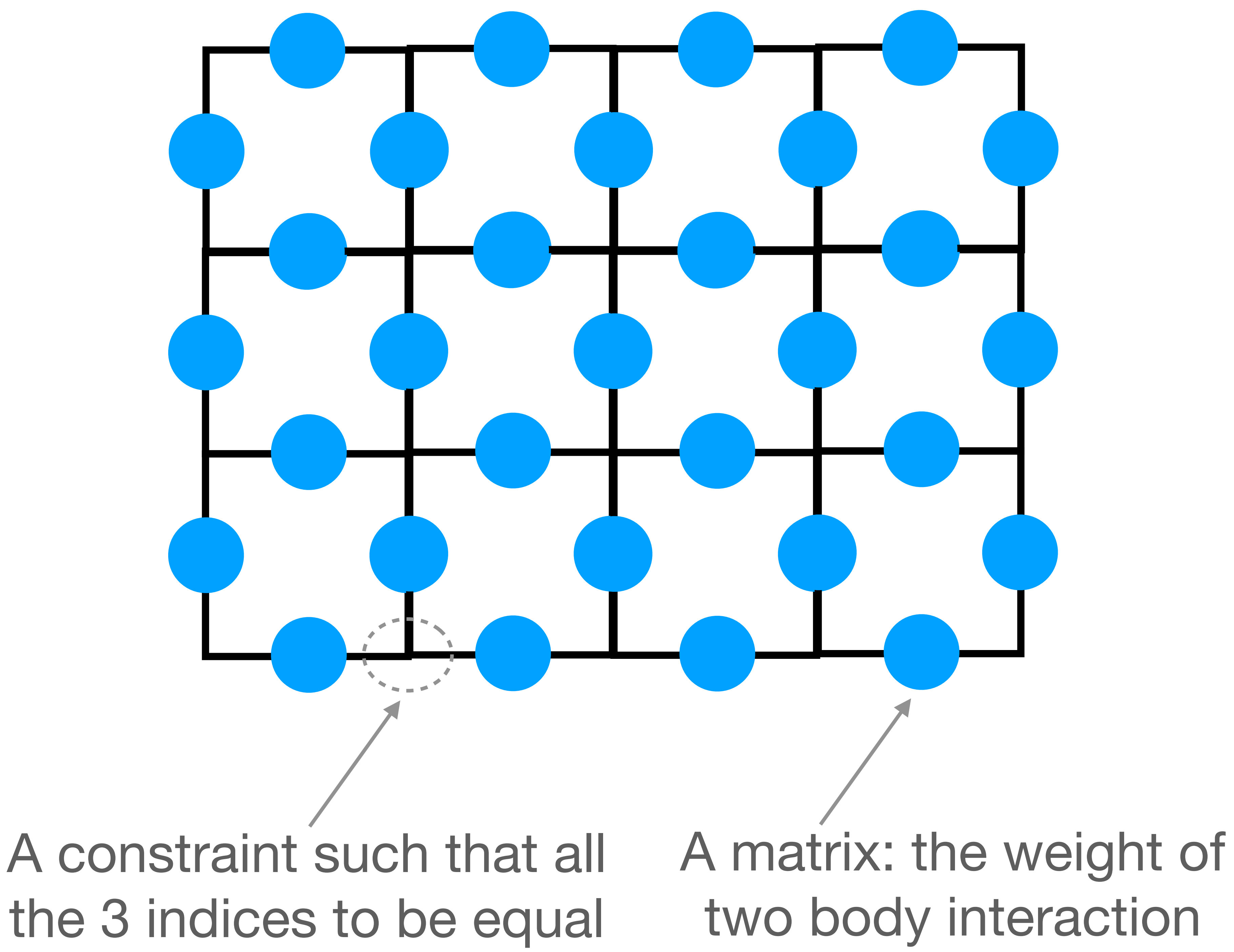
=>



$= \delta_{ab}\delta_{bc}\delta_{cd}$

Motivation

Open the black box of tensor networks (TN)



Partition function of 2D Ising model as tensor network

It's hard to answer the following questions from the picture of TN directly

Why is it exactly solvable?

Why free fermion (Jordan-Wigner transformation)?

How to understand Kramers-Wannier duality pictorially?

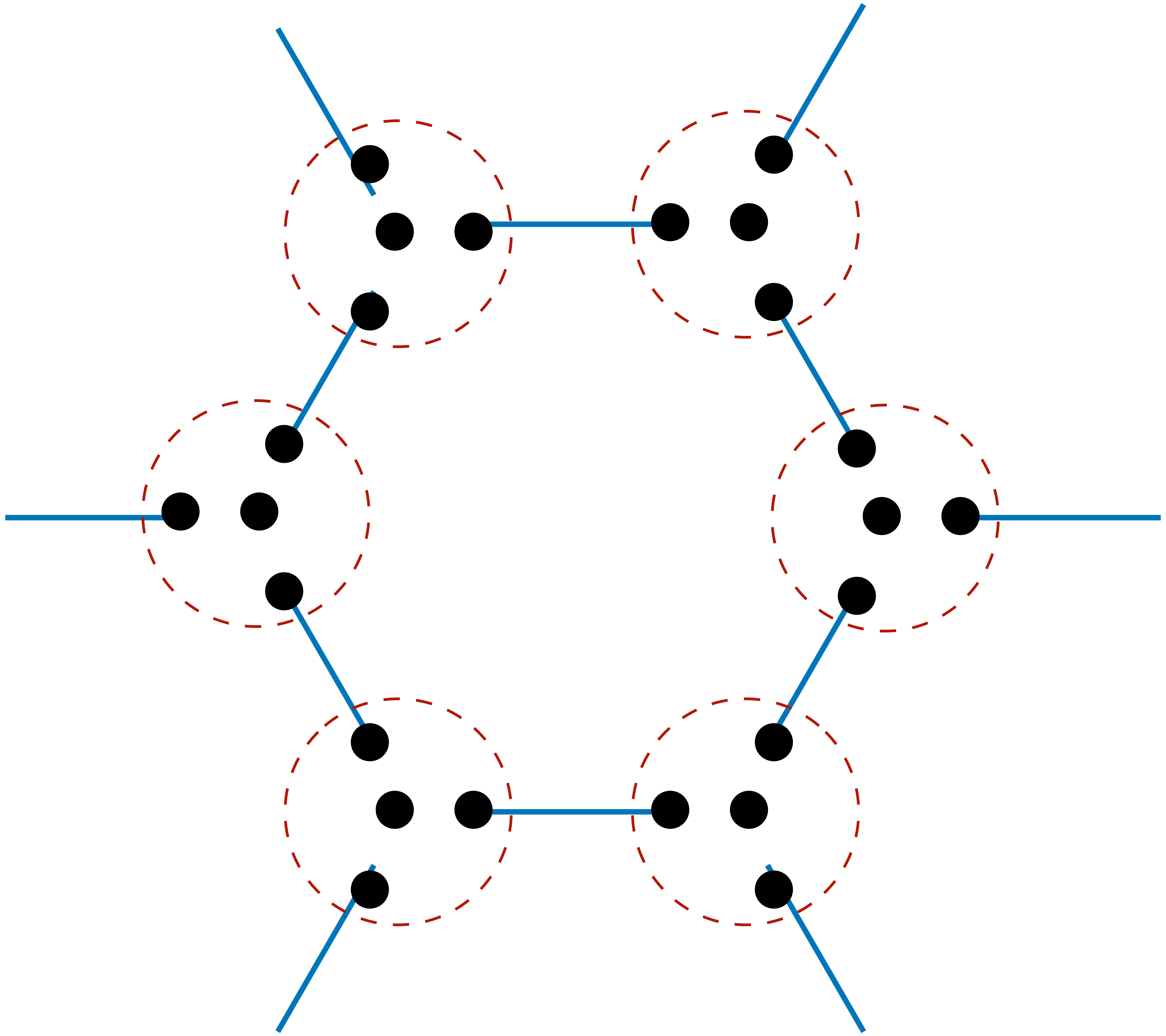
Idea: fractionalizing the spins

inner structure of lines and box in TN from picture directly

open the black box

Motivation

Fractionalizing a qubit



Kitaev's honeycomb model:

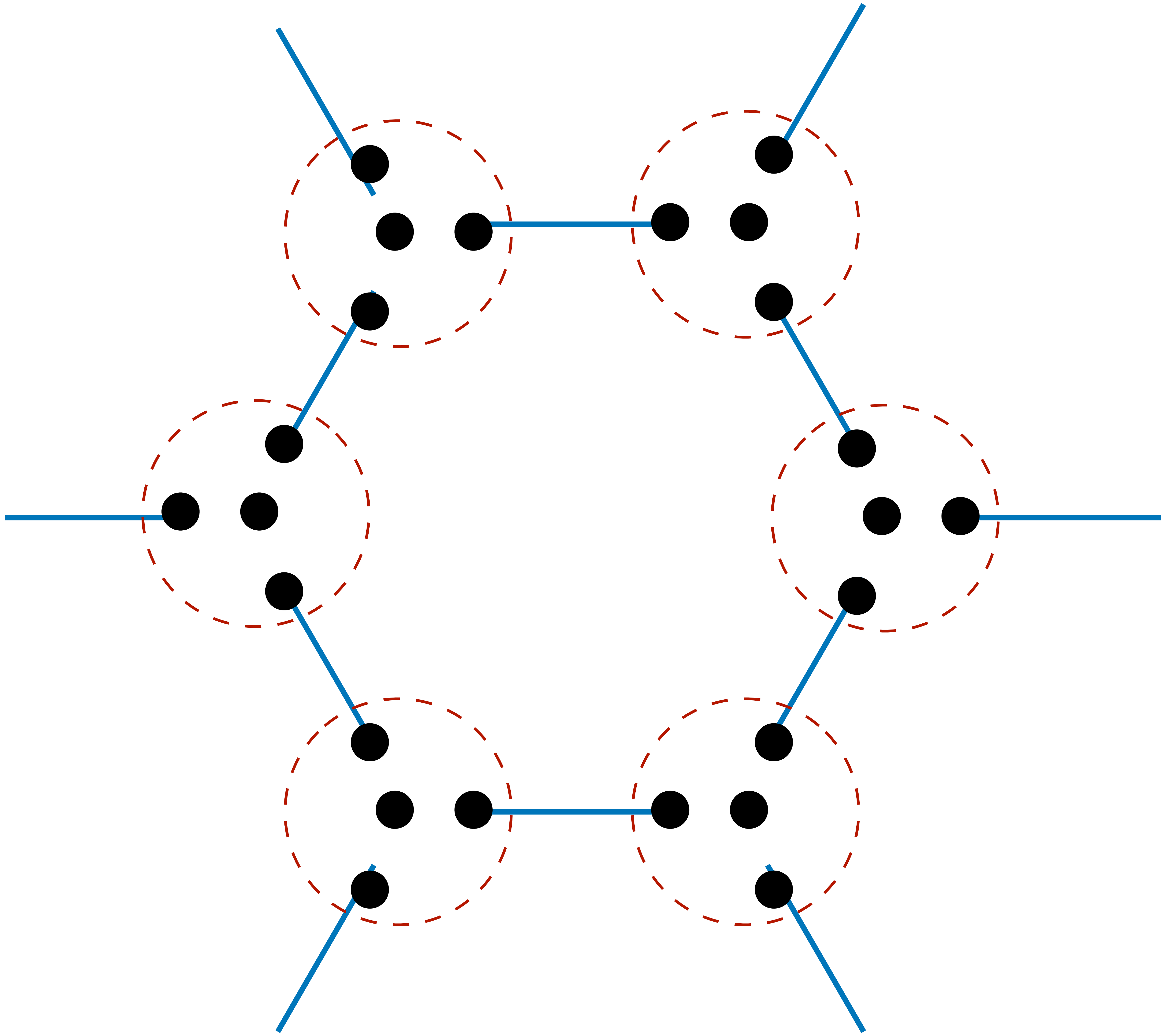
1 spin <=> 4 Majorana modes

Kitaev, Alexei. "Anyons in an exactly solved model and beyond." *Annals of Physics* 321.1 (2006): 2-111.

- *Can we generalize this approach to tensor network?*
- *Relations to other solvable models in statistical physics, quantum information and computer science?*

Motivation

Fractionalizing a qubit



The answer is affirmative:

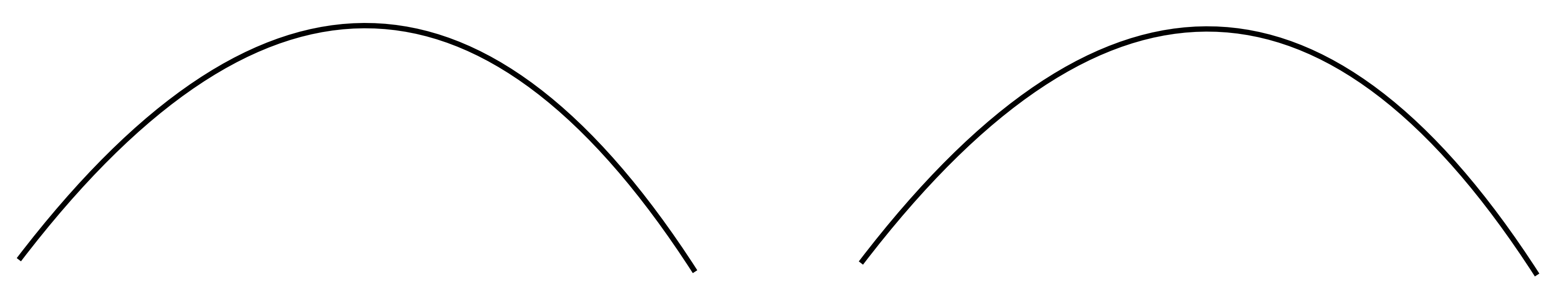
- Open the black box of tensor network, topological characterization
- Understanding 2D Ising model pictorially: Yang-Baxter equation, Jordan-Wigner transformation, Kramers-Wannier duality
- Generalizing Kramers-Wannier duality, new solvable models
- A unified framework for classically simulable quantum circuits: Clifford circuits & matchgate, and their generalization

Motivation

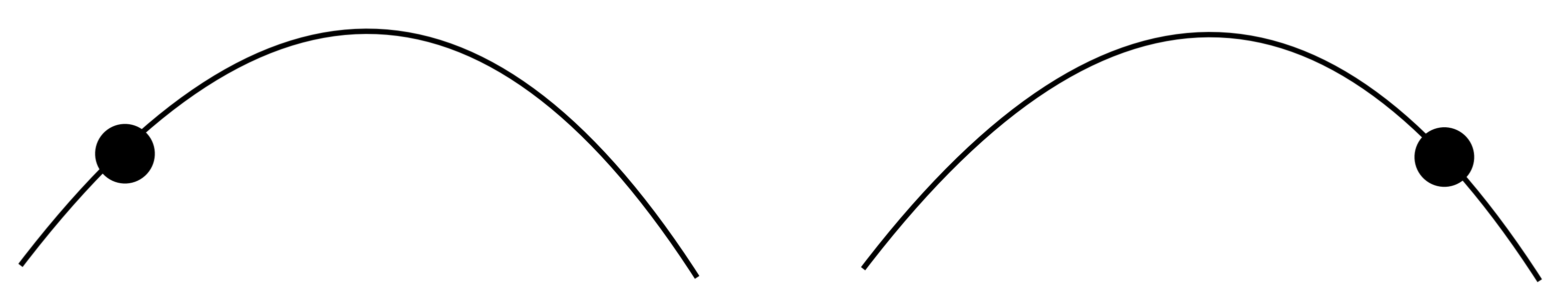
Fractionalizing a qubit

4 Majorana zero modes (2 ground states of Kitaev's Majorana chain)

*with total even fermion parity, **Quon language on Z_2**, also known as sparse encoding*



$|0\rangle$



$|1\rangle$

or equivalently, under the constraint $-\gamma_1\gamma_2\gamma_3\gamma_4 = 1$

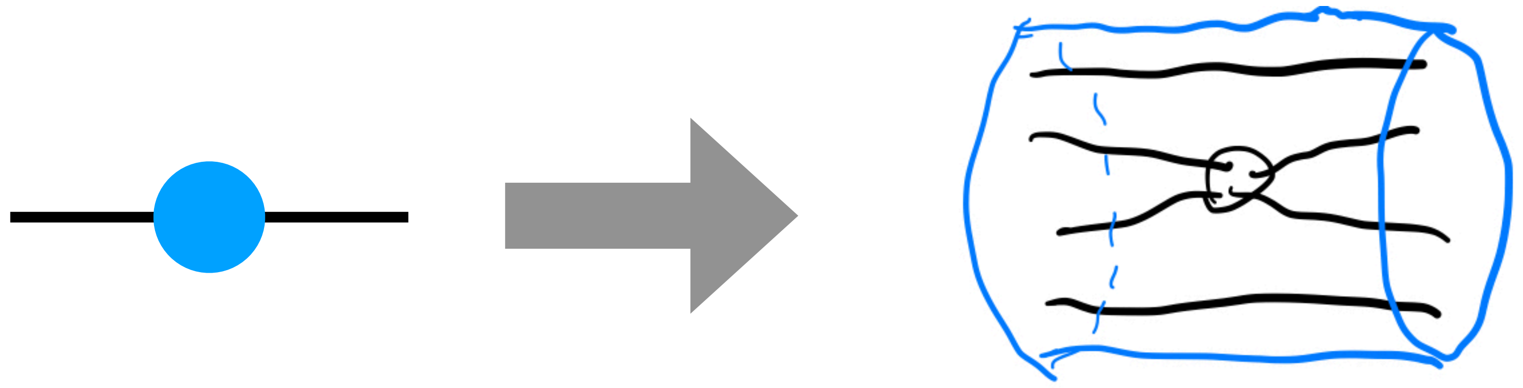
Liu, Zhengwei, Alex Wozniakowski, and Arthur M. Jaffe. "Quon 3D language for quantum information." *Proceedings of the National Academy of Sciences* 114.10 (2017): 2497-2502.

Sarma, Sankar Das, Michael Freedman, and Chetan Nayak. "Majorana zero modes and topological quantum computation." *npj Quantum Information* 1.1 (2015): 1-13.

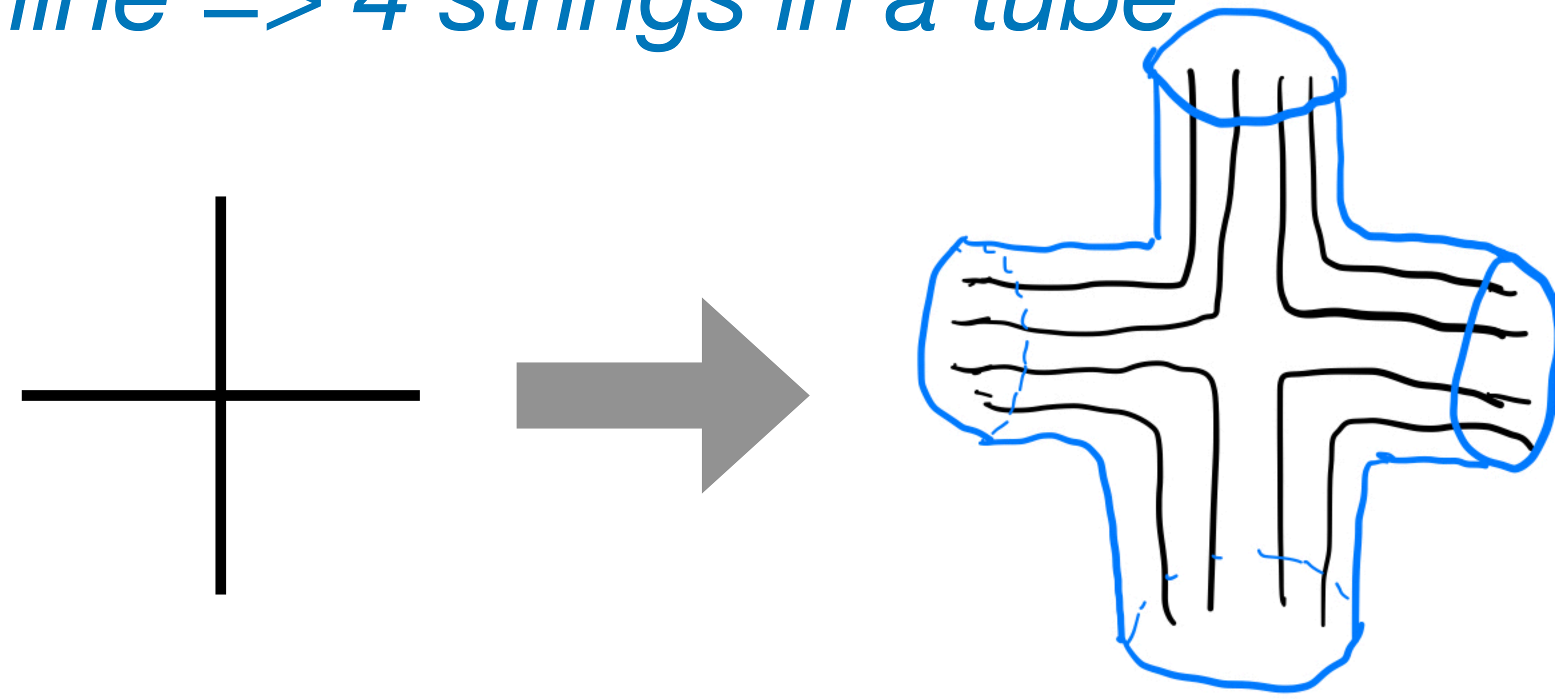
Motivation

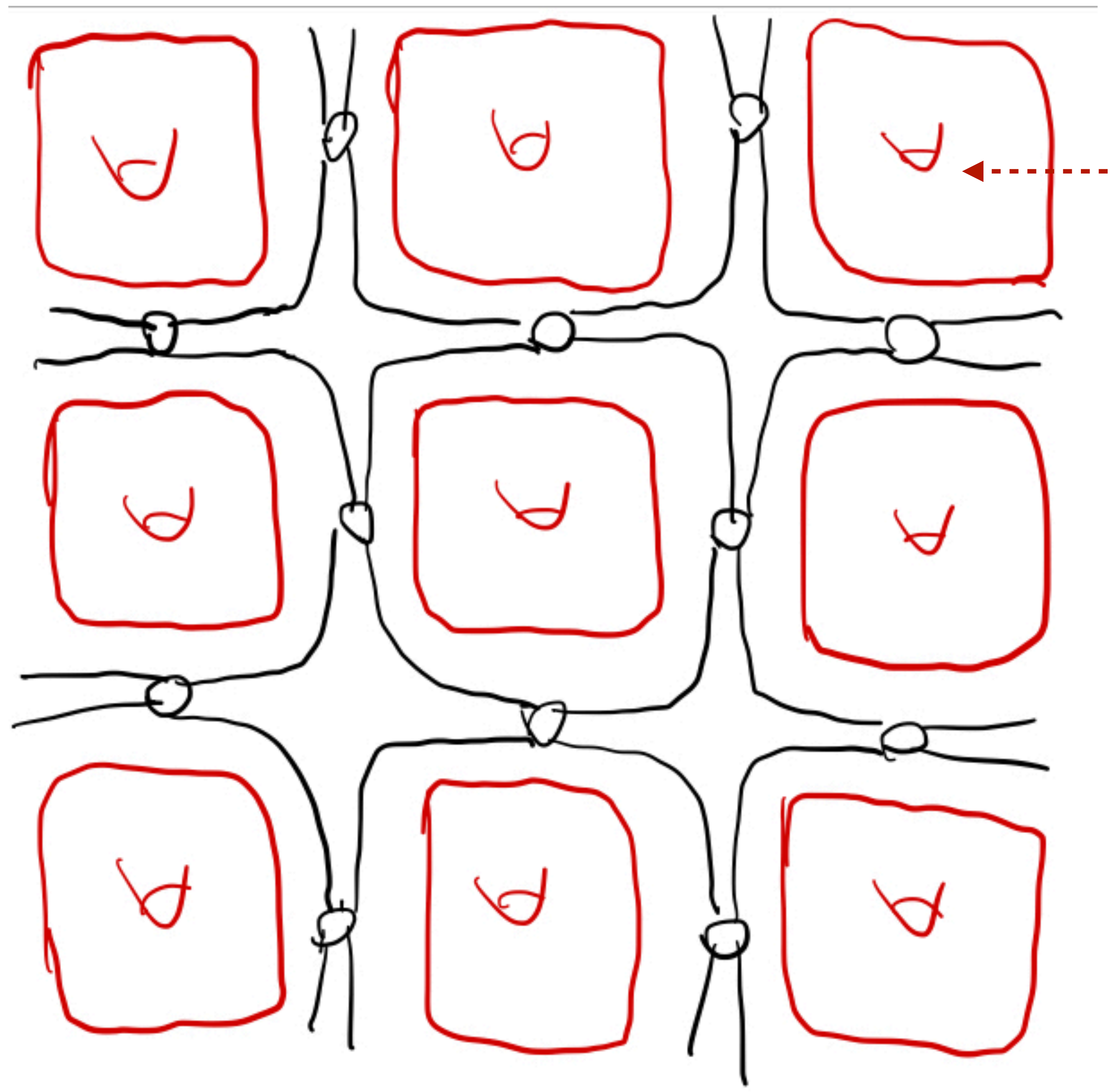
Fractionalizing qubits in tensor network

Ising model / Kramers-Wannier duality in Quon language



1 line => 4 strings in a tube



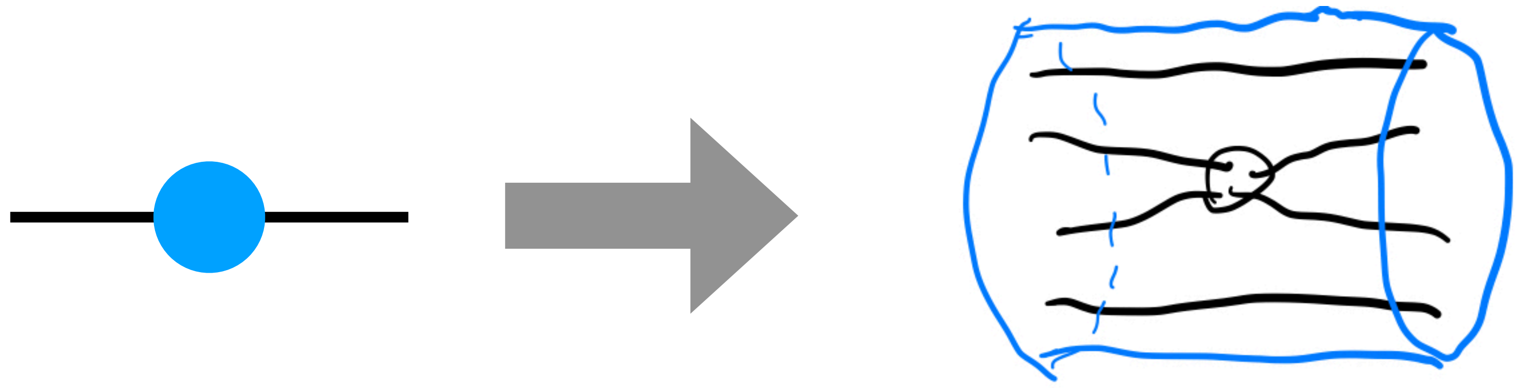


Representing genus

Motivation

Fractionalizing qubits in tensor network

Ising model / Kramers-Wannier duality in Quon language



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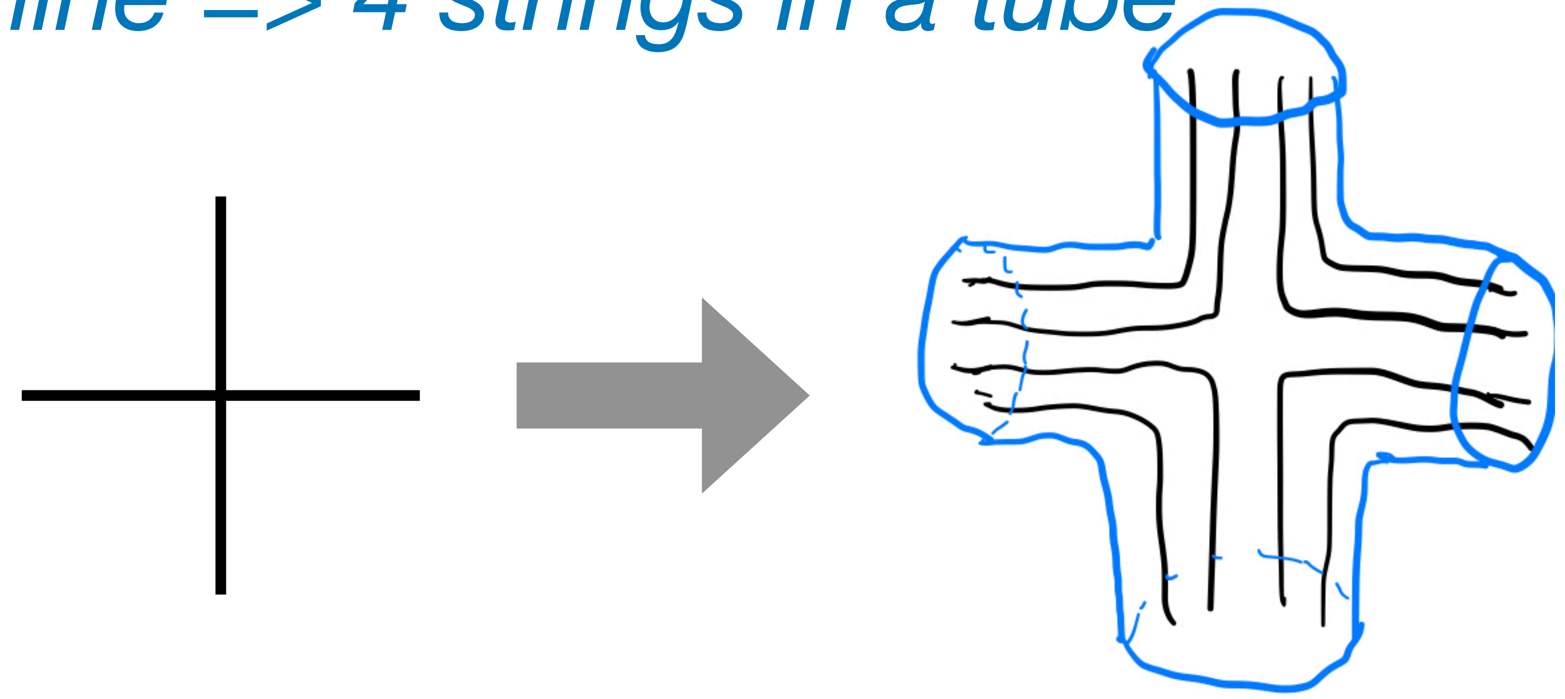
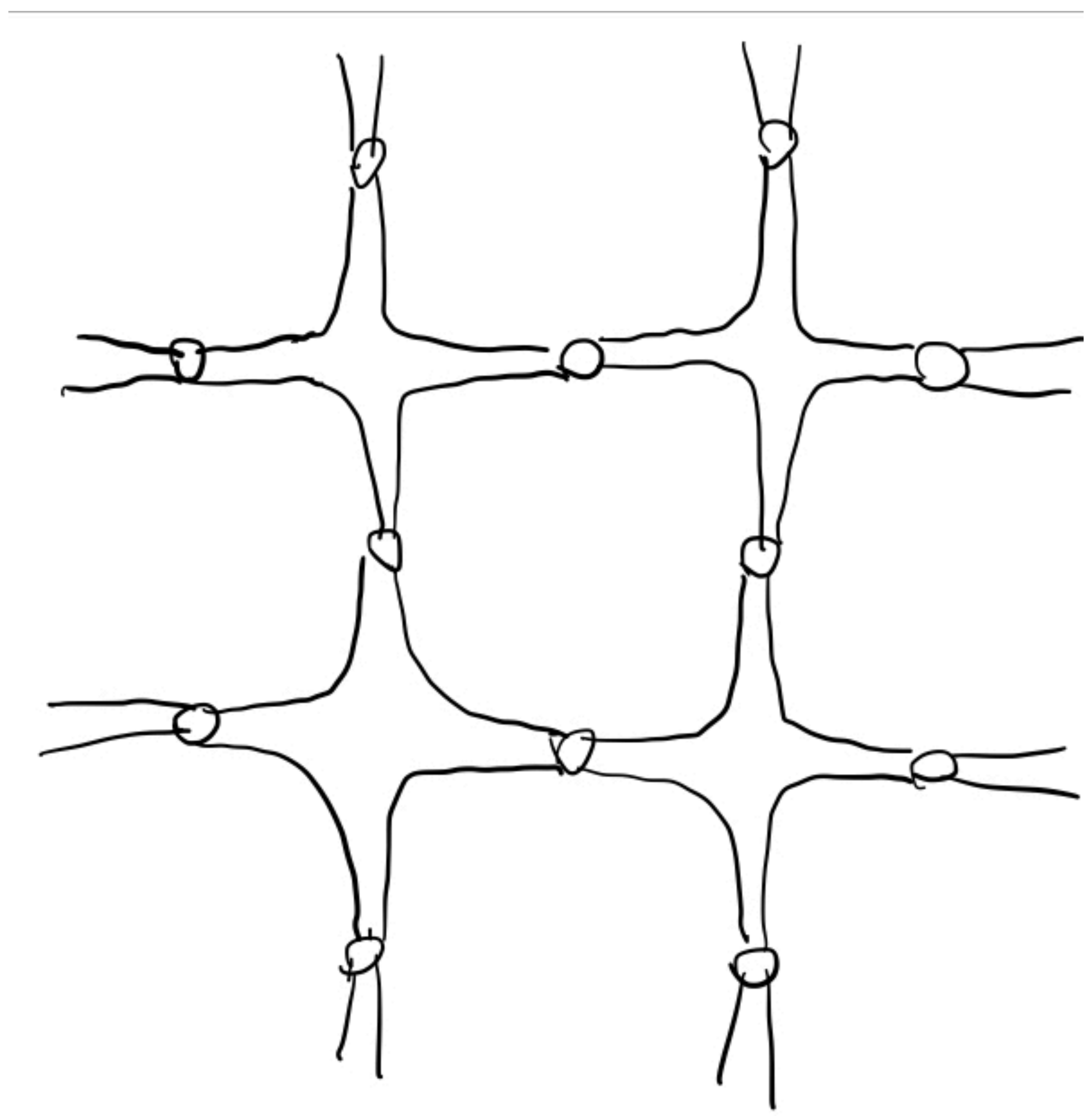
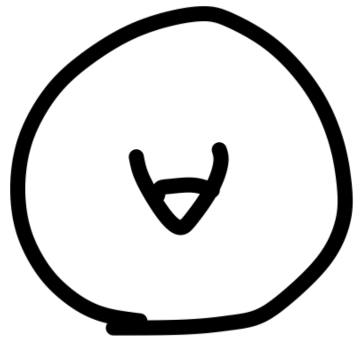




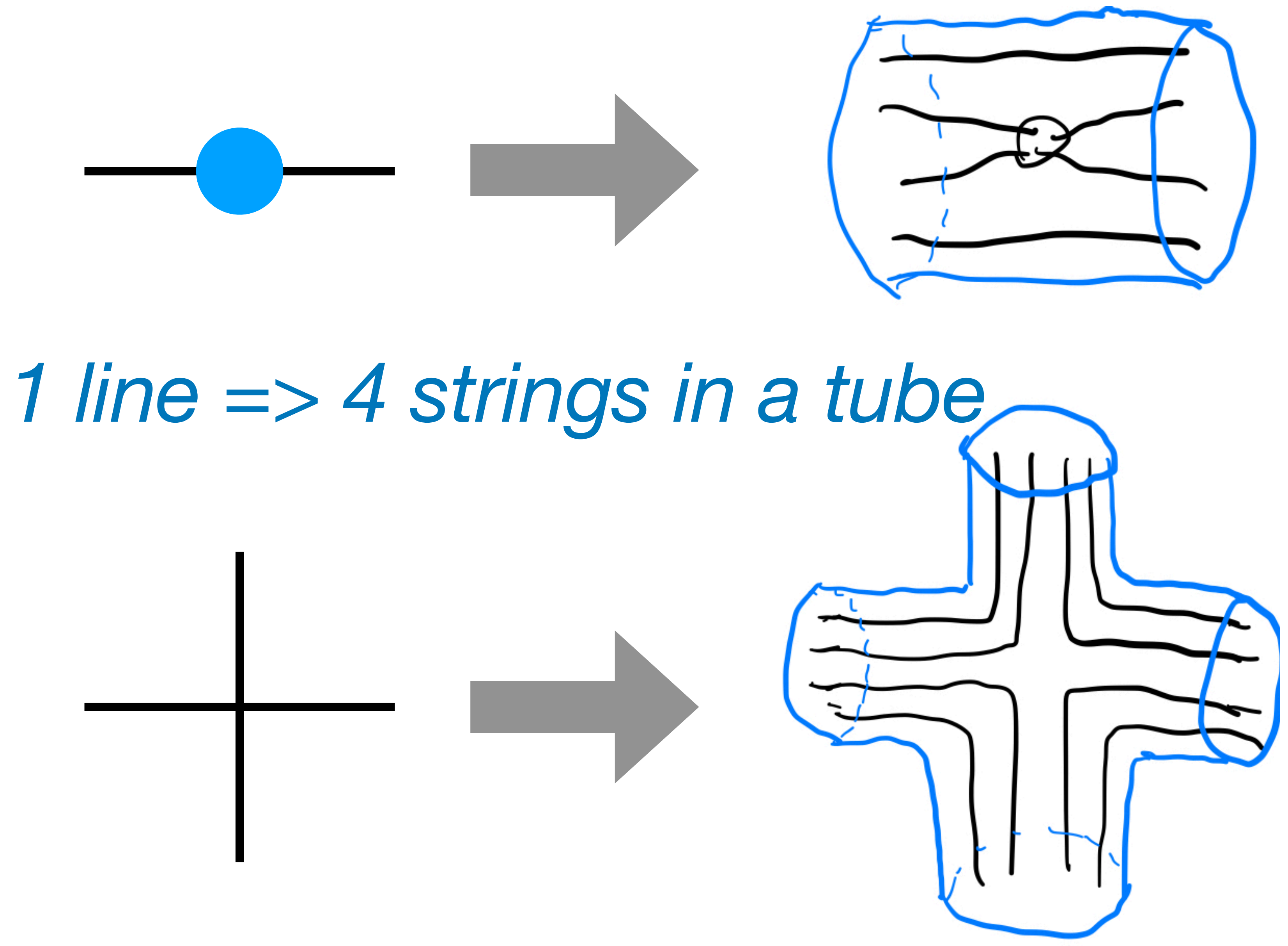
String-genus relation

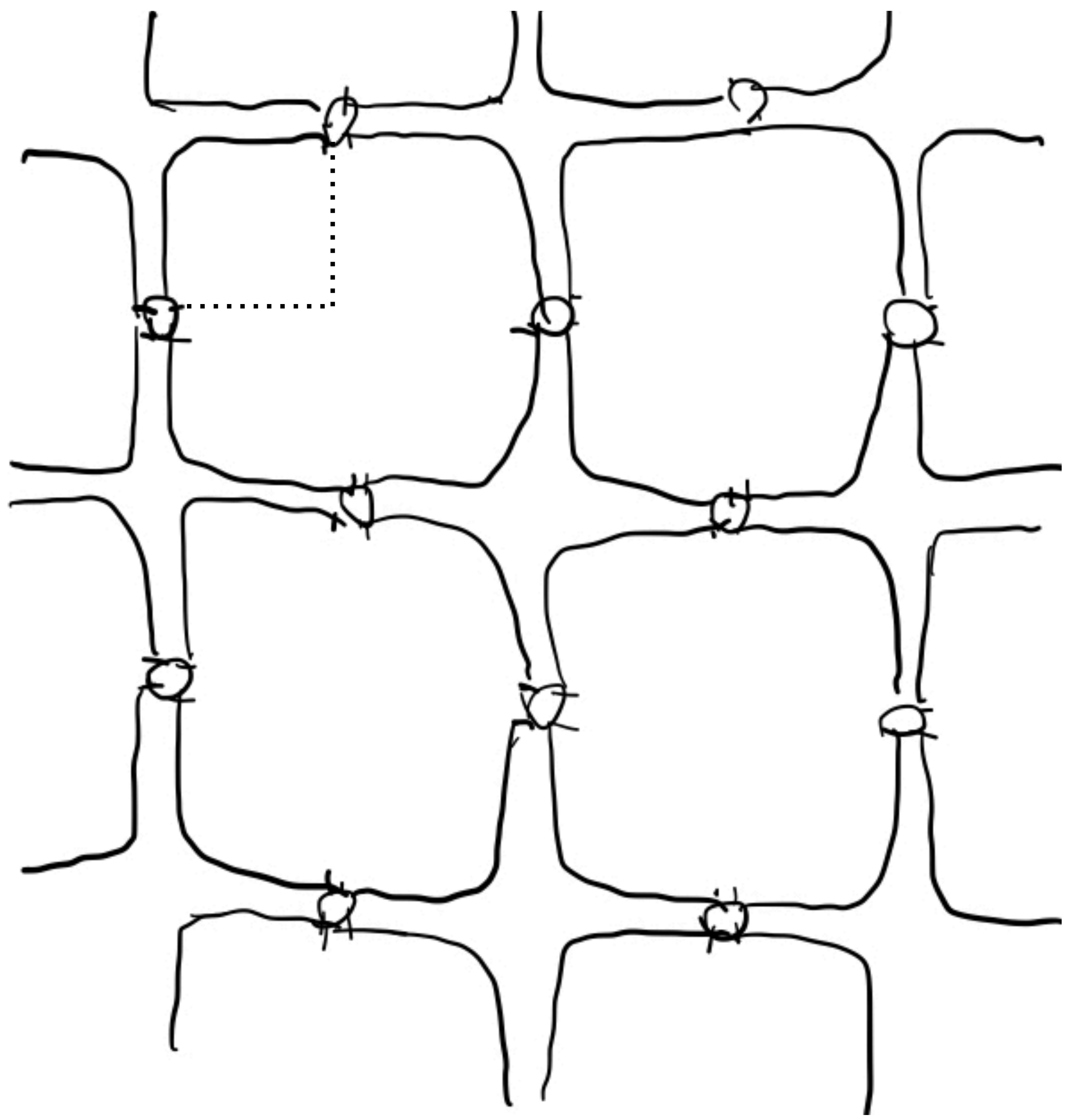
remove 

Motivation

Fractionalizing qubits in tensor network

Ising model / Kramers-Wannier duality in Quon language





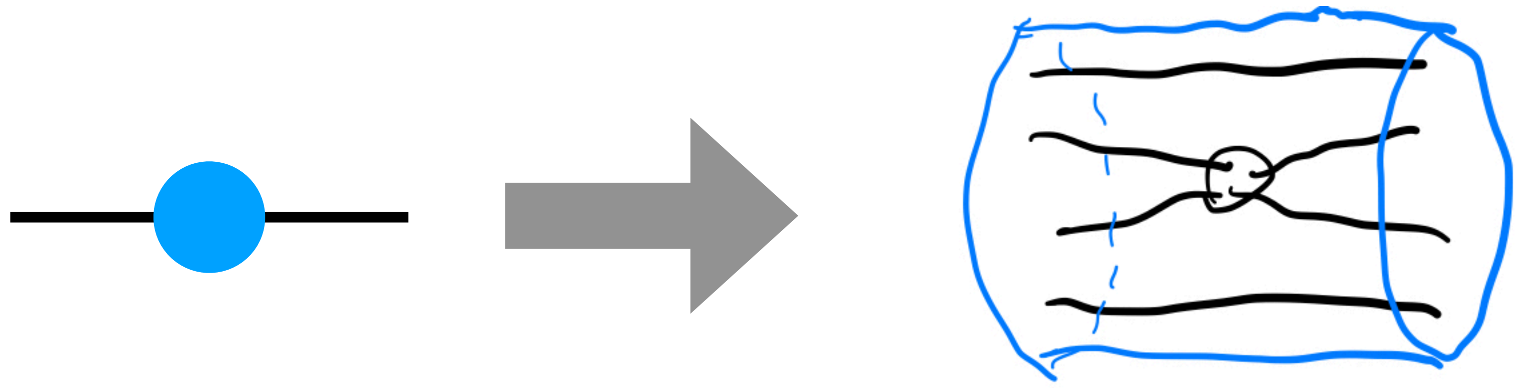
Reshape

(free-fermion)

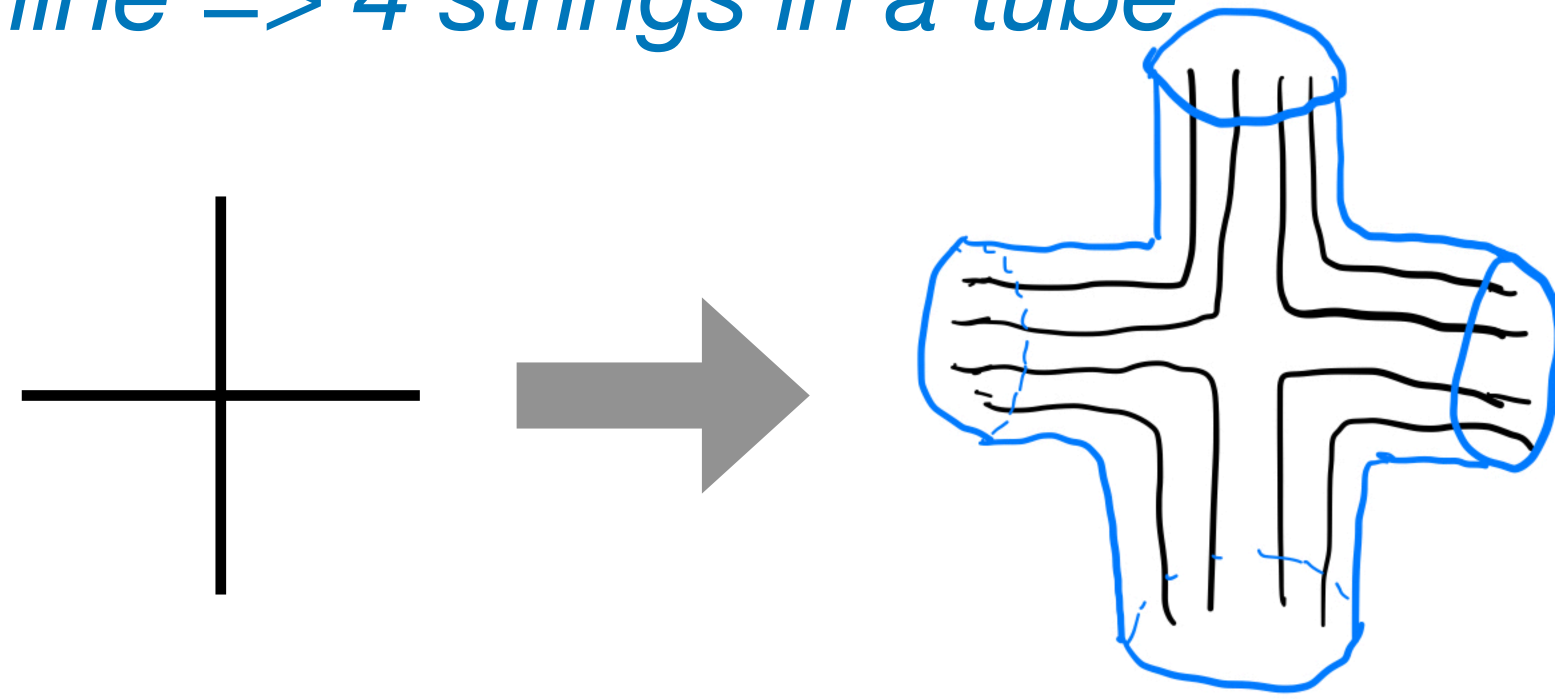
Motivation

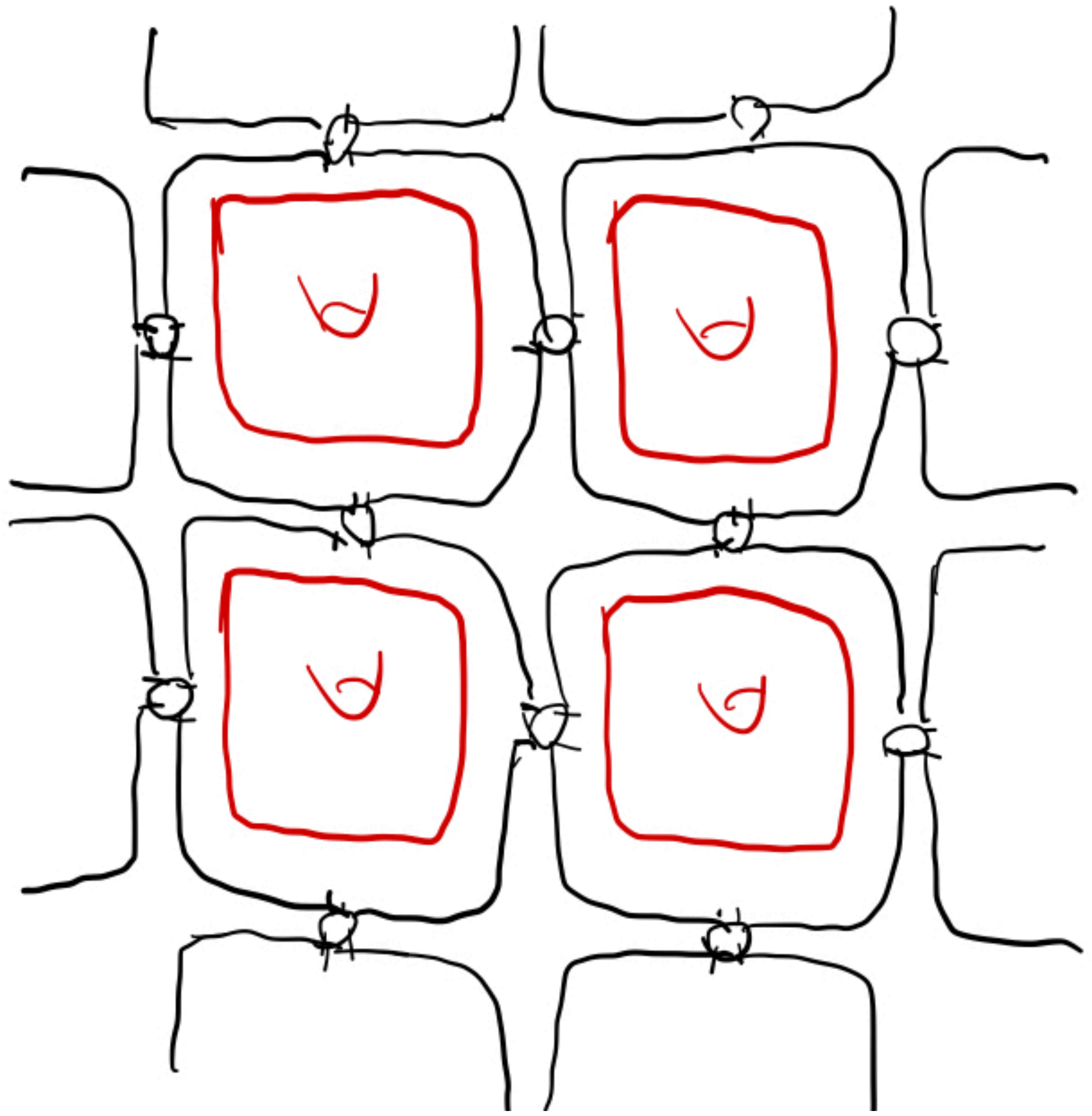
Fractionalizing qubits in tensor network

Ising model / Kramers-Wannier duality in Quon language

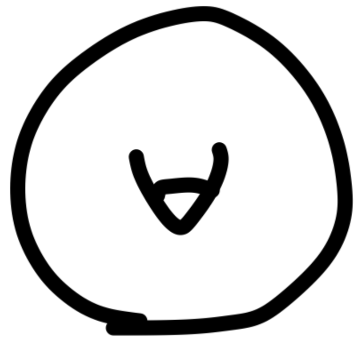


1 line => 4 strings in a tube





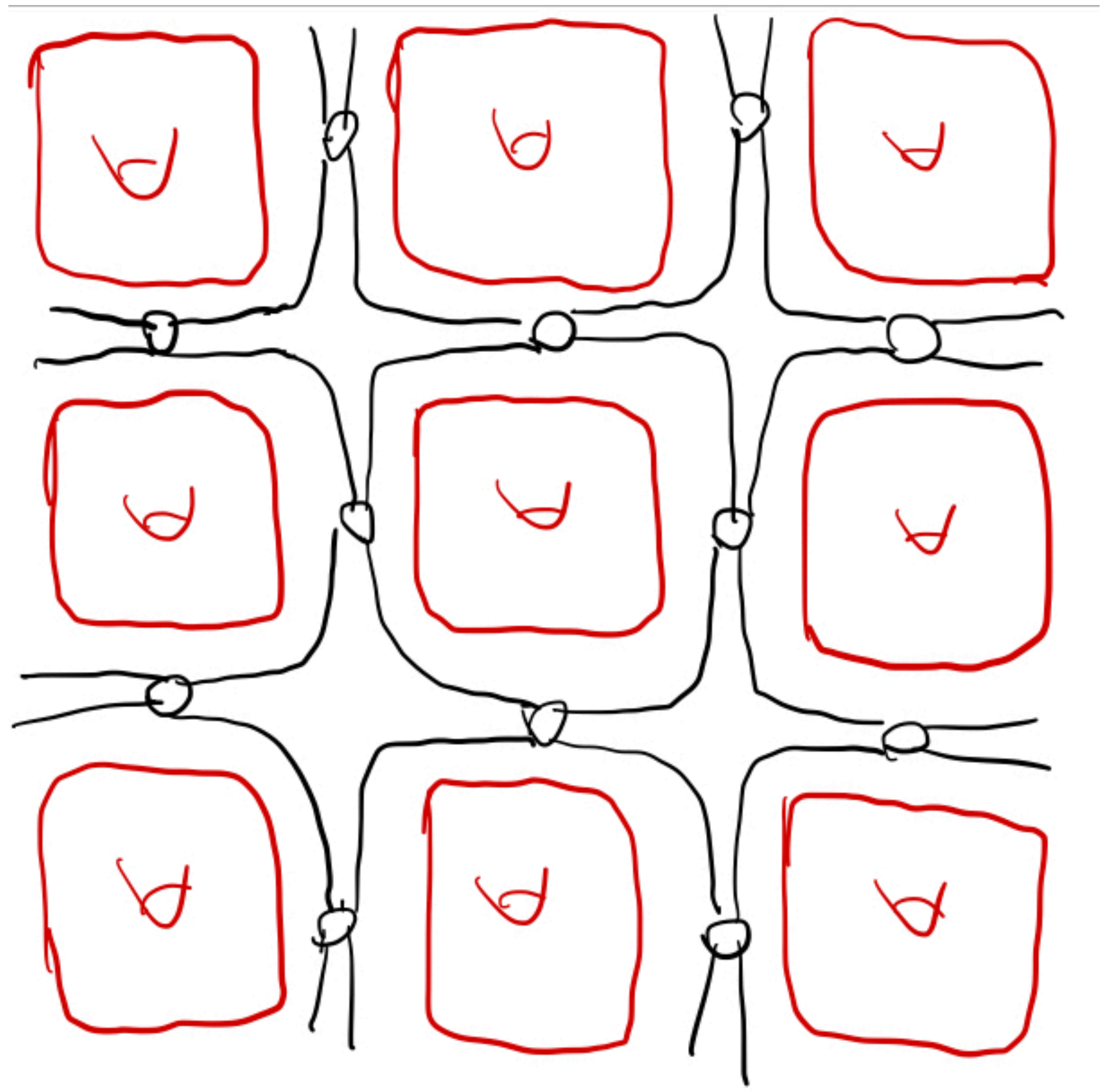
String-genus relation

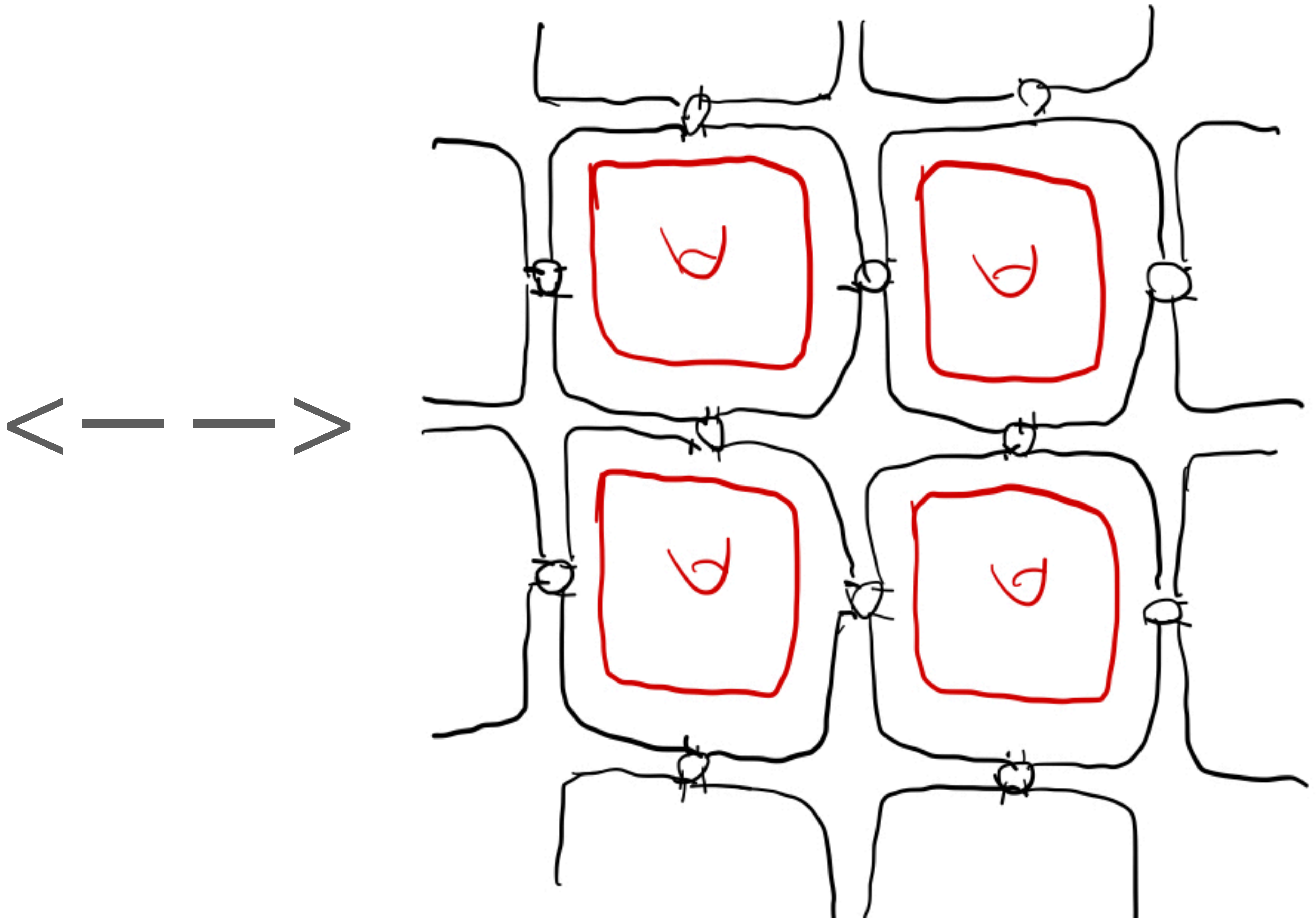
Add 

Motivation

Fractionalizing qubits in tensor network

Ising model / Kramers-Wannier duality in Quon language



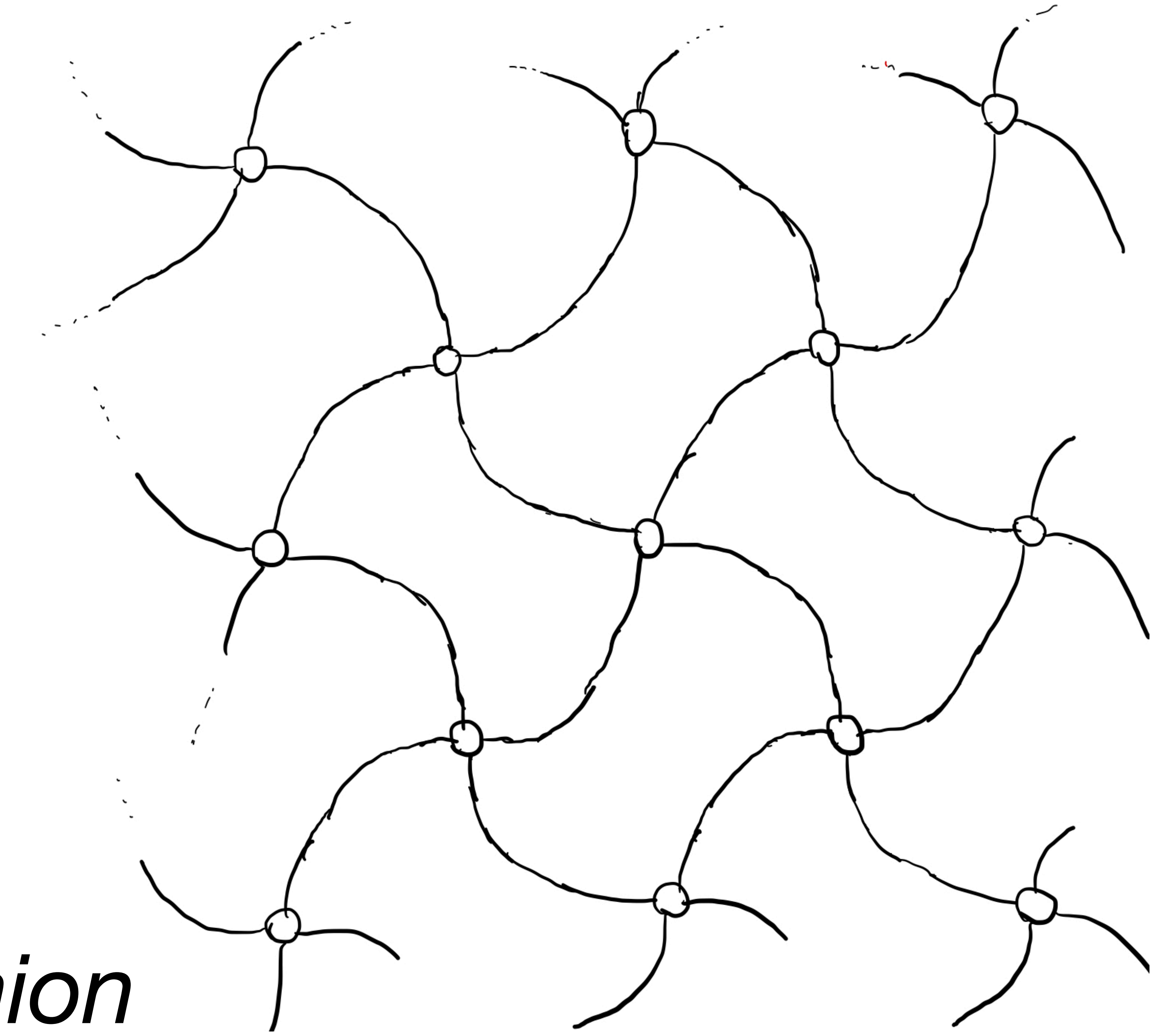


KW duality: shift the positions of string-genus

Motivation

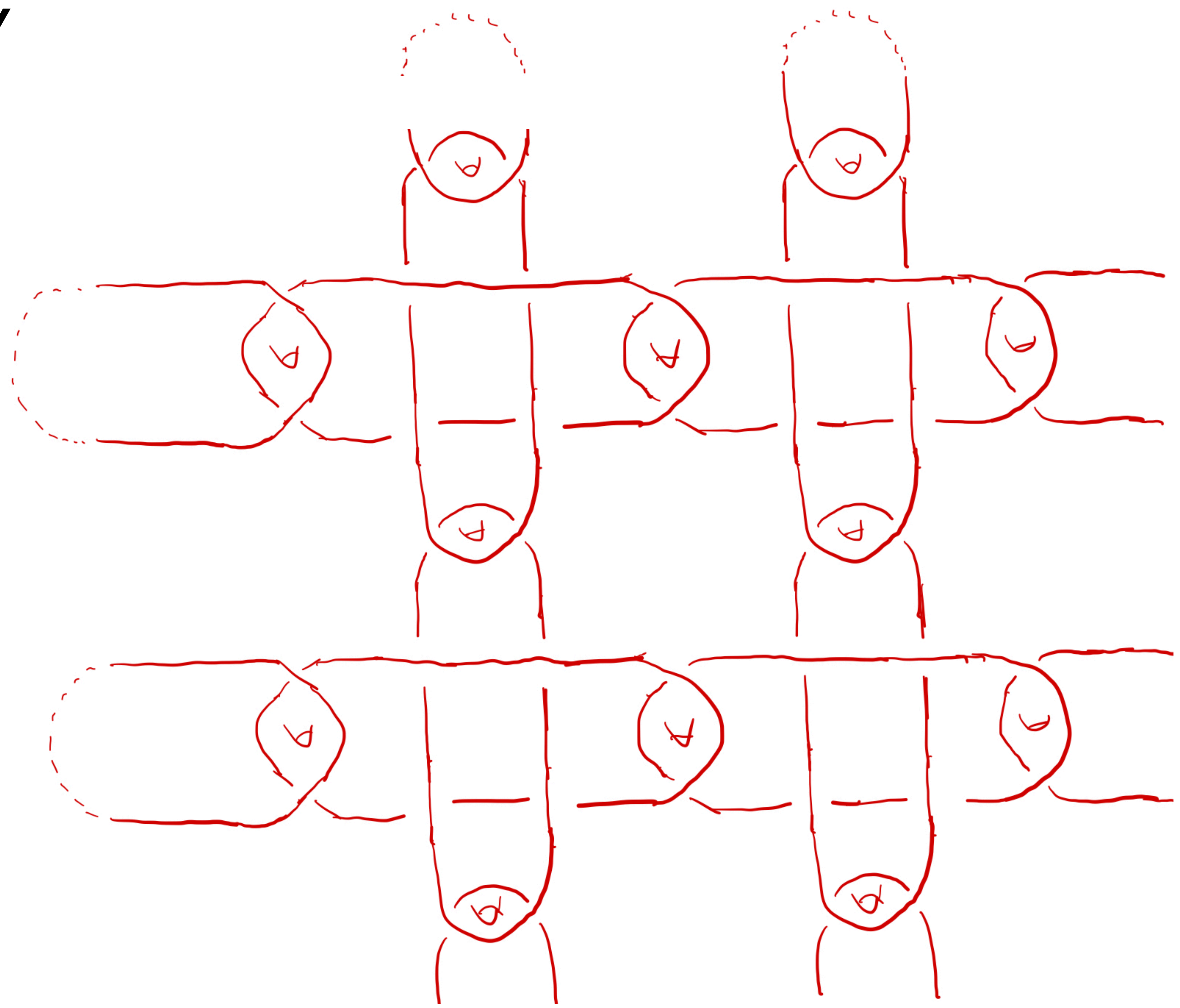
Fractionalizing qubits in tensor network

Generalization of Kramers-Wannier duality



+

nested string-genus

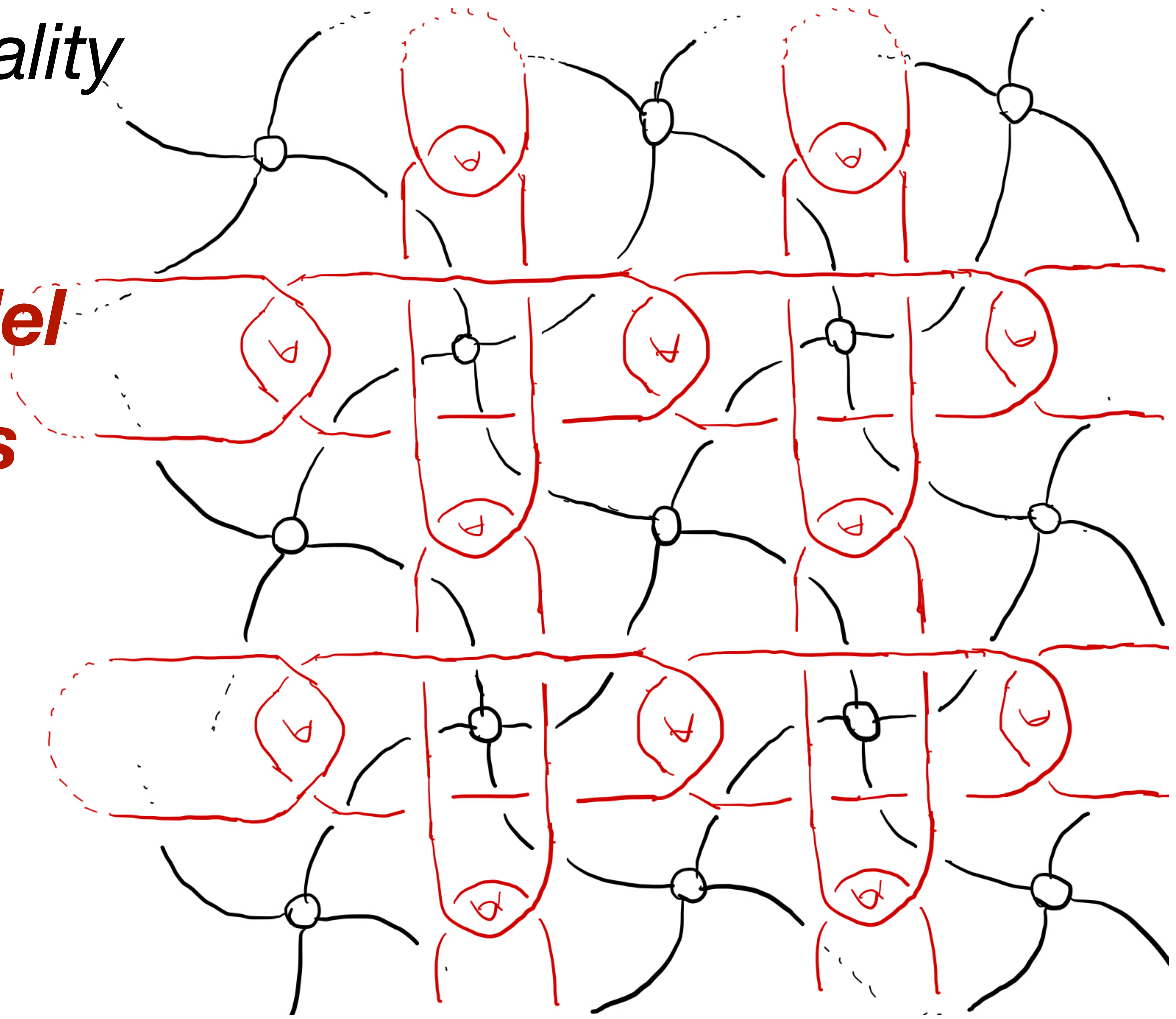


Not make sense so far, the rest of the talk is to make it clear

Motivation

Fractionalizing qubits in tensor network

Generalization of Kramers-Wannier duality



involving magnetic field in Ising model

involving interacting-fermionic terms

Not make sense so far, the rest of the talk is to make it clear

Motivation

Fractionalizing qubits in tensor network

Generalization of Kramers-Wannier duality

Even more complicated

Not make sense so far, the rest of the talk is to make it clear

Outline for the rest of the talk

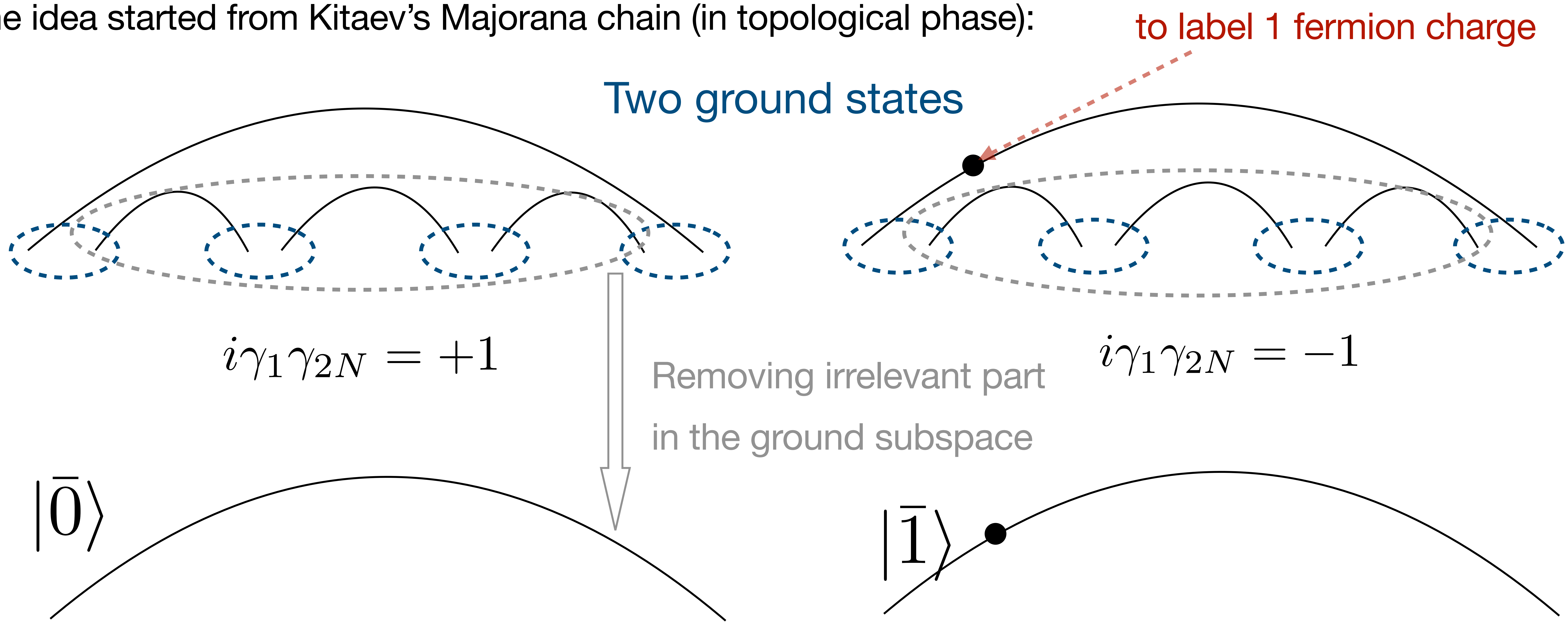
- Pictorial rules from sparse encoding of Majorana zero modes: Z_2 Quon language
- Topological characterization of Ising model on planar graph without magnetic field:
  - Star-Triangle relation, Jordan-Wigner transformation, Kramers-Wannier duality
- **New solvable class:**
  - **Generalizing Kramers-Wannier duality in a topological way**
  - **Interpretation: Ising model with magnetic field / interacting fermion operator**
- Other applications:
  - Matchgate & Clifford circuit and their combination / algorithm: like untying knots
  - a representation suitable for coding, quantum circuit compiler

Basic Rules

Basic Rules

Fractionalizing a qubit

The idea started from Kitaev's Majorana chain (in topological phase):

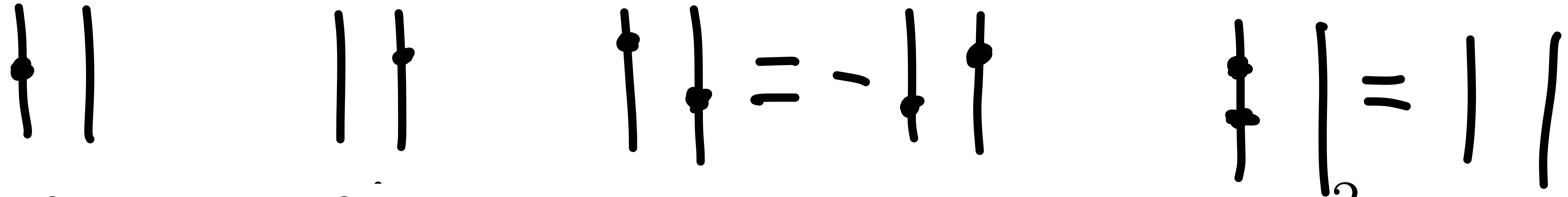


Kitaev, A. Yu. "Unpaired Majorana fermions in quantum wires." *Physics-Uspekhi* 44.10S (2001): 131.

Basic Rules

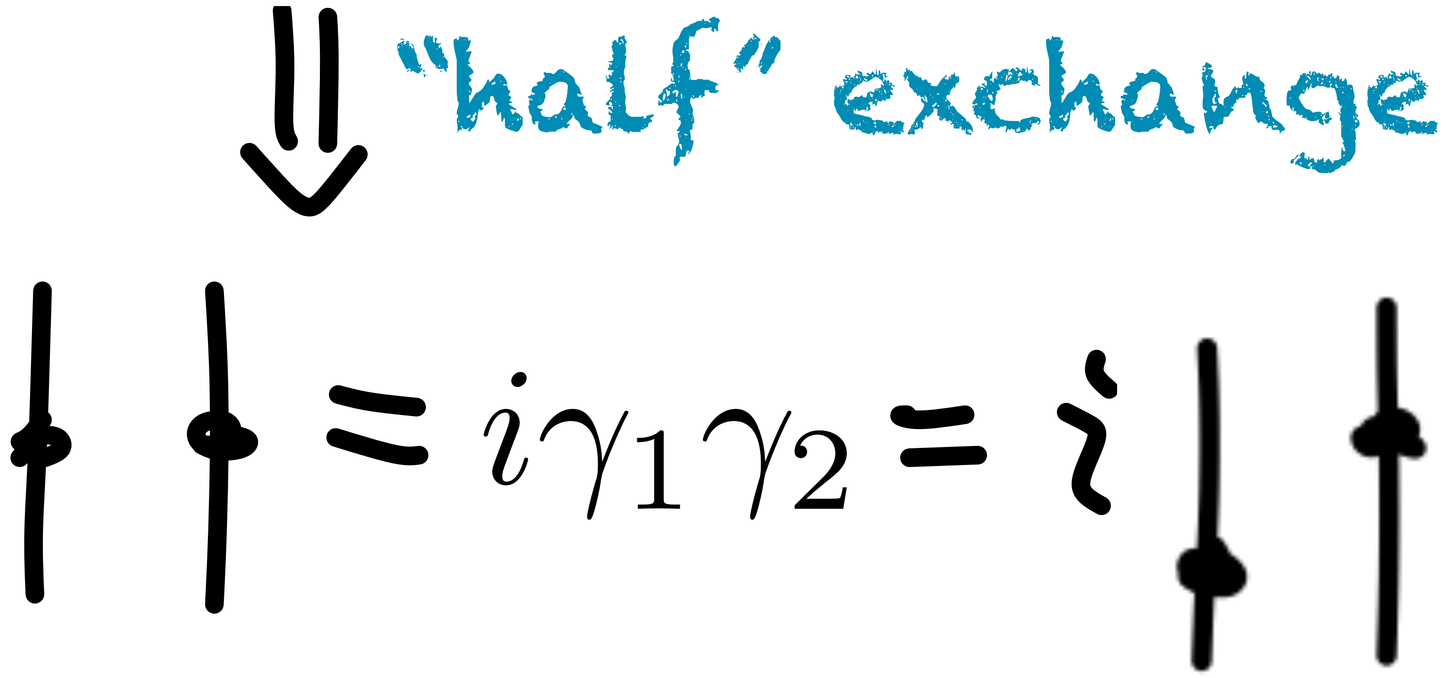
Starting from dense encoding (2 Majorana modes)

Strings as Majorana modes / charges as Majorana operators



$\gamma_1 \qquad \gamma_2 \qquad \gamma_1\gamma_2 = -\gamma_2\gamma_1 \qquad \gamma_1^2 = 1$

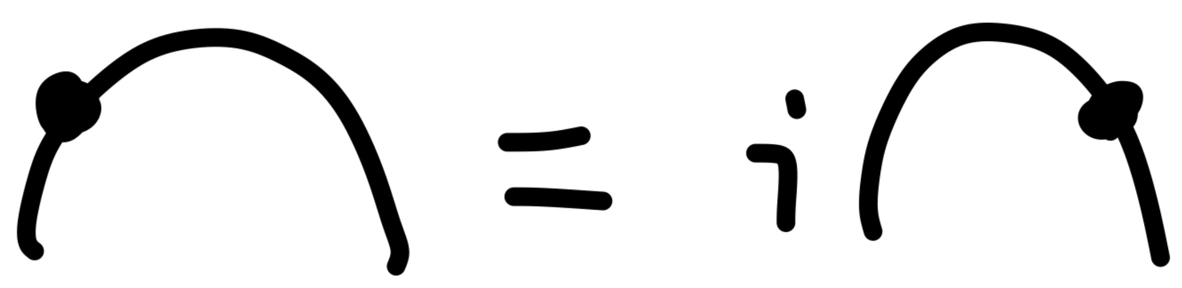
$\{\gamma_i, \gamma_j\} = 2\delta_{ij}$



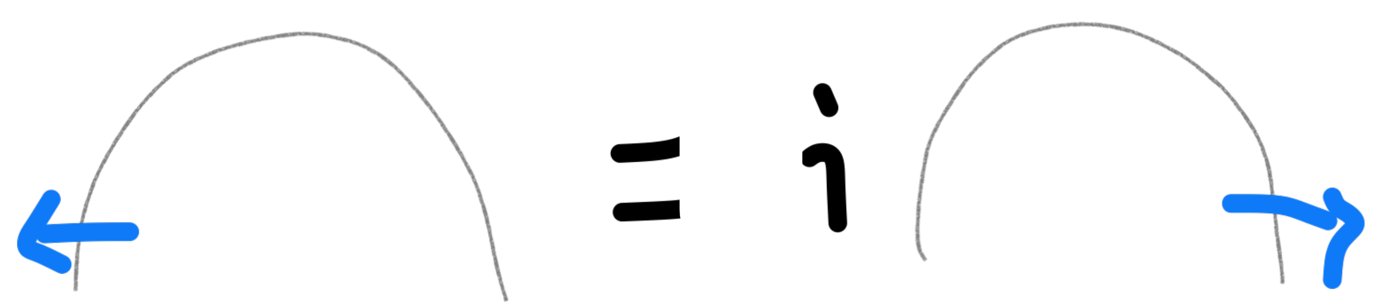
Basic Rules

Starting from dense encoding (2 Majorana modes)

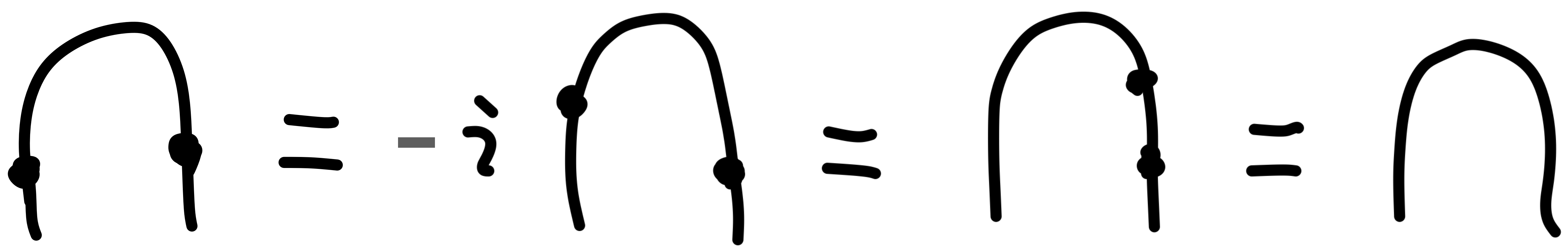
Strings as Majorana modes / charges as Majorana operators



"rotate" the "fermion charge" by 180°



create a pair of "Majorana fermions" from vacuum

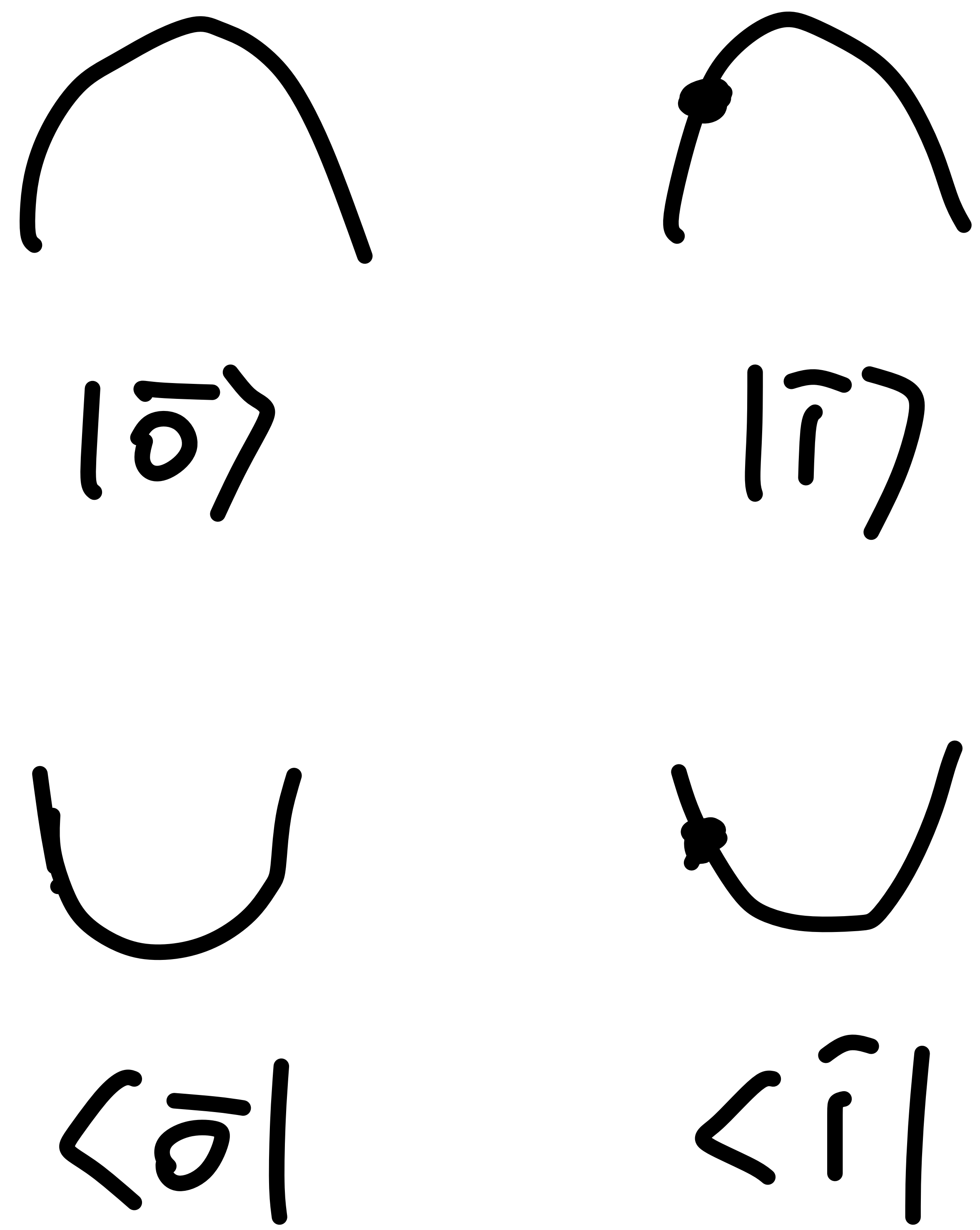


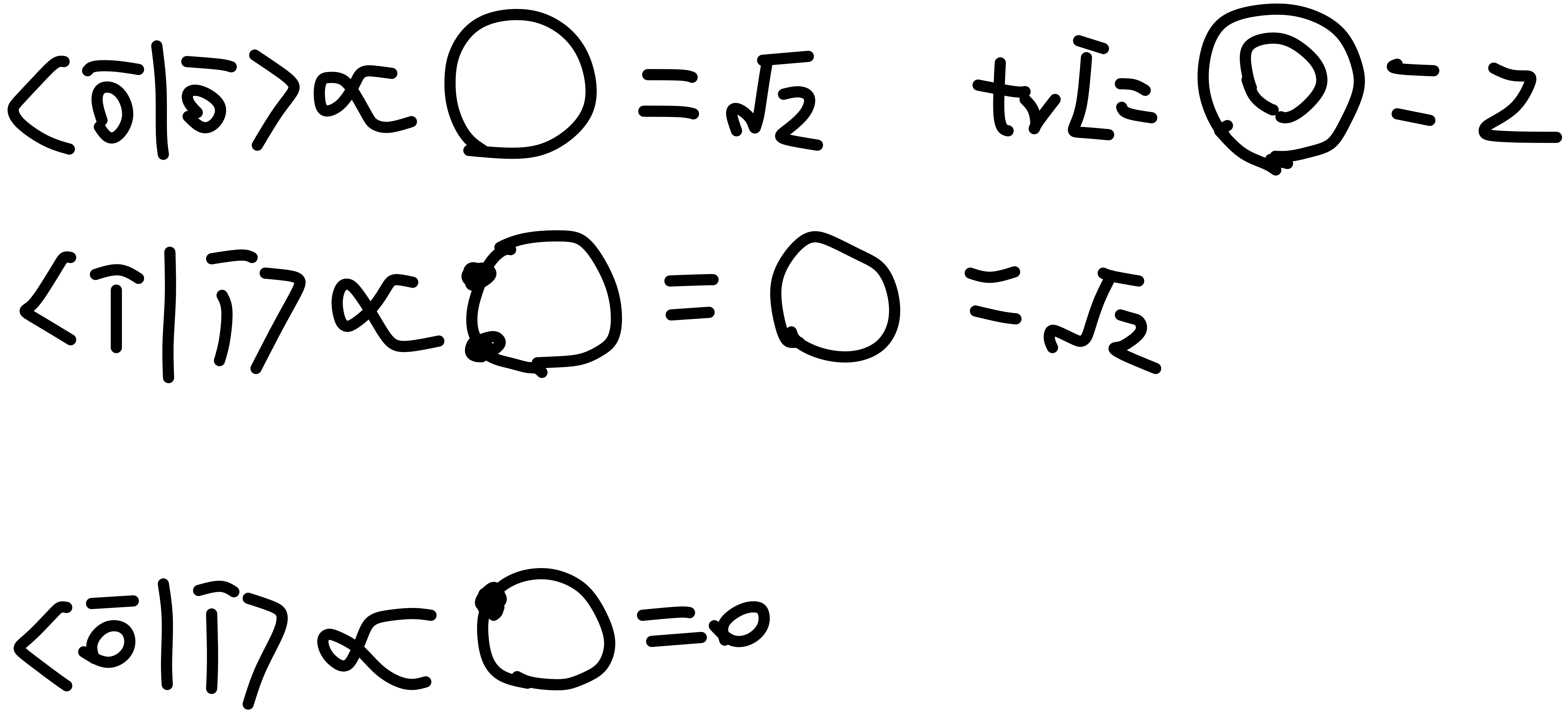
Jaffe, Arthur, Zhengwei Liu, and Alex Wozniakowski. "Holographic software for quantum networks." *Science China Mathematics* 61.4 (2018): 593-626.

Basic Rules

Starting from dense encoding (2 Majorana modes)

Qubits and inner product



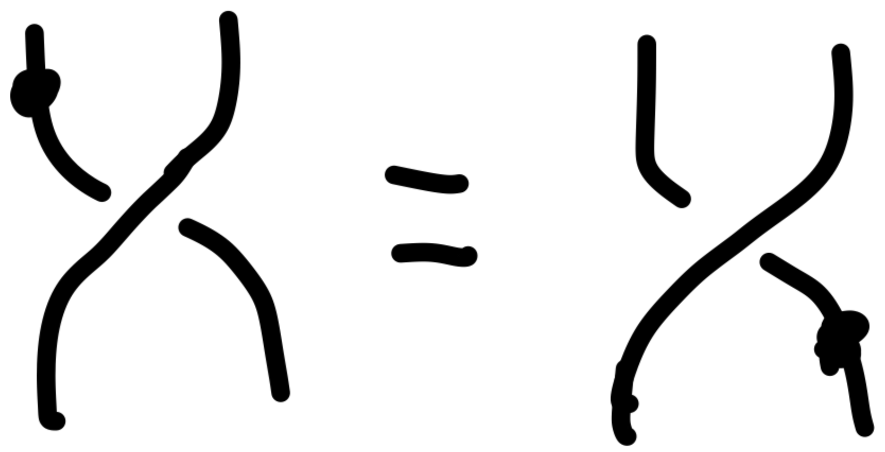


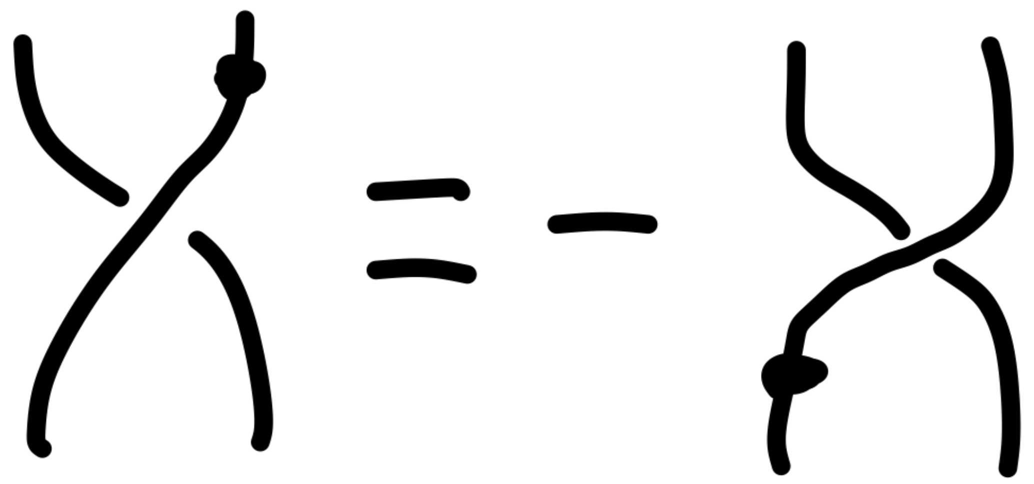
Basic Rules

Starting from dense encoding (2 Majorana modes)

Braiding

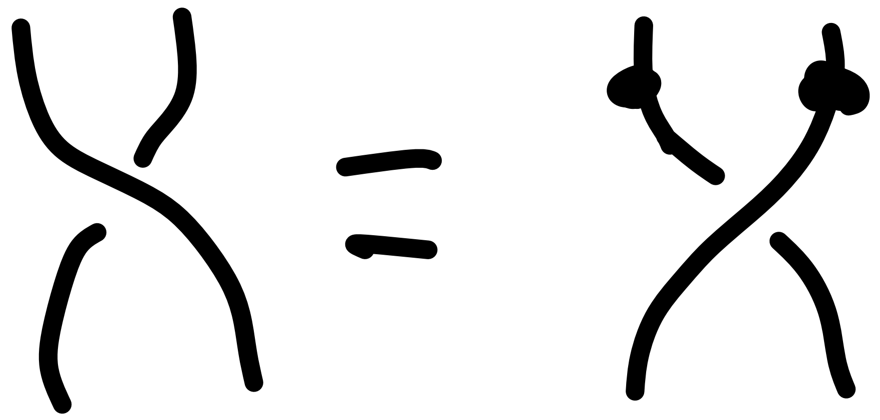
charges can almost freely move !





$$\gamma_1 \to \gamma_2$$
$$\gamma_2 \to -\gamma_1$$

braiding of Majorana fermion

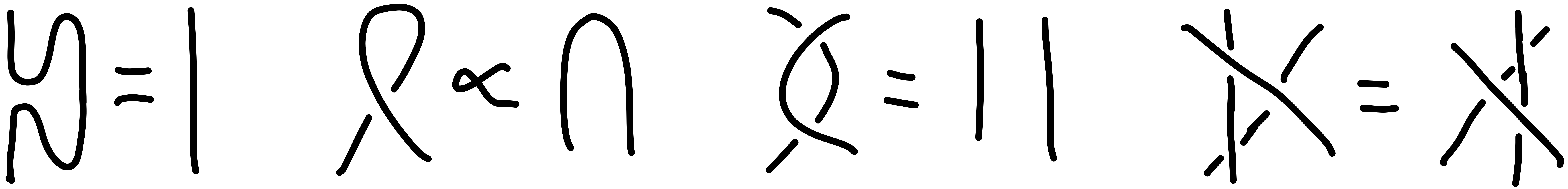


exchange the two braidings up to extra two charges

Basic Rules

Starting from dense encoding (2 Majorana modes)

Isotopy and Redemeister moves



Because of world line of extreme points in Kitaev's Majorana chain

However, easier than ordinary knots:

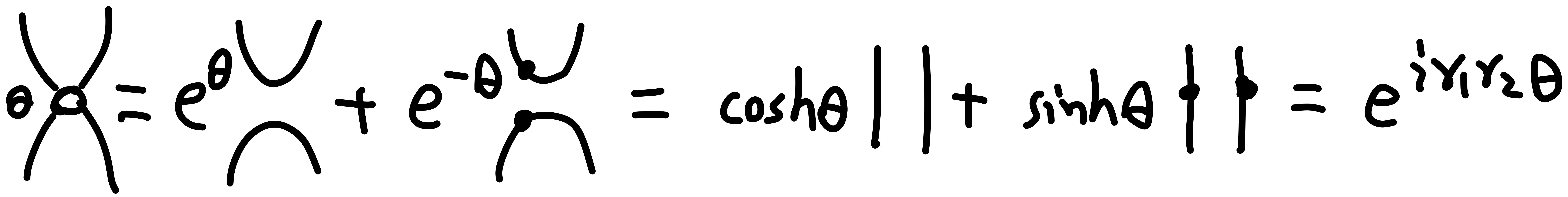
up to almost freely-movable charges, all the diagram are trivial "links"

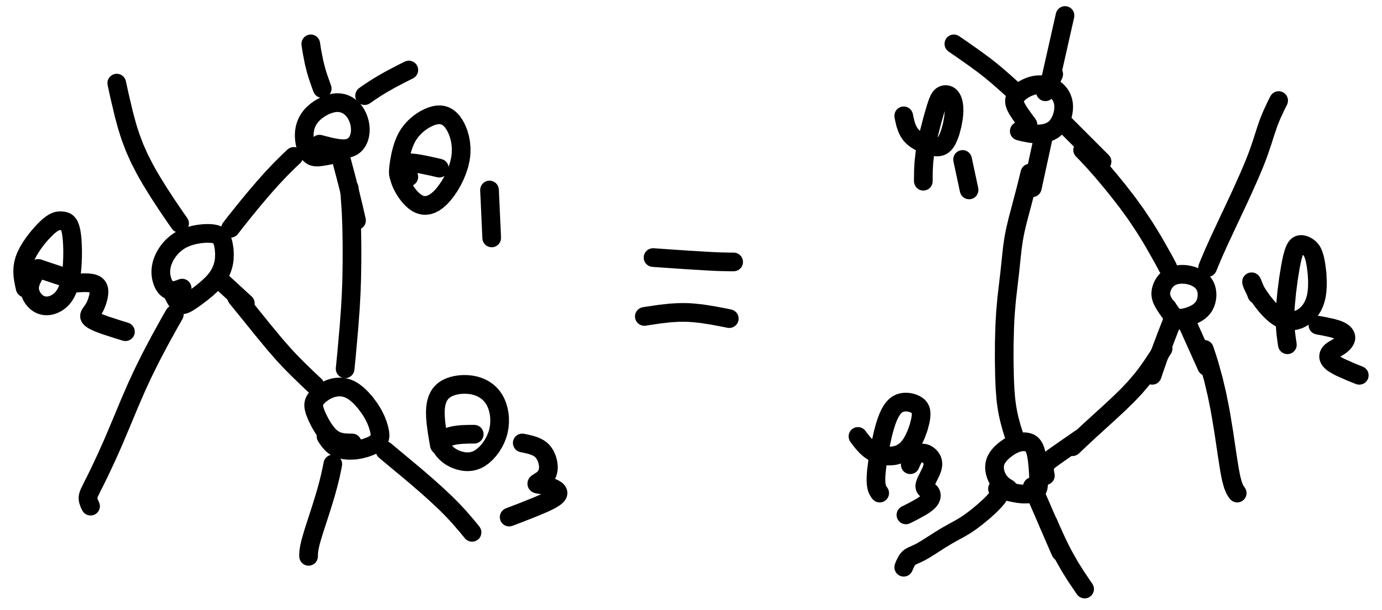
Basic Rules

Starting from dense encoding (2 Majorana modes)

General superposition (parameterized crossing)

Yang Baxter equation as Euler decomposition





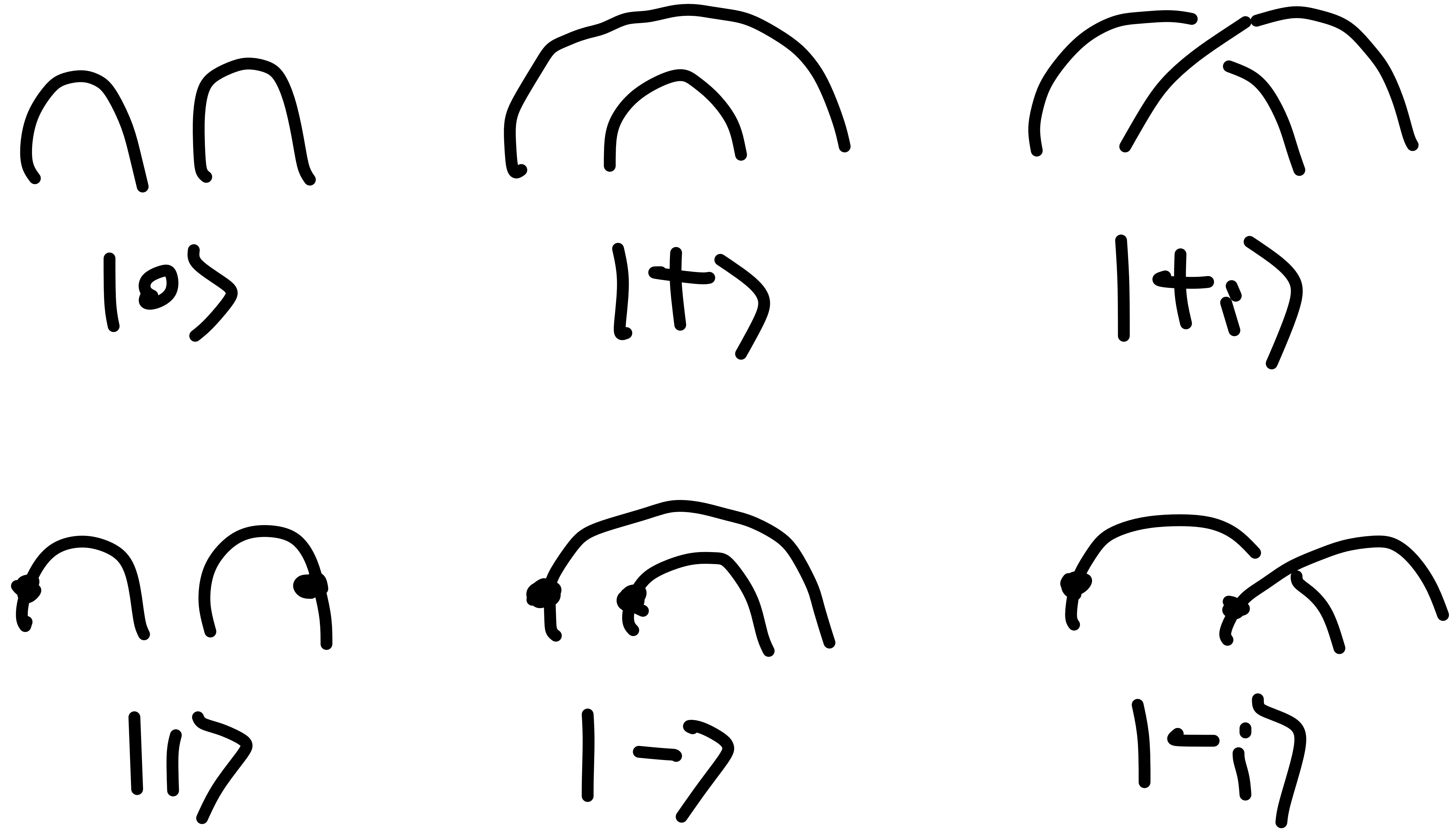
$i\gamma_1\gamma_2,\ i\gamma_2\gamma_3,\ i\gamma_1\gamma_3$ spinor representation of so(3) generators

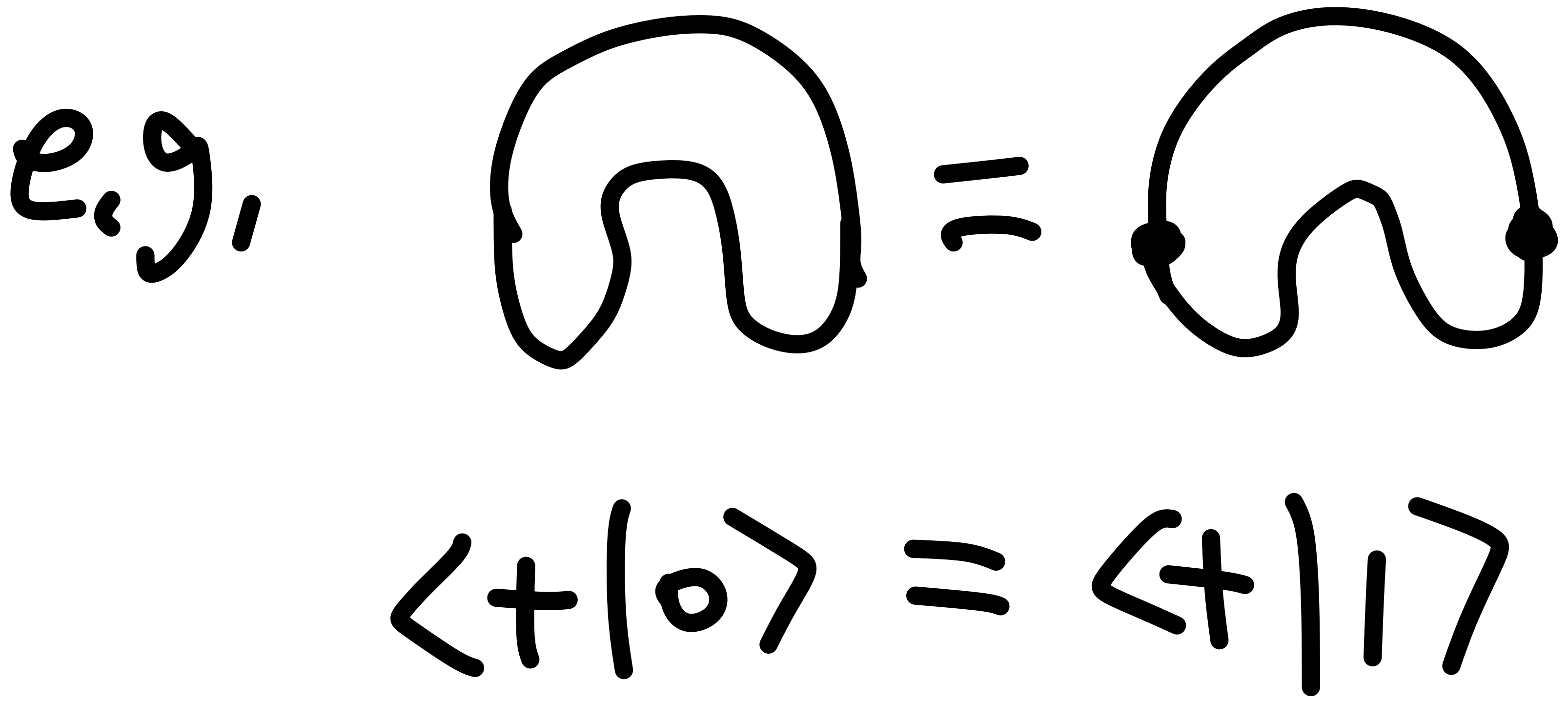
2 ways of Euler decomposition as Yang-Baxter equation !

Basic Rules

Sparse encoding of 4 Majorana modes (Quon on Z_2)

Qubits





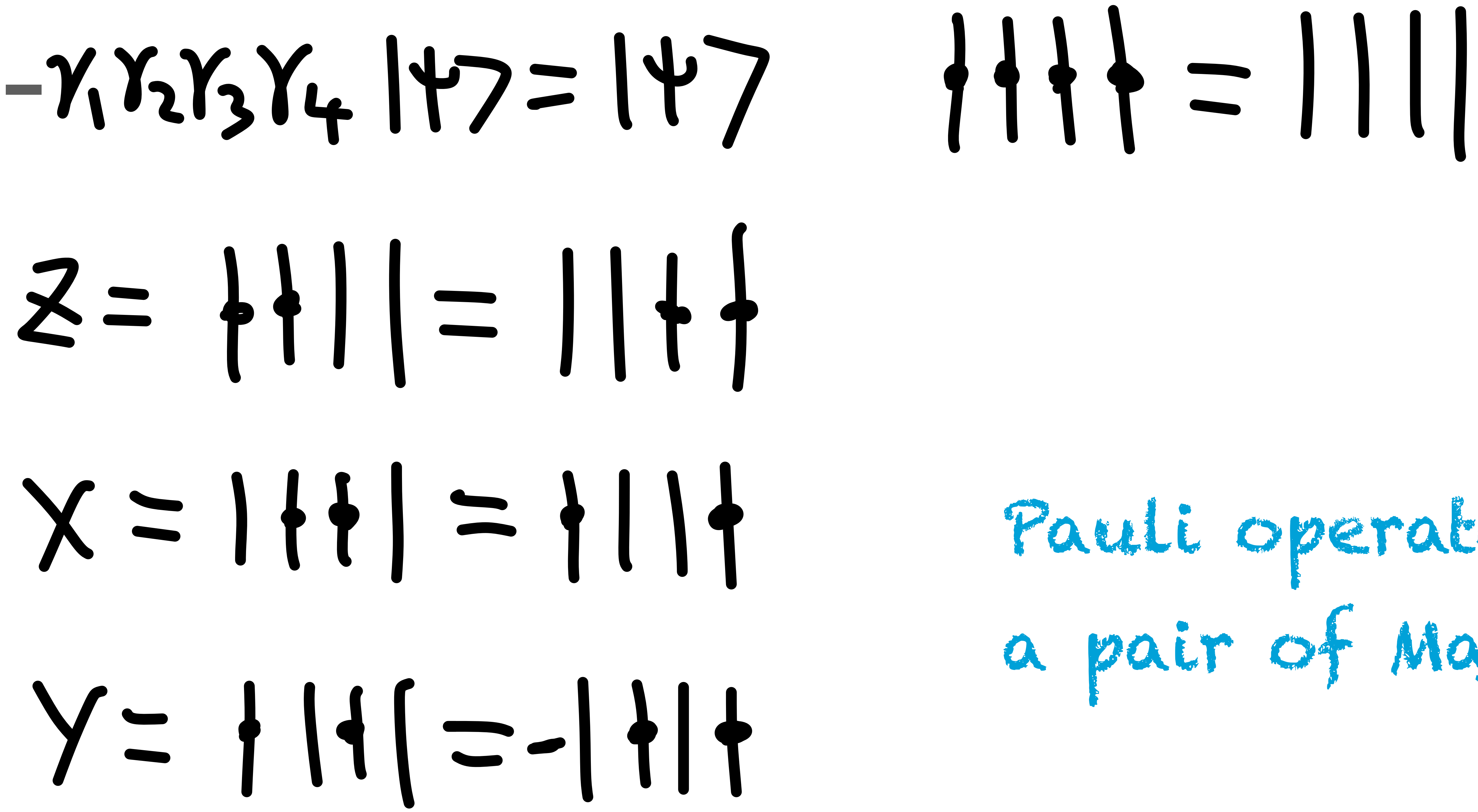
The reason why it is |+⟩

Sarma, Sankar Das, Michael Freedman, and Chetan Nayak. "Majorana zero modes and topological quantum computation." *npj Quantum Information* 1.1 (2015): 1-13.

Basic Rules

Sparse encoding of 4 Majorana modes (Quon on Z_2)

Kitaev's encoding of Pauli operators



Kitaev, Alexei. "Anyons in an exactly solved model and beyond." *Annals of Physics* 321.1 (2006): 2-111.

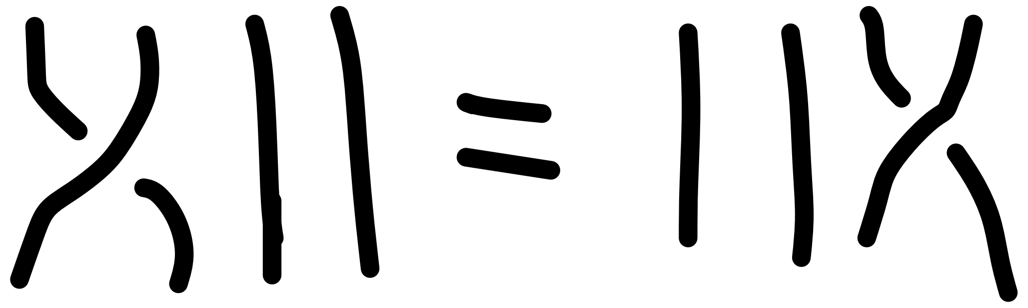
Pauli operators:

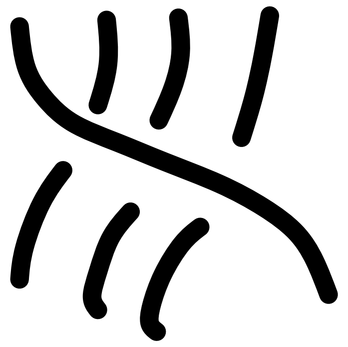
a pair of Majorana operators / charges

Basic Rules

Sparse encoding of 4 Majorana modes (Quon on Z_2)

Other single qubit Clifford gates

S =  $= \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

H =  $= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

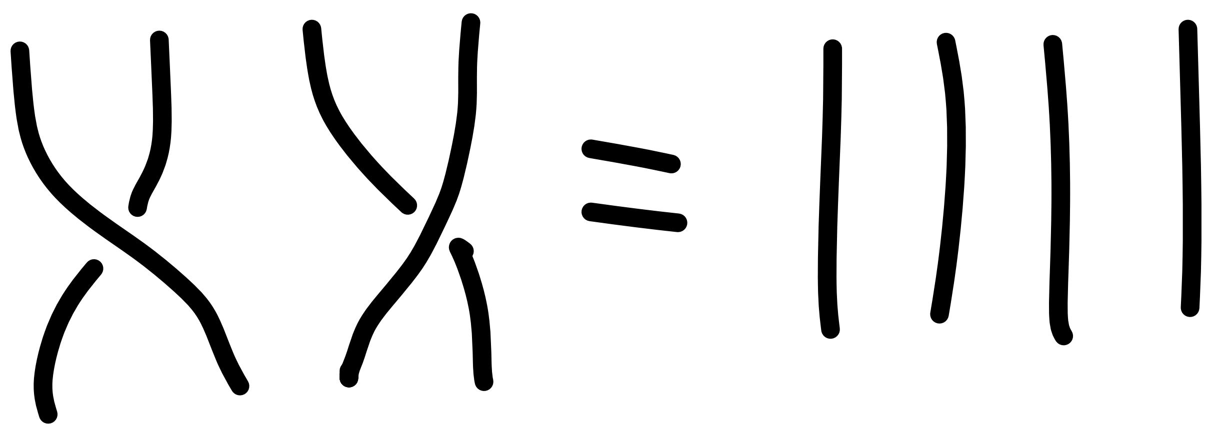
e.g.,

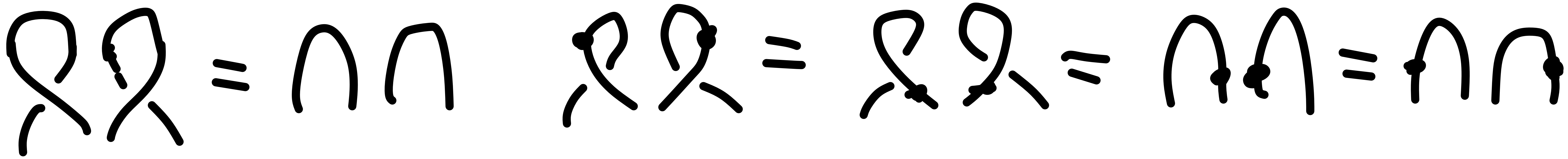
$H|1\rangle = |-\rangle$

Basic Rules

Sparse encoding of 4 Majorana modes (Quon on Z_2)

A redundant degree of freedom





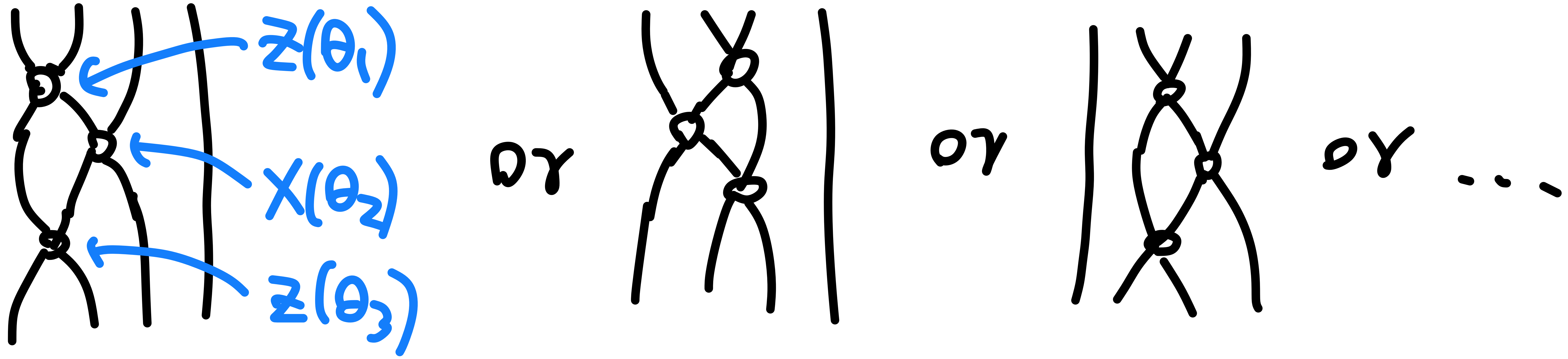
originate from the sparse encoding

important for evaluating Clifford circuits

Basic Rules

Sparse encoding of 4 Majorana modes (Quon on Z_2)

General single qubit gate SU(2)



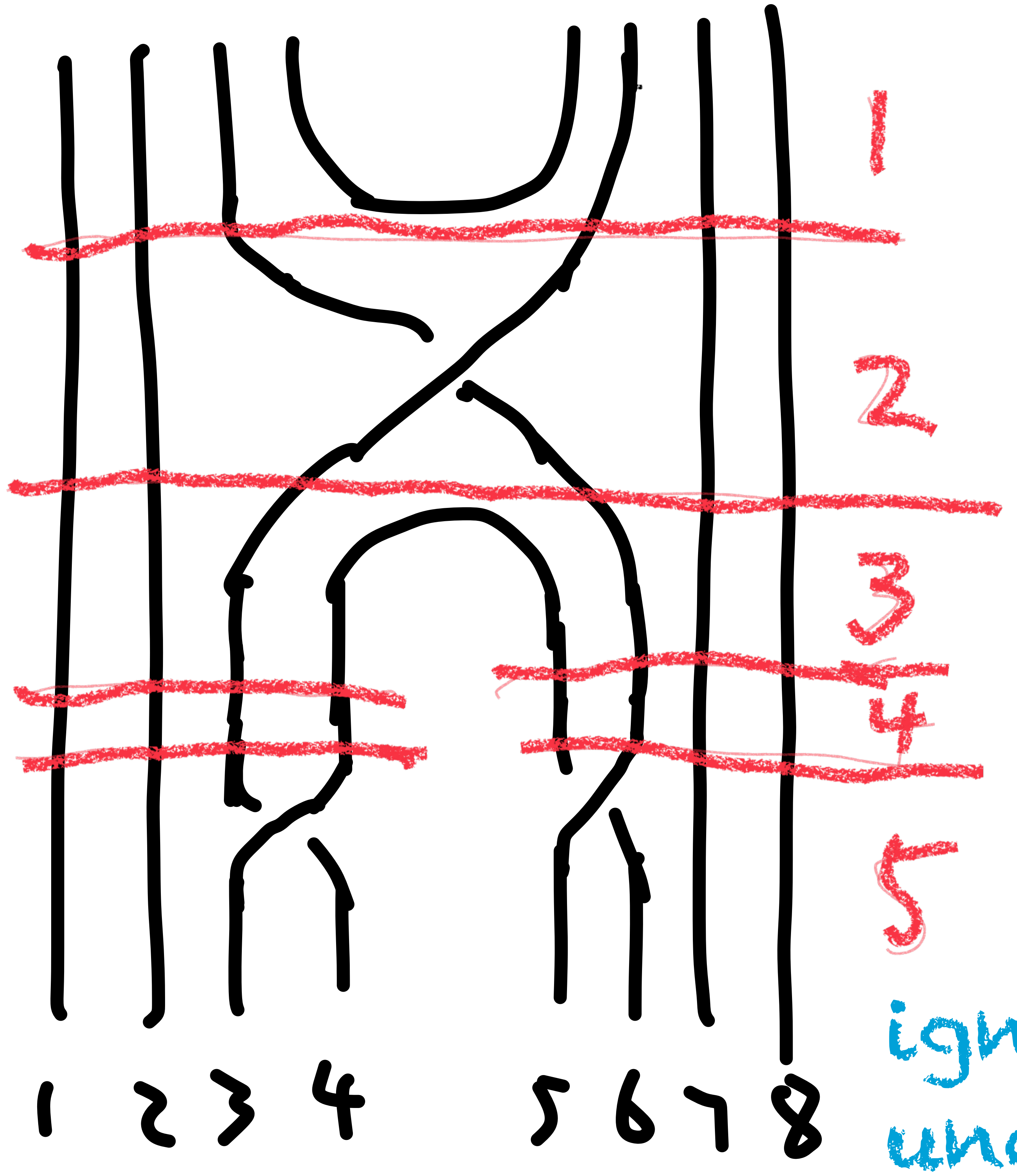
3 modes contain complete information

1 modes is redundant

Basic Rules

Sparse encoding of 4 Majorana modes (Quon on Z_2)

Control-Z (operational meaning as Majorana zero modes)



1. measure $i\gamma_4\gamma_5 = +1$
2. braid modes 3,6
3. prepare states with $i\gamma_4\gamma_5 = +1$
4. measure parity for 1,2,3,4 and 5,6,7,8
5. braid modes 3,4 and 5,6

ignore correction due to undesired measurement result

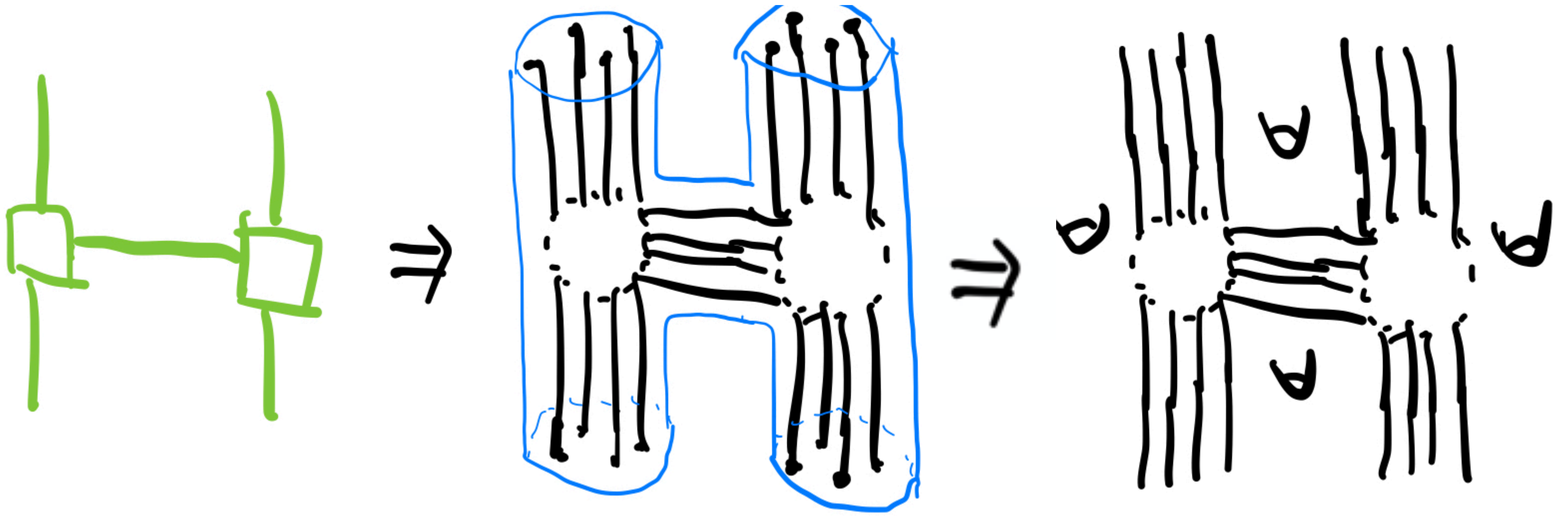
Sarma, Sankar Das, Michael Freedman, and Chetan Nayak. "Majorana zero modes and topological quantum computation." *npj Quantum Information* 1.1 (2015): 1-13.

Basic Rules

Sparse encoding of 4 Majorana modes (Quon on Z_2)

Multi-qubits and Genus

Graph (TN) => 3D Manifold (4 strings in 1 tube)



a convenient way to draw

parity constraints for each 4 modes (drawn in 1 tube)

strings living in 3-manifold with genus

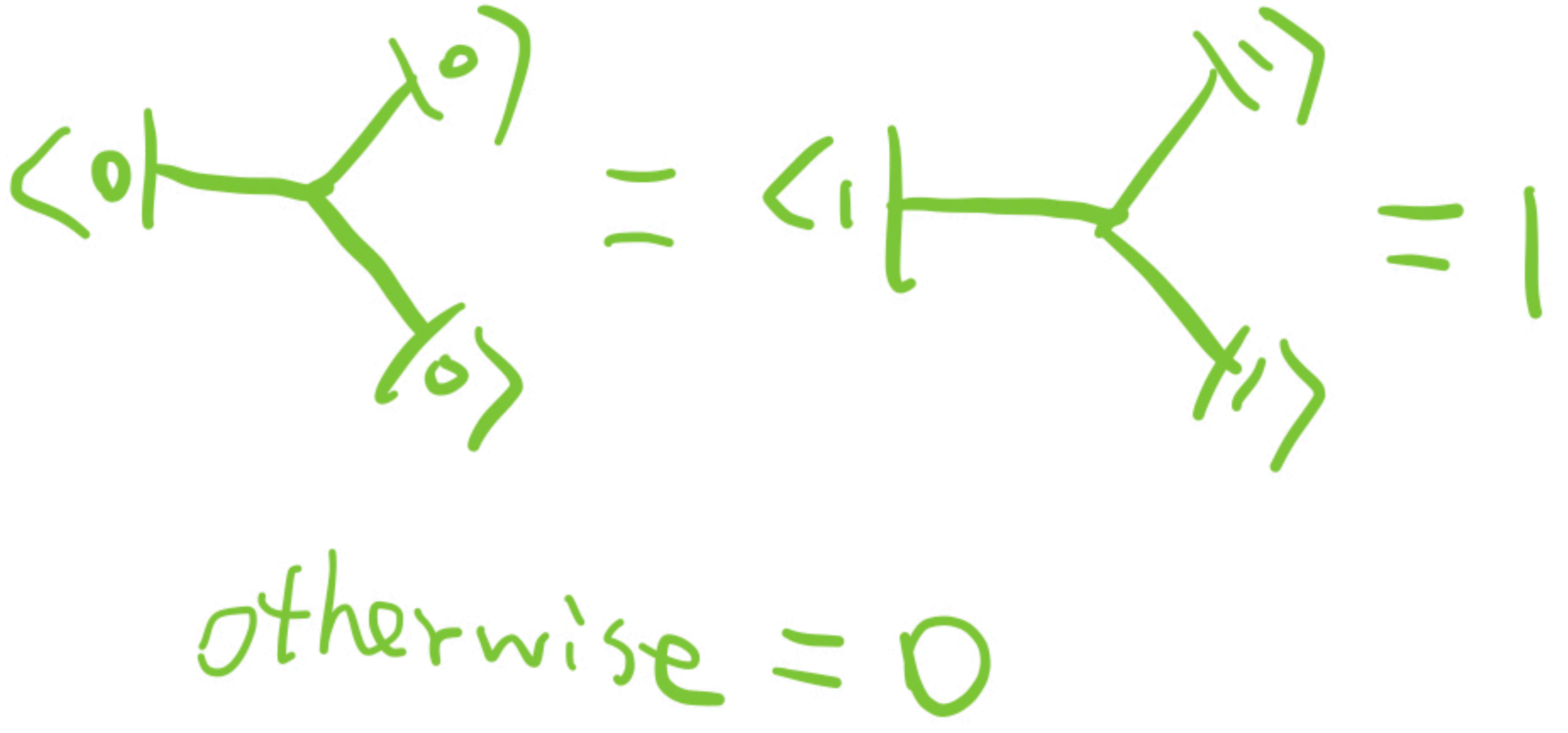
Liu, Zhengwei, Alex Wozniakowski, and Arthur M. Jaffe. "Quon 3D language for quantum information." *Proceedings of the National Academy of Sciences* 114.10 (2017): 2497-2502.

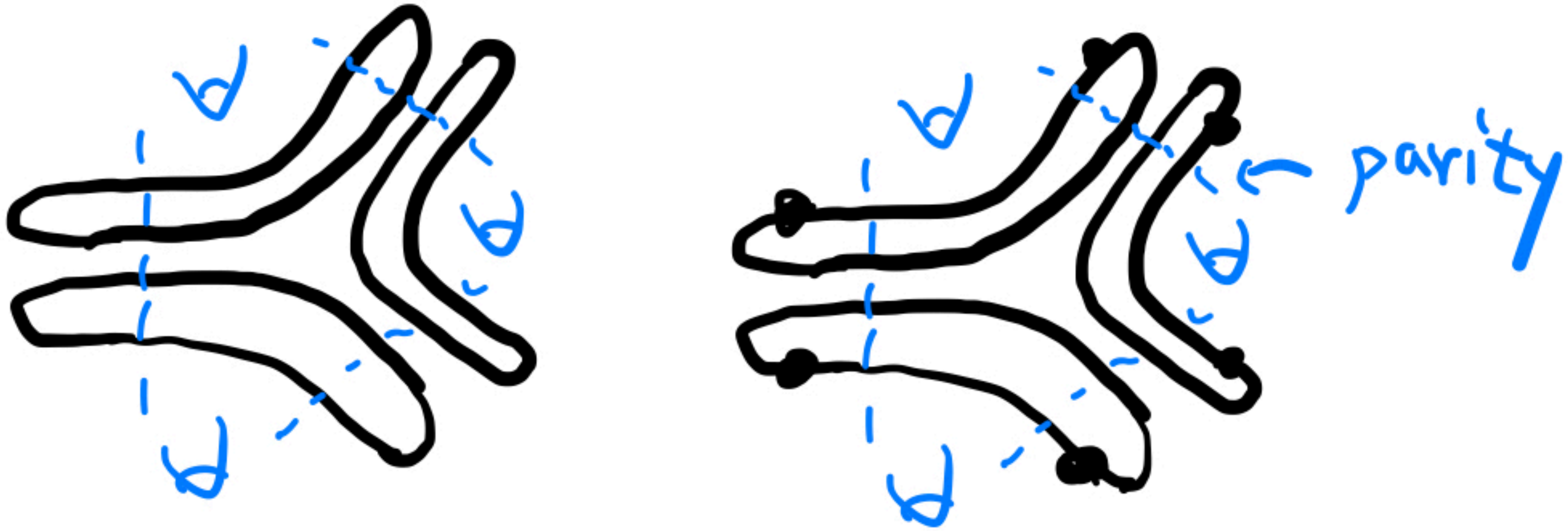
Liu Z. Quon language: Surface algebras and fourier duality. Communications in Mathematical Physics. 2019 Mar 1;366(3):865-94.

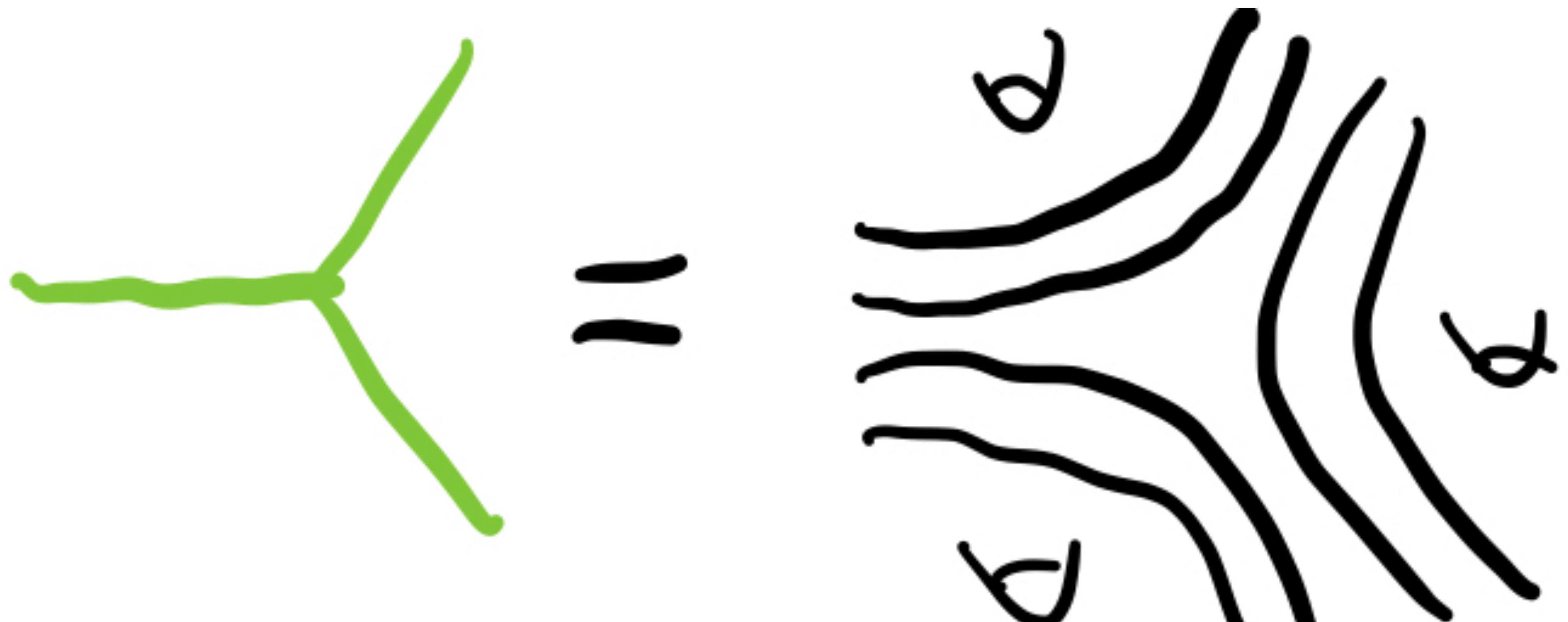
Basic Rules

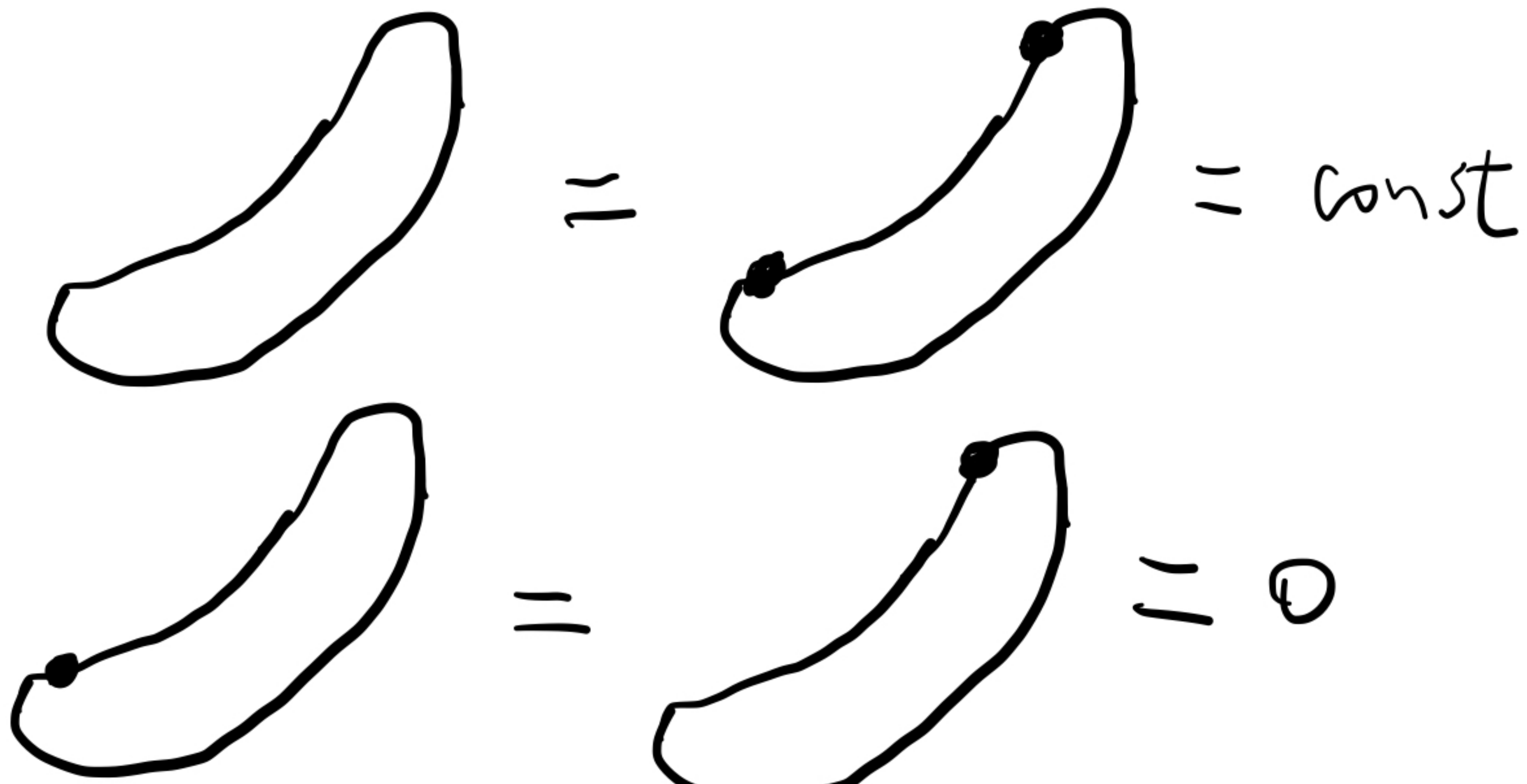
Sparse encoding of 4 Majorana modes (Quon on Z_2)

Copy tensor





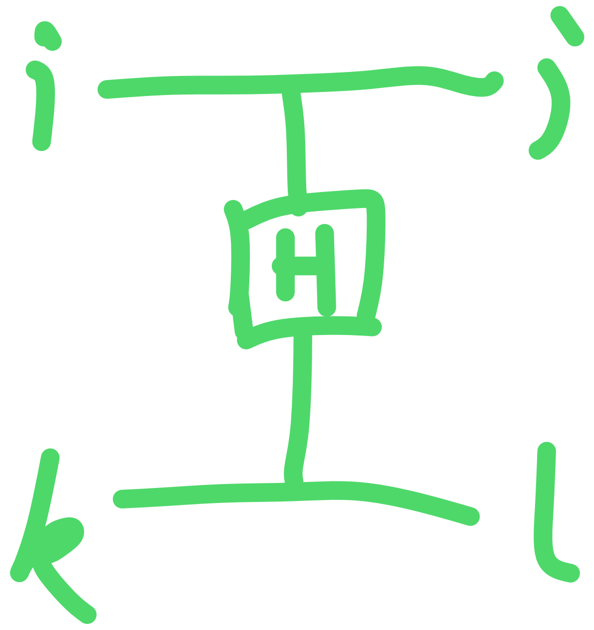




Basic Rules

Sparse encoding of 4 Majorana modes (Quon on Z_2)

Control-Z (from copy tensor and Hadamard gate)



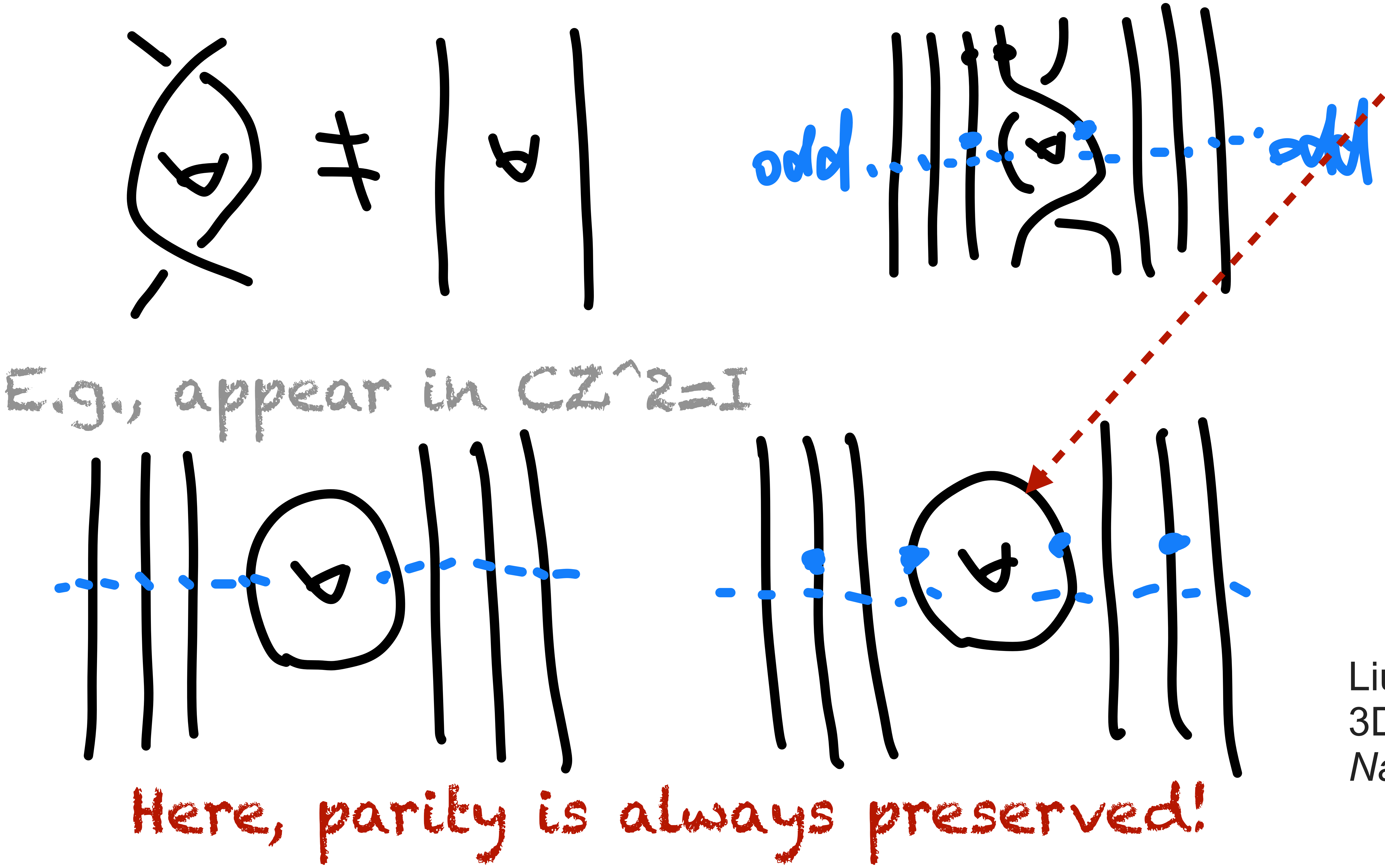
$$\propto \delta_{ij}\delta_{kl}(-1)^{ik} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

control-Z

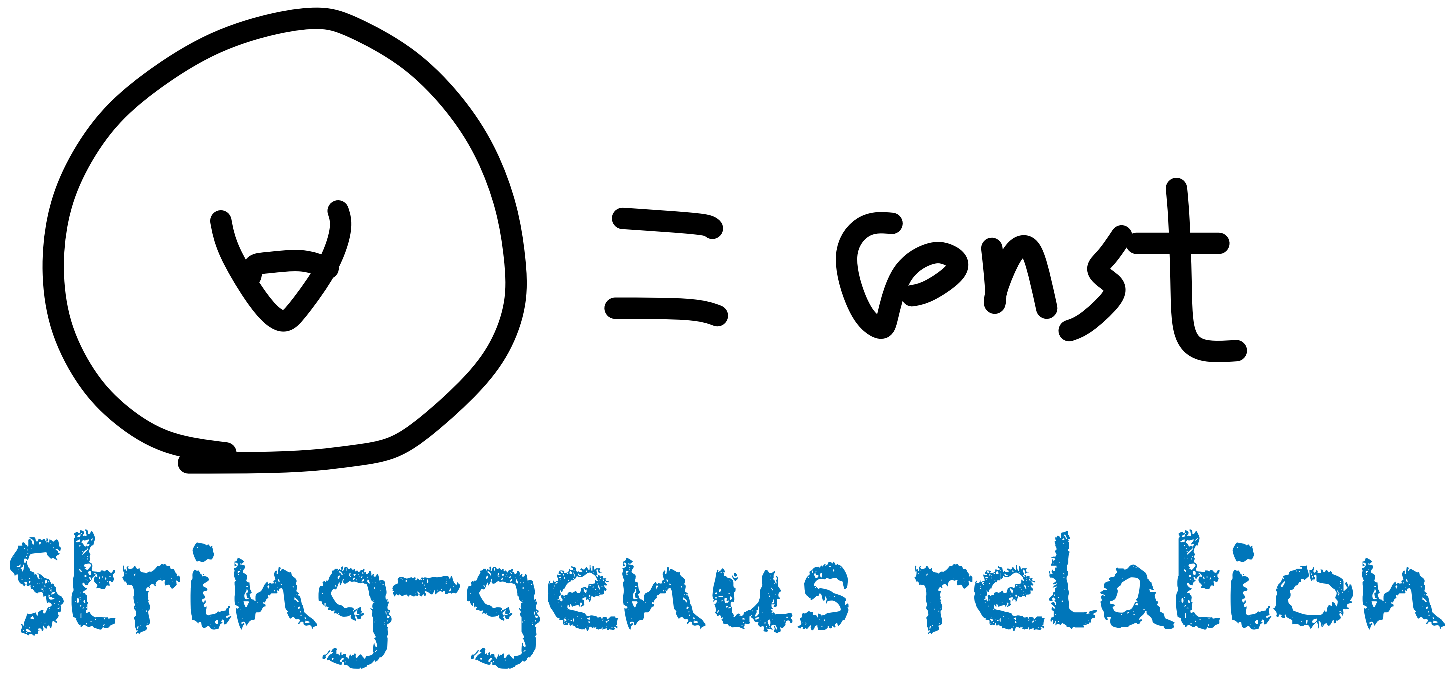
Basic Rules

Sparse encoding of 4 Majorana modes (Quon on Z_2)

String-genus relation



prepare and measure:
could be removed as long as it preserves sparse encoding at any time slice

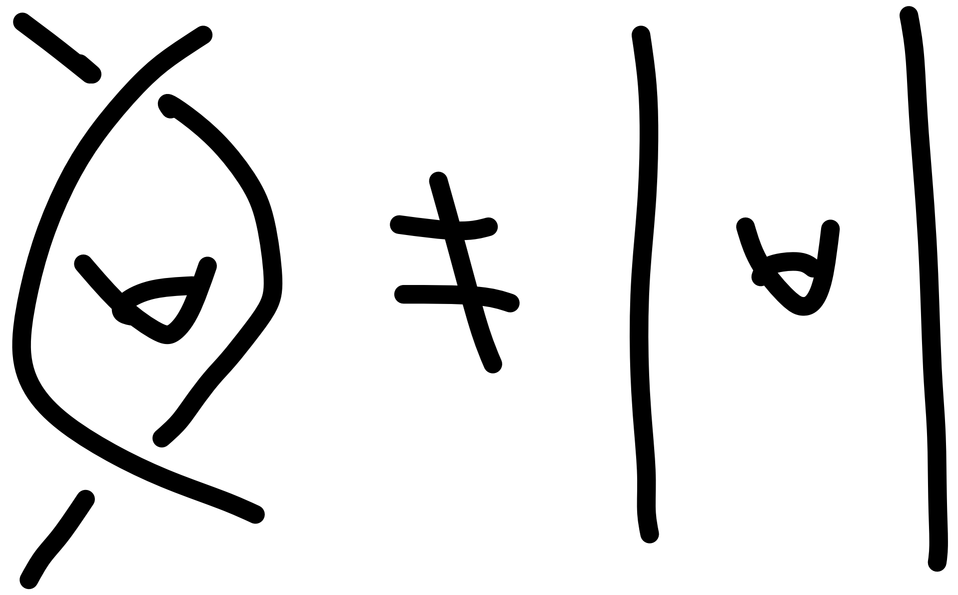


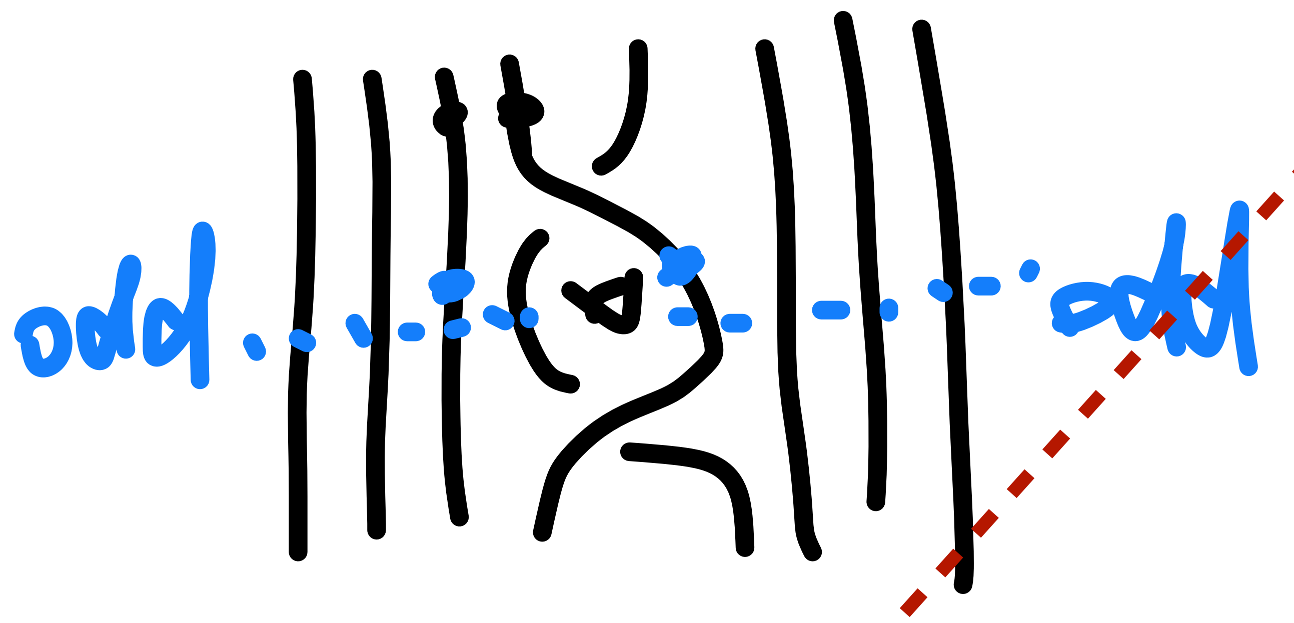
Liu, Zhengwei, Alex Wozniakowski, and Arthur M. Jaffe. "Quon 3D language for quantum information." *Proceedings of the National Academy of Sciences* 114.10 (2017): 2497-2502.

Basic Rules

Sparse encoding of 4 Majorana modes (Quon on Z_2)

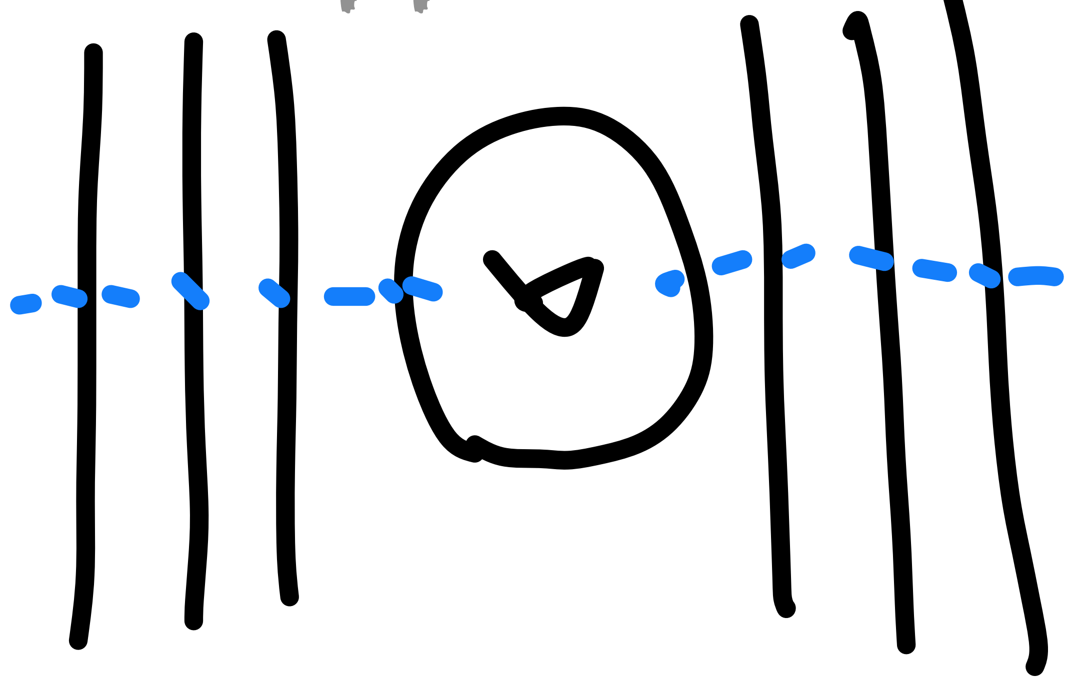
String-genus relation

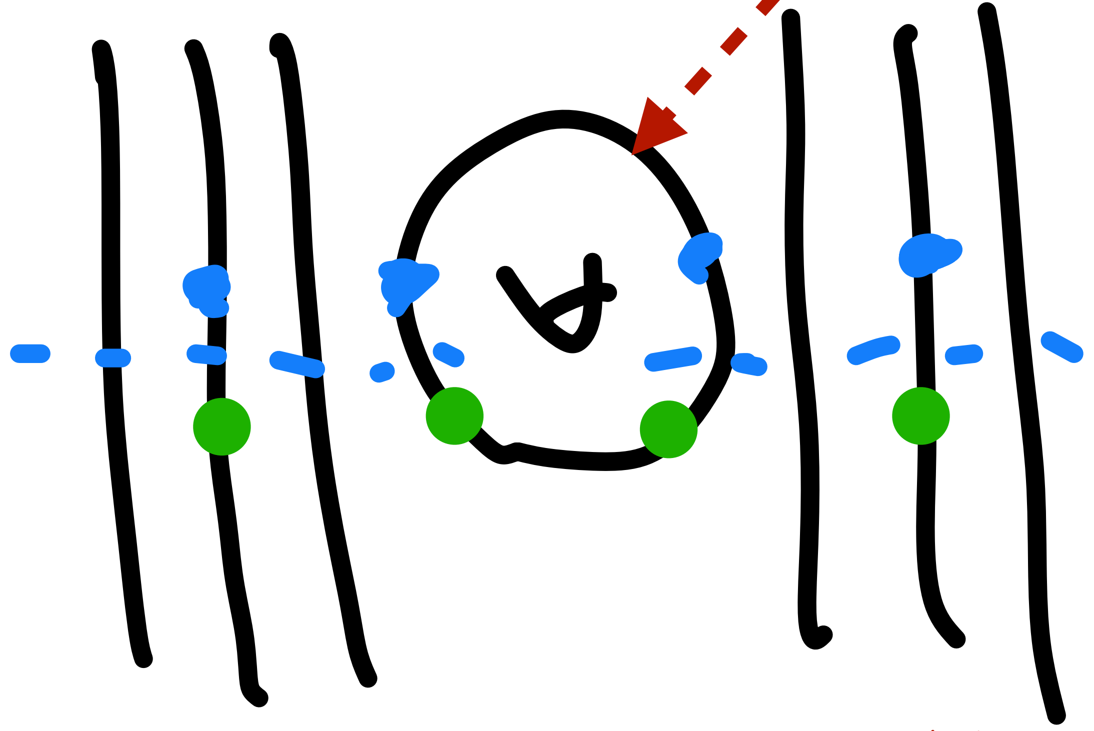




prepare and measure:
could be removed as long as it preserves sparse encoding at any time slice

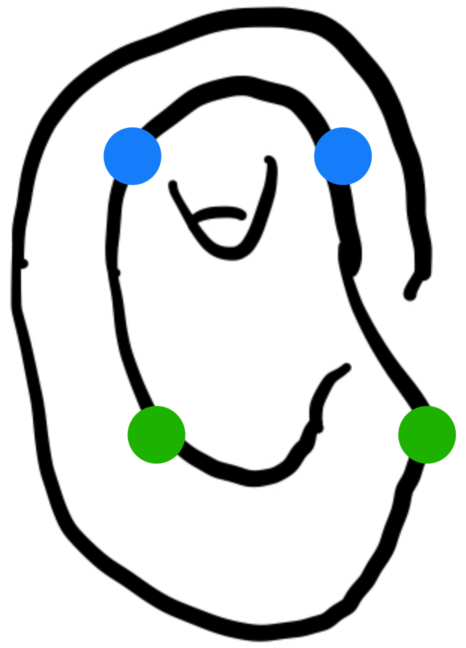
E.g., appear in CZ^2=I





Here, parity is always preserved!

not works



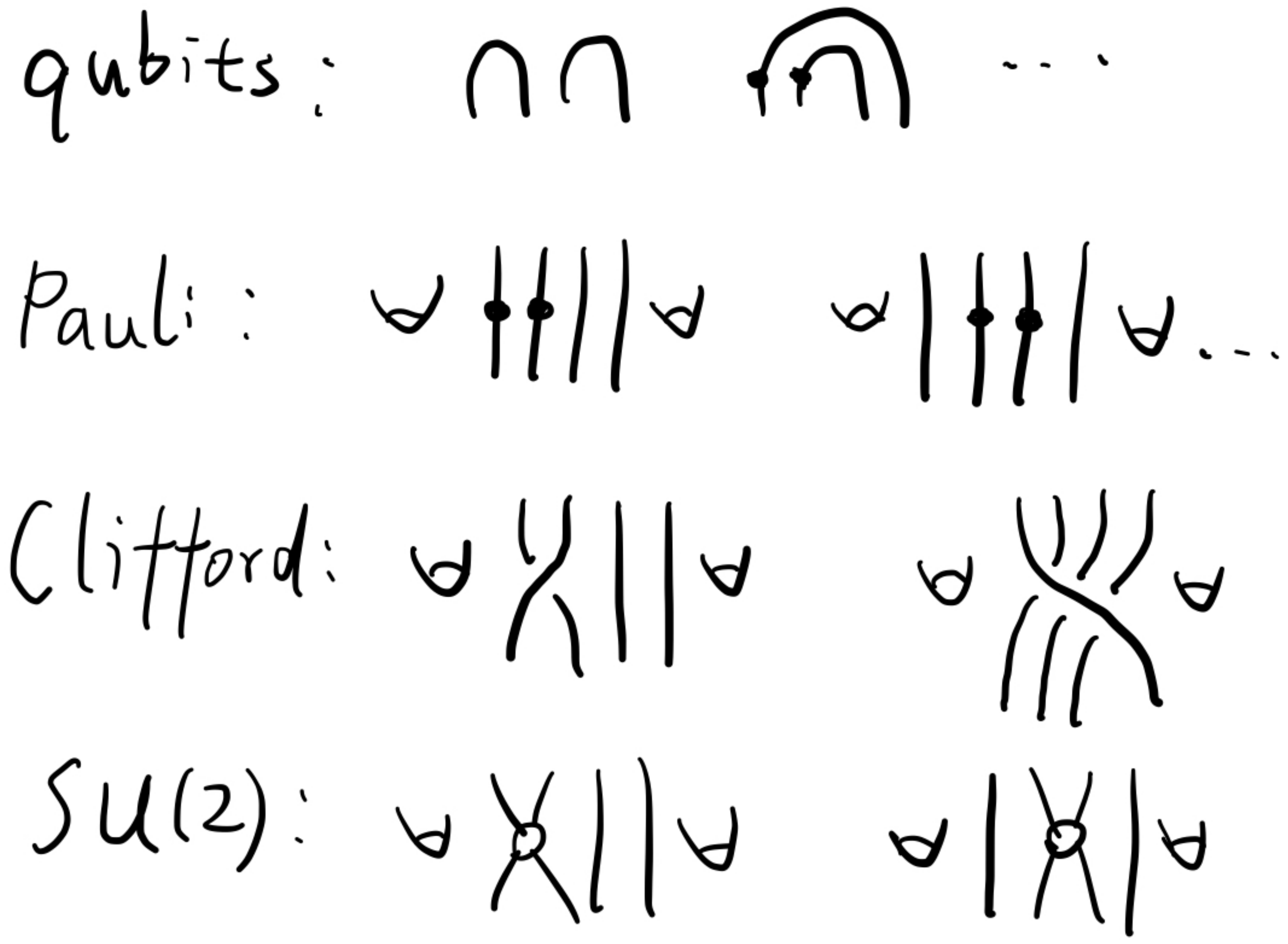
cannot balance the parities aside

Basic Rules

Summary

Basic elements

can also be drawn in 3D



universal for any TN

1 line in TN <=>

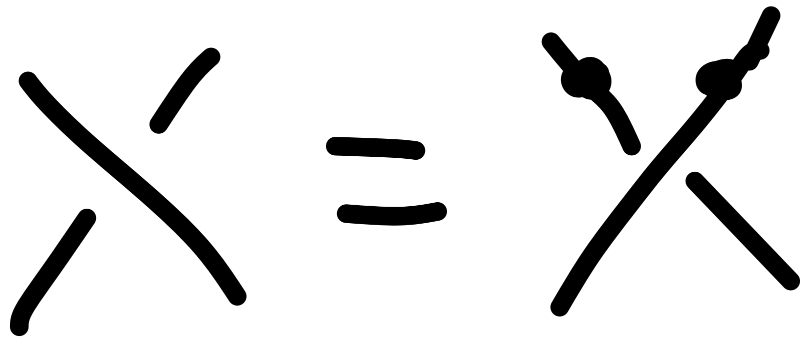
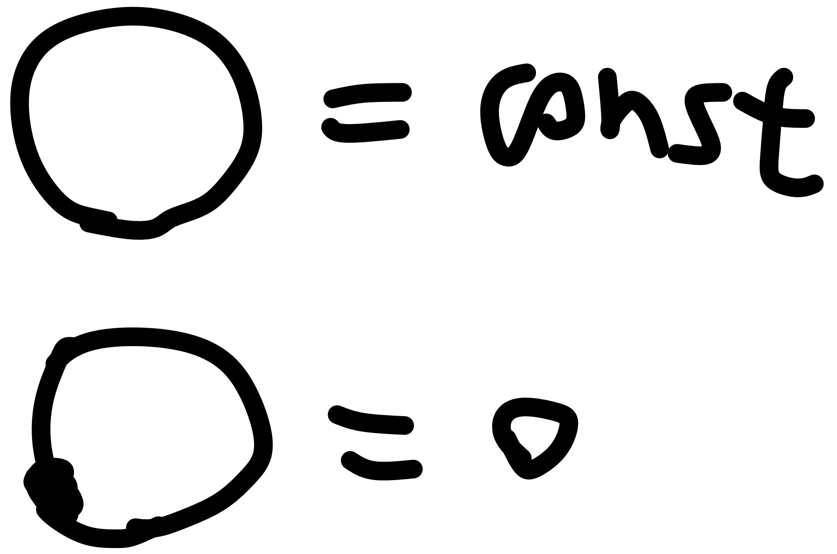
1 tube in Quon

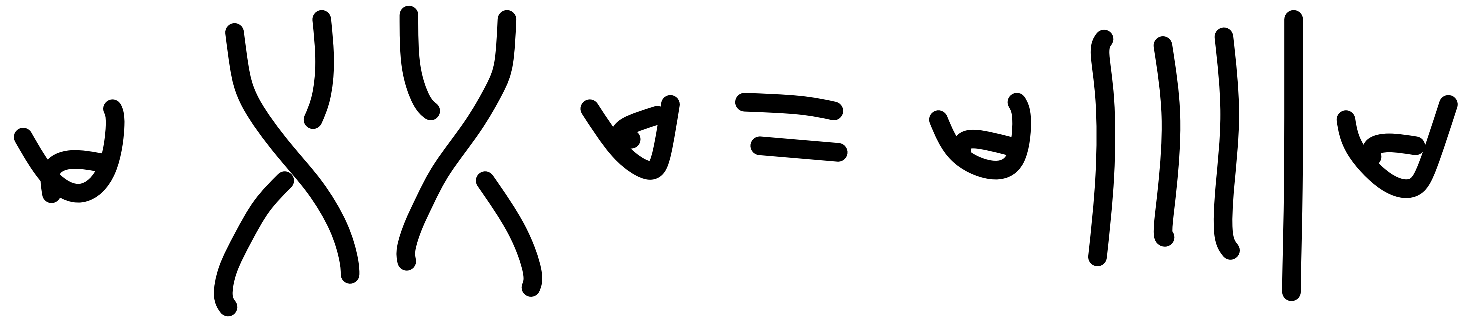
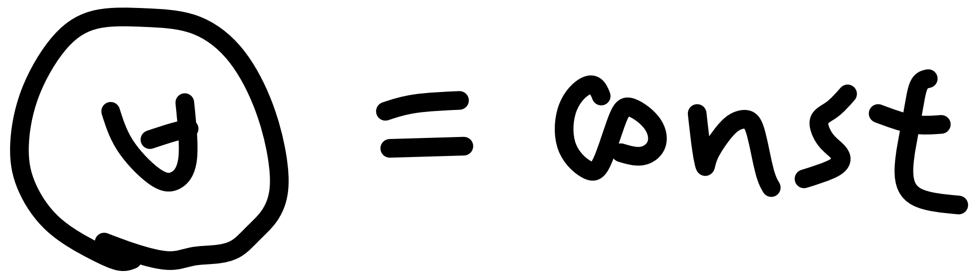
Basic Rules

Summary

Which makes it more powerful than tensor network: fractionalization, topological rules, string genus relation

Evaluation rules

1. Isotopy moves (world line of Majorana zero modes)
2. charges moves (Majorana fermion)
3. 
4. 

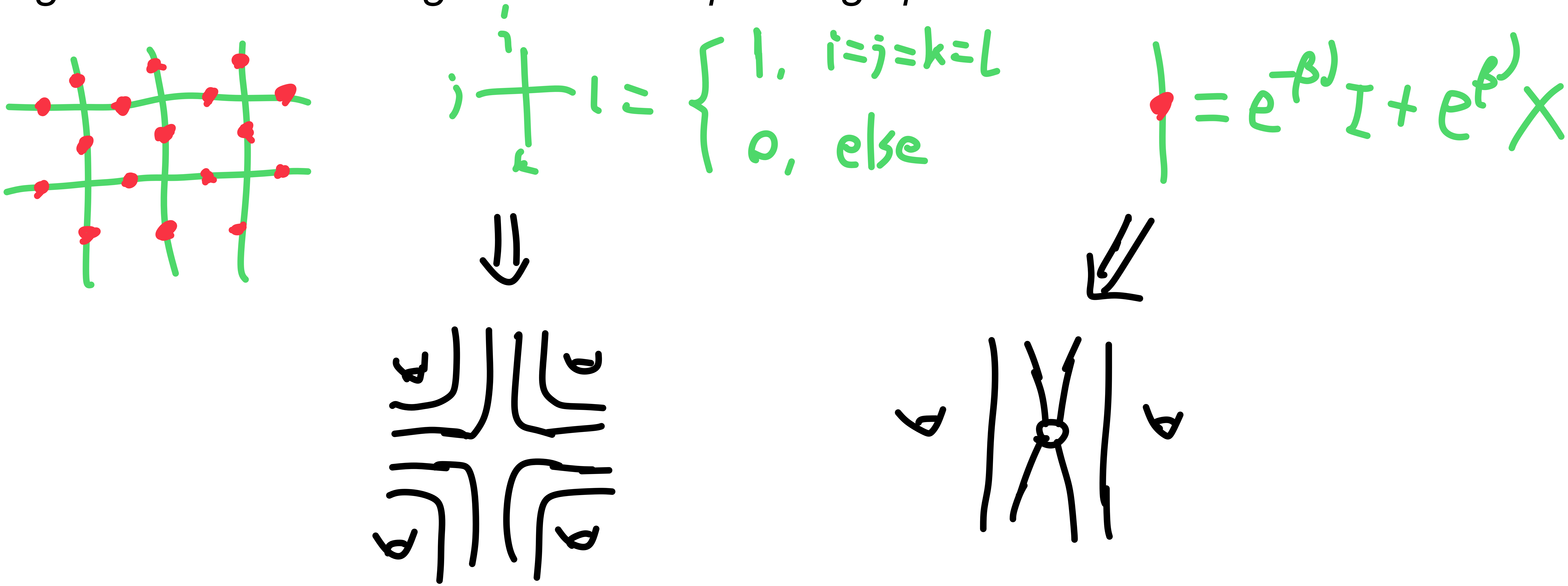
5. 
6. 

2D Ising model:

- Kramers-Wannier duality
- Star-triangle relation
- Jordan-Wigner transformation

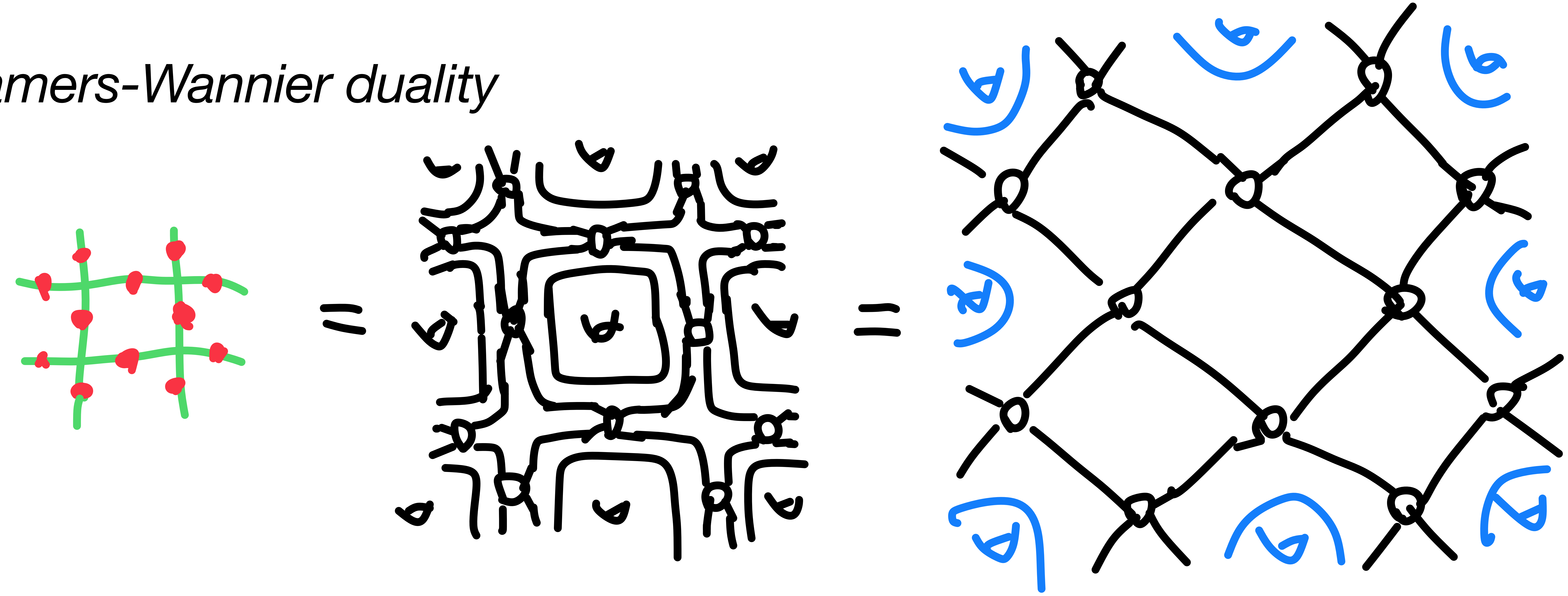
Ising model

Ising model without magnetic field on planar graph

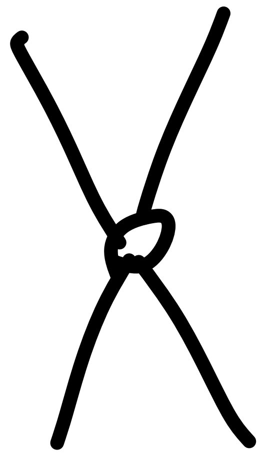


Ising model

Kramers-Wannier duality

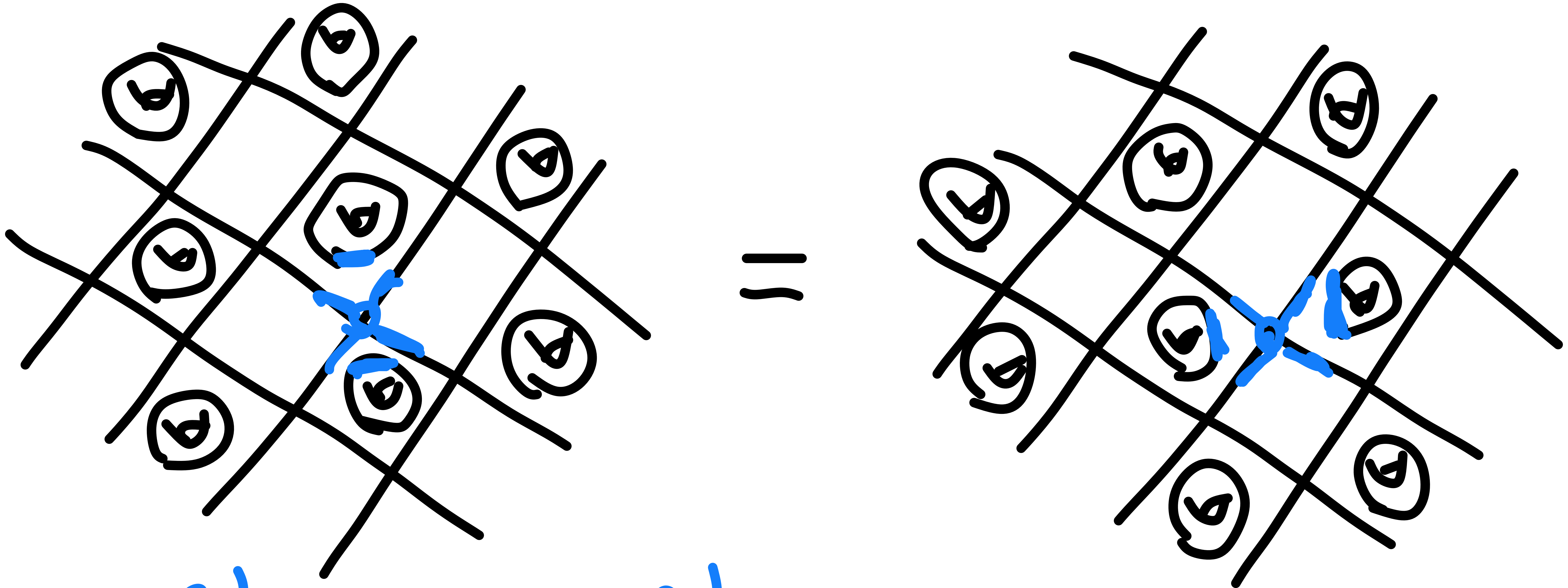


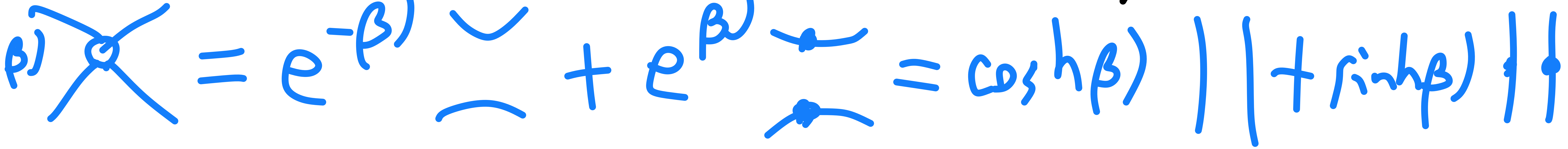
no genus remained in the bulk, works for any planar graph

that's why free fermion  $= e^{i\gamma_a\gamma_b\theta}$ quadratic operator

Ising model

Kramers-Wannier duality



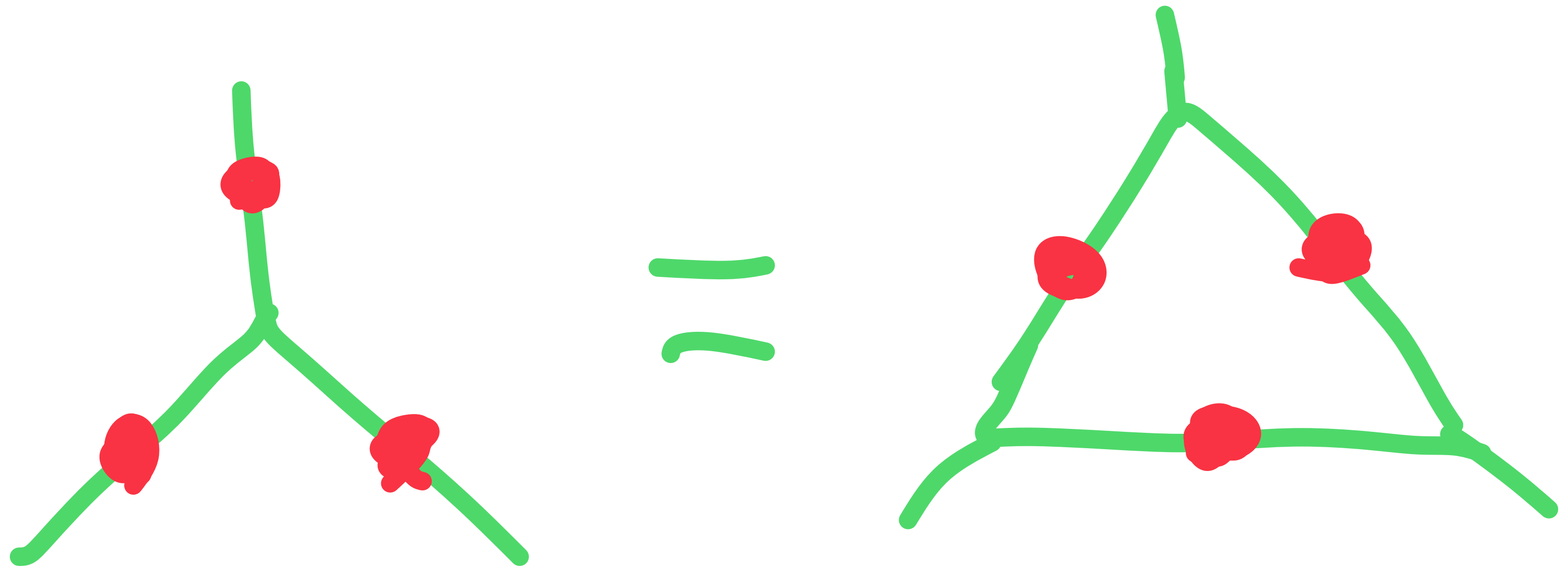


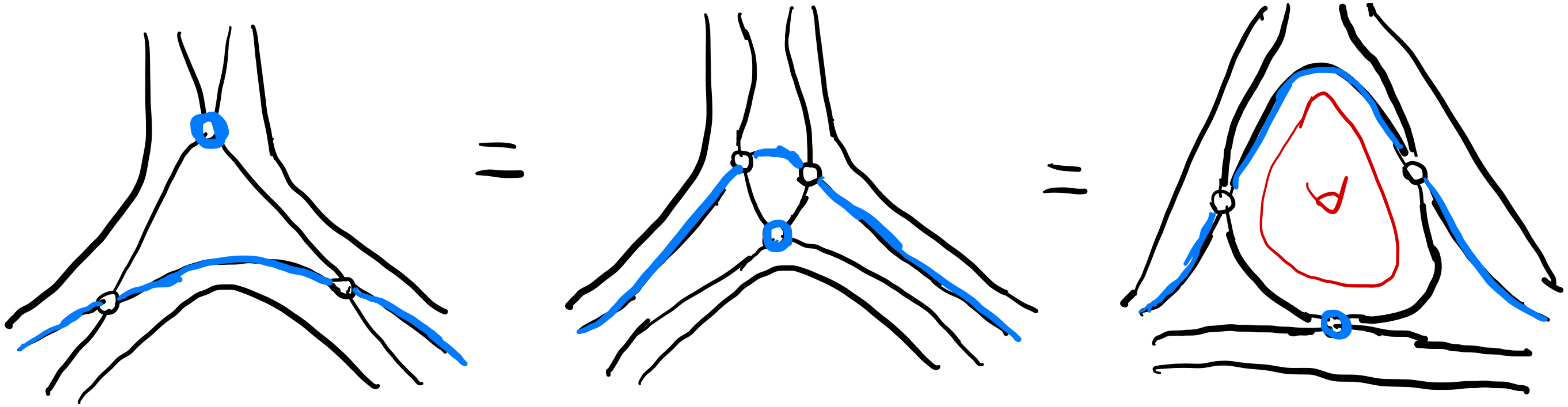
Kramers-Wannier duality:

adding string-genus at different positions

Ising model

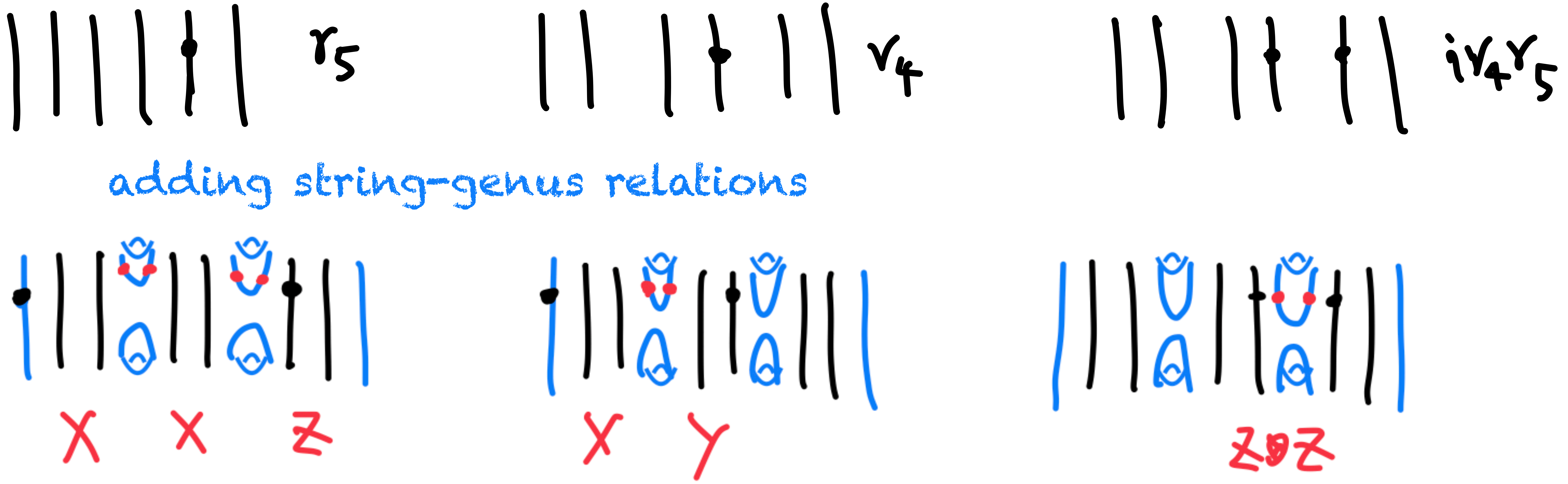
Yang-Baxter equation and star-triangle relation





Ising model

Jordan-Wigner transformation

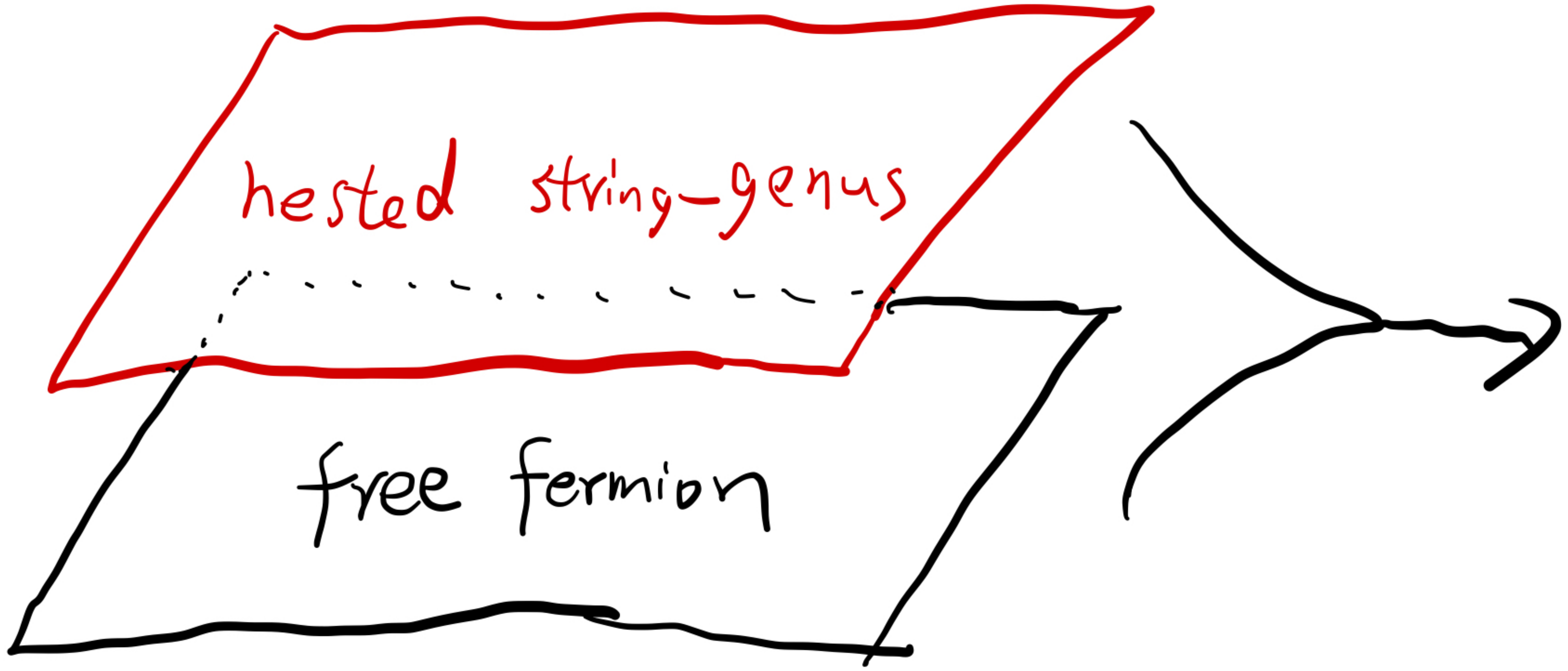


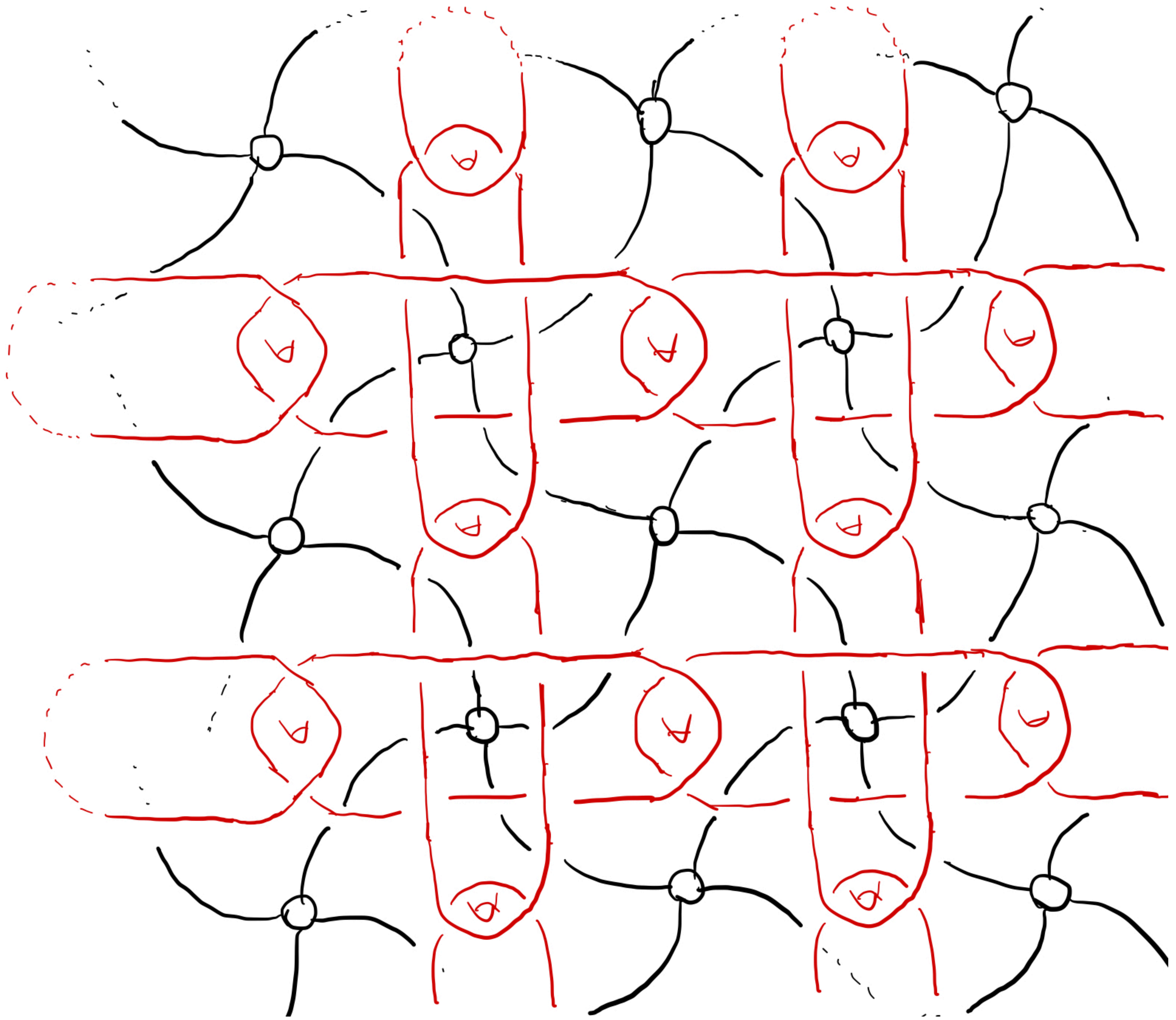
Jordan-Wigner transformation: creating a pair of charges around a genus
To free fermion: simply locally-removable string-genus relations

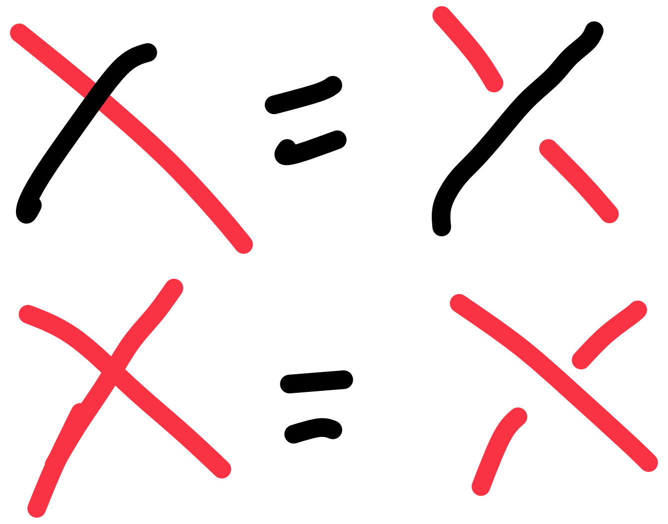
Generalizing Kramers-Wannier duality

New solvable TN

Extending Kramers-Wannier duality in a topological way

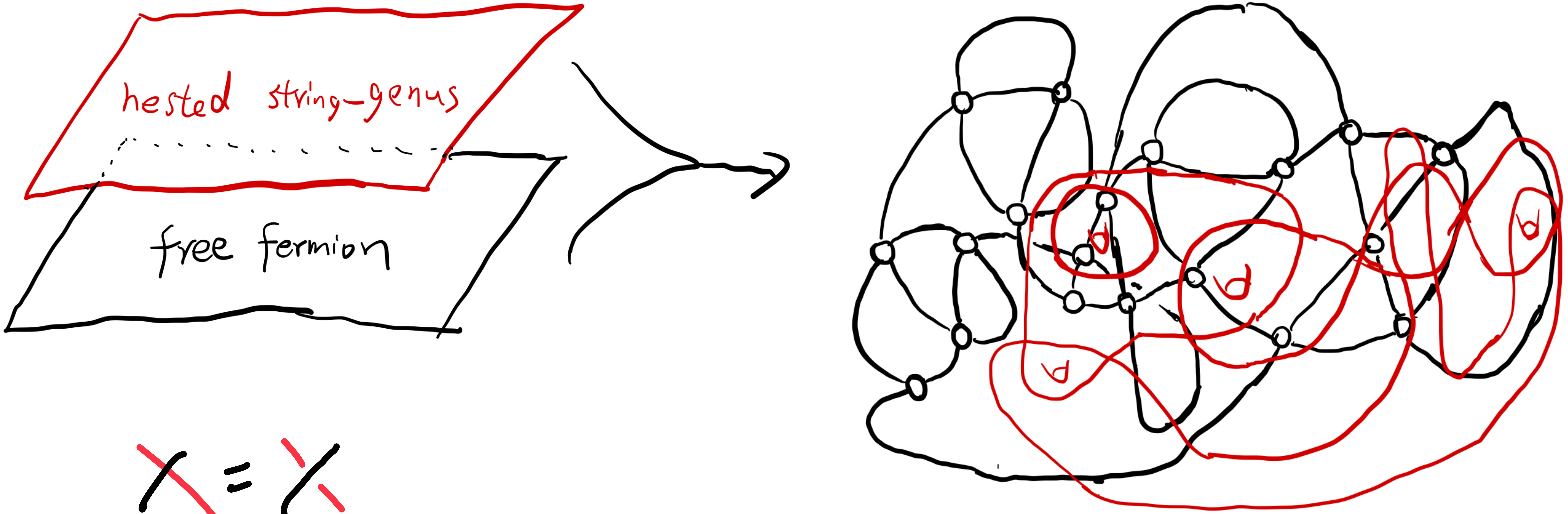


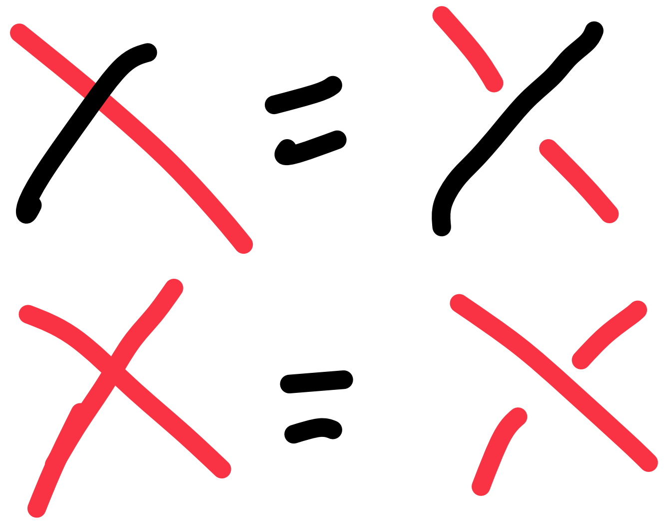




New solvable TN

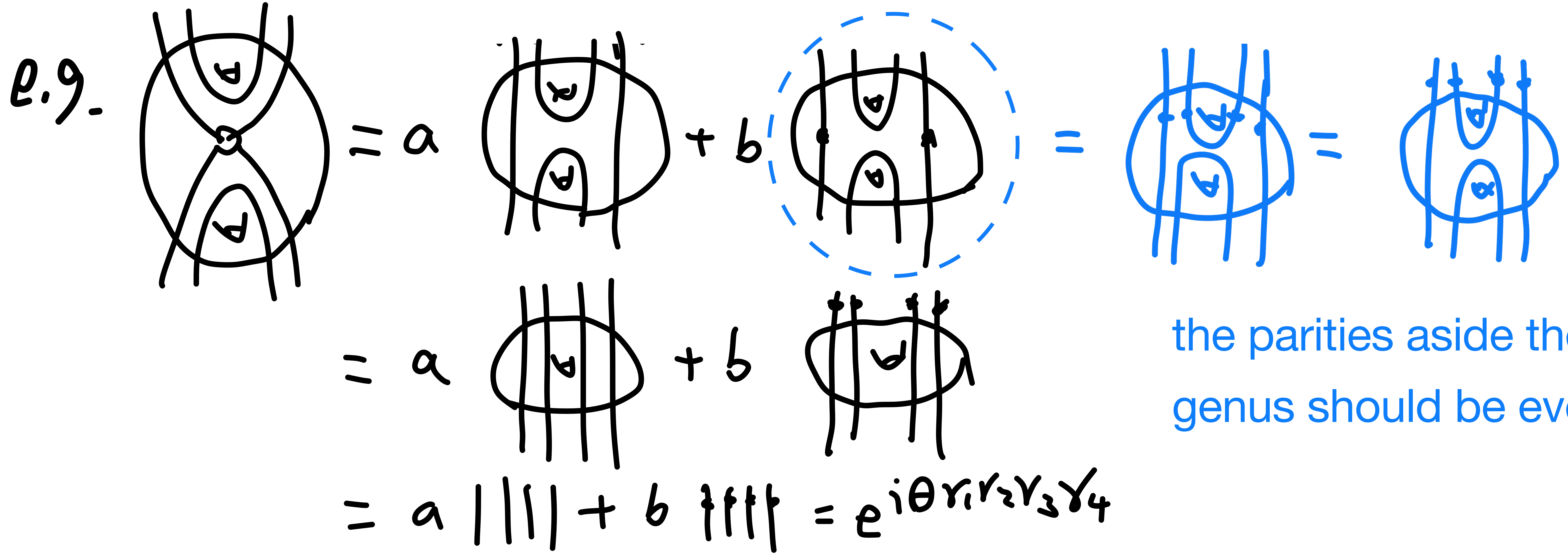
Extending Kramers-Wannier duality in a topological way





New solvable TN

non-Gaussian fermion interaction terms

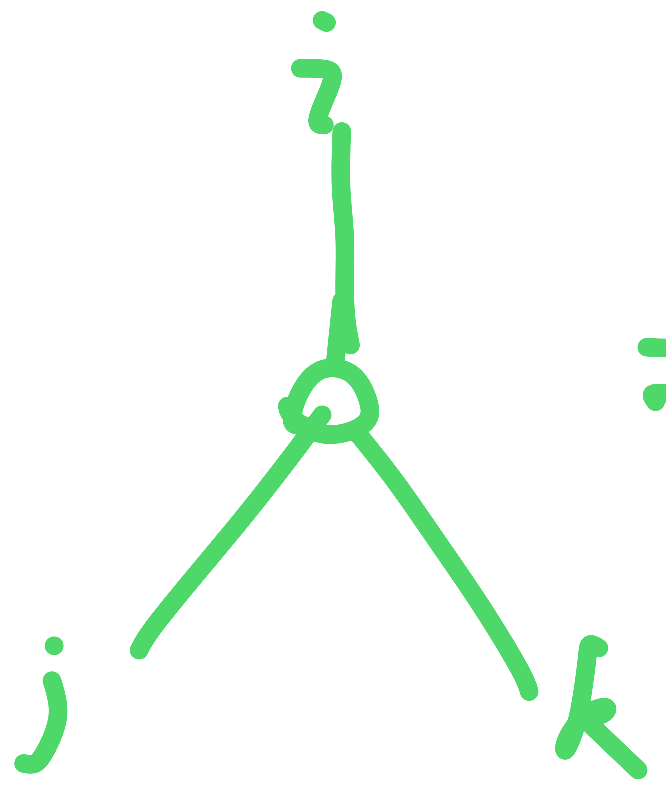


the parities aside the genus should be even

Other results

Matchgate circuits

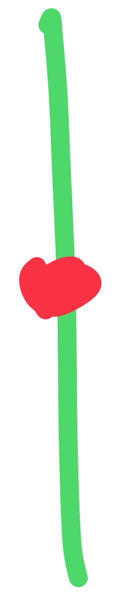
Dimer model / counting perfect matching



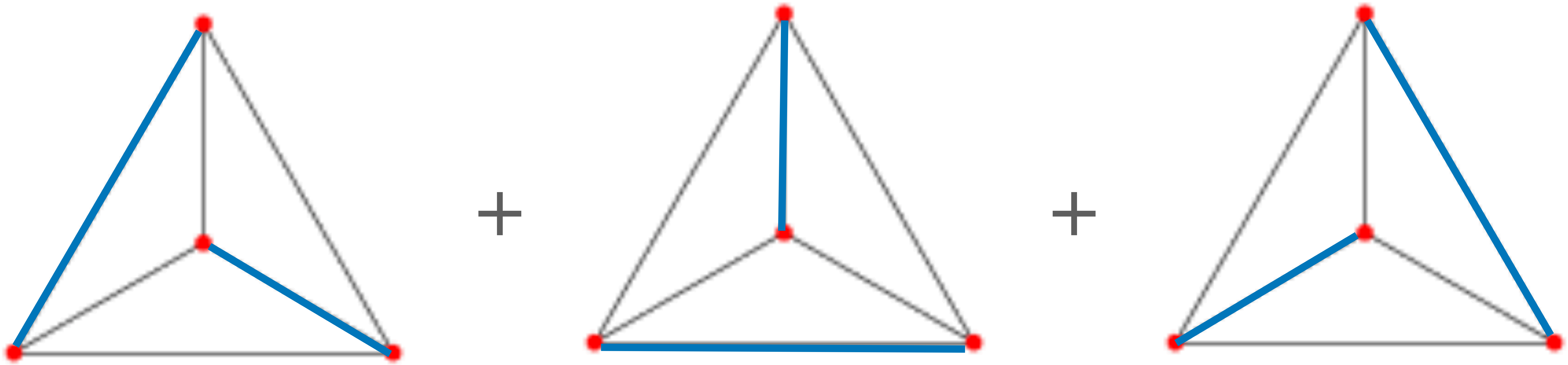
$$= \begin{cases} 1, & i+j+k=1 \\ 0, & \text{else} \end{cases}$$

W-states

weighted PM

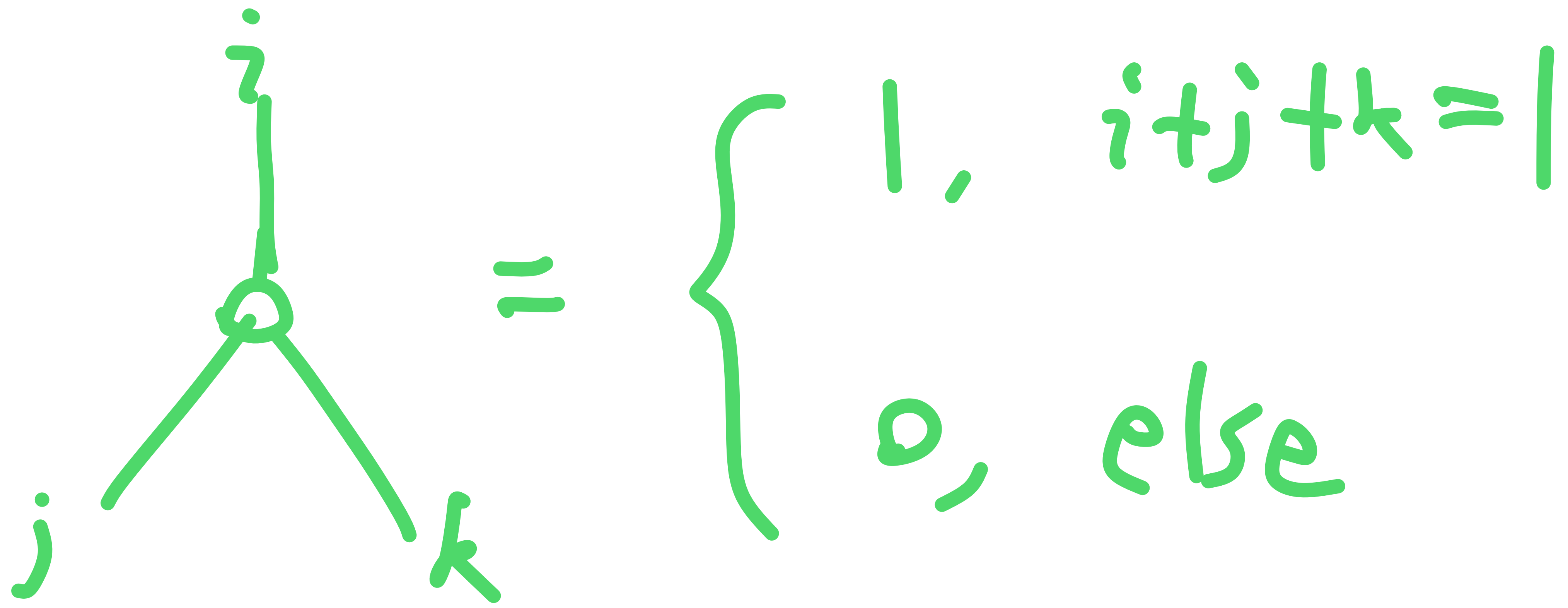


$$= \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix}$$

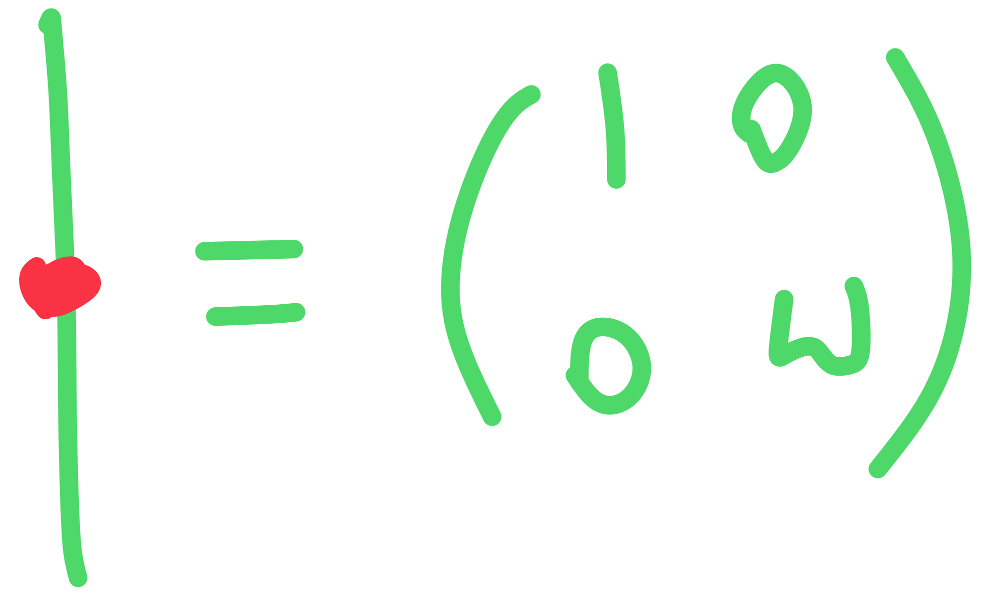


Matchgate circuits

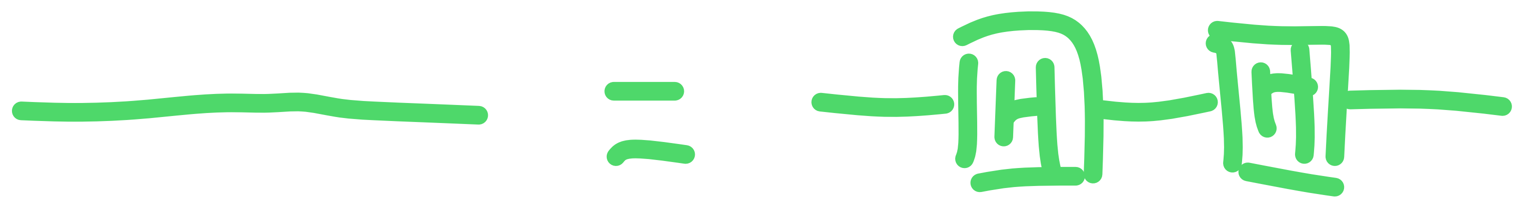
Dimer model / #PM

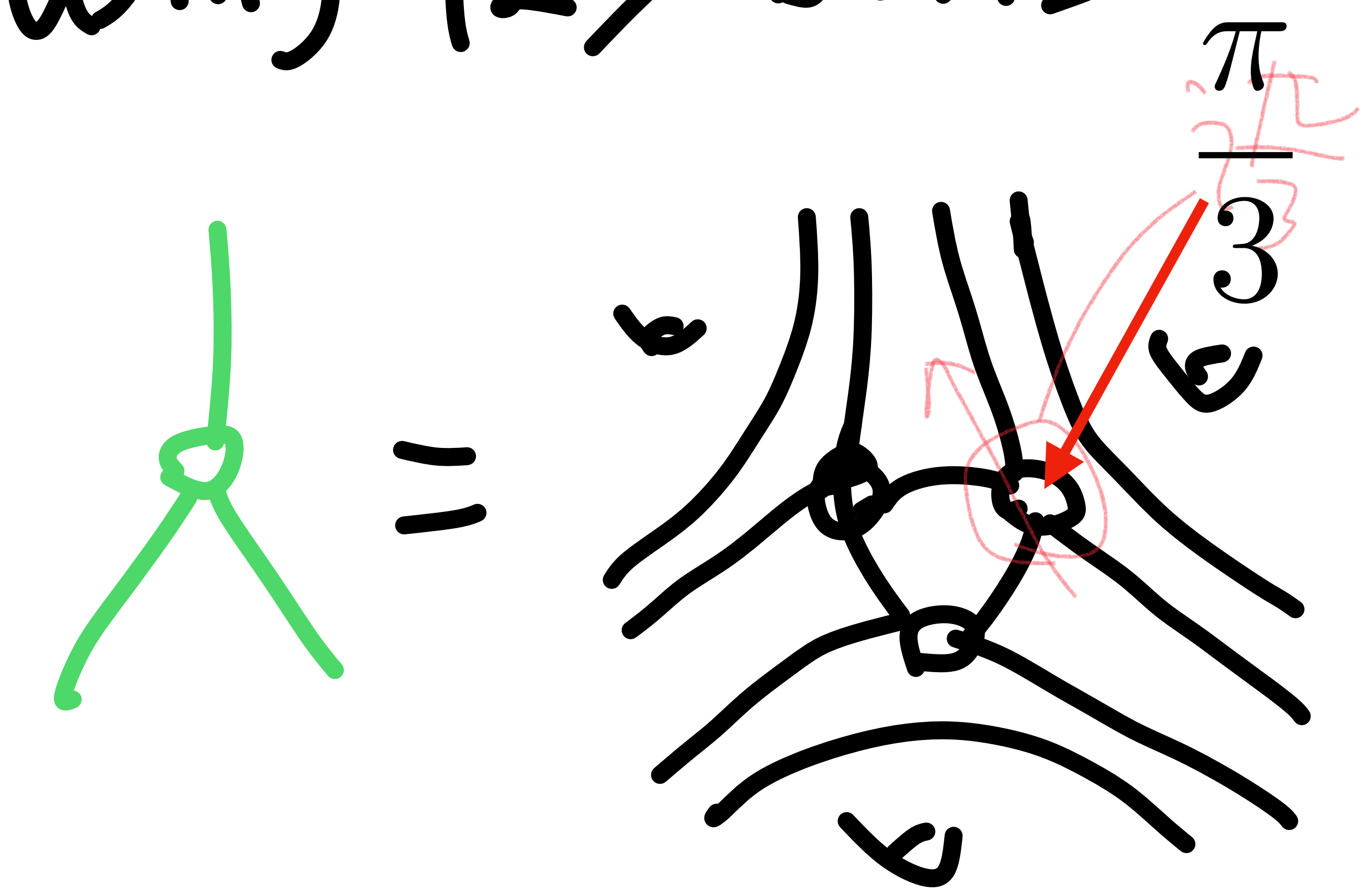


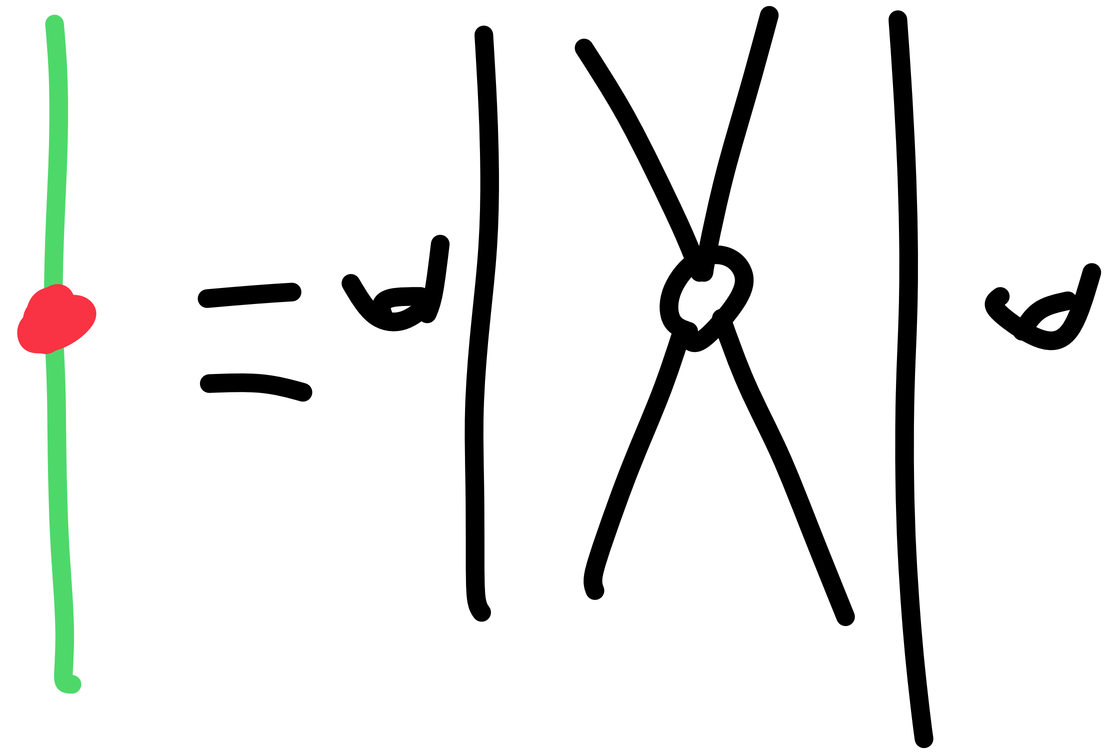
weighted PM



using |±⟩ basis

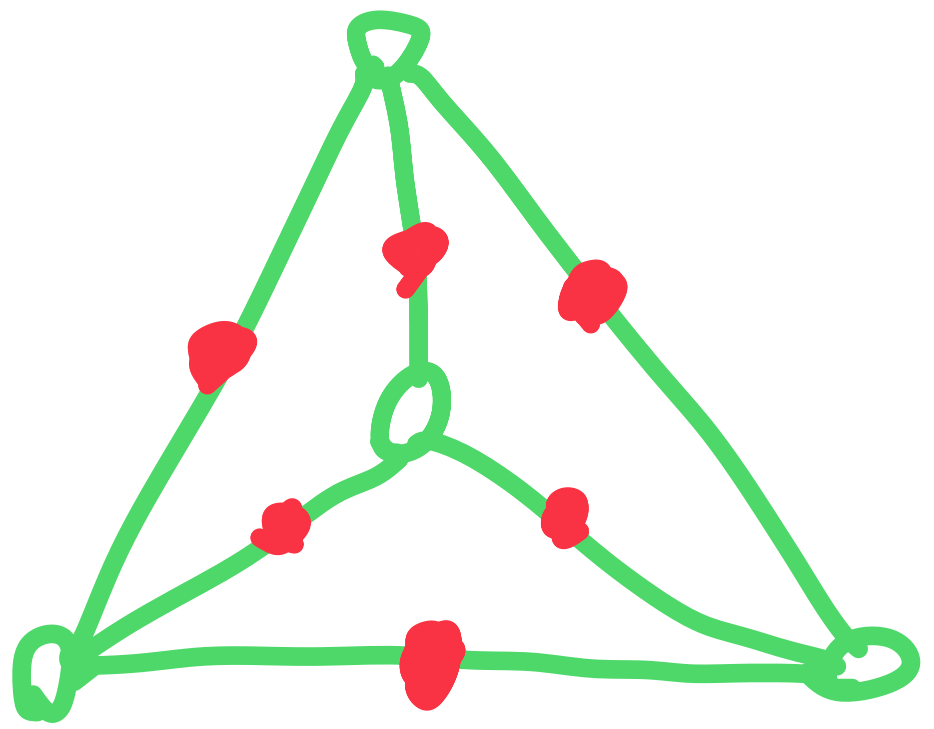




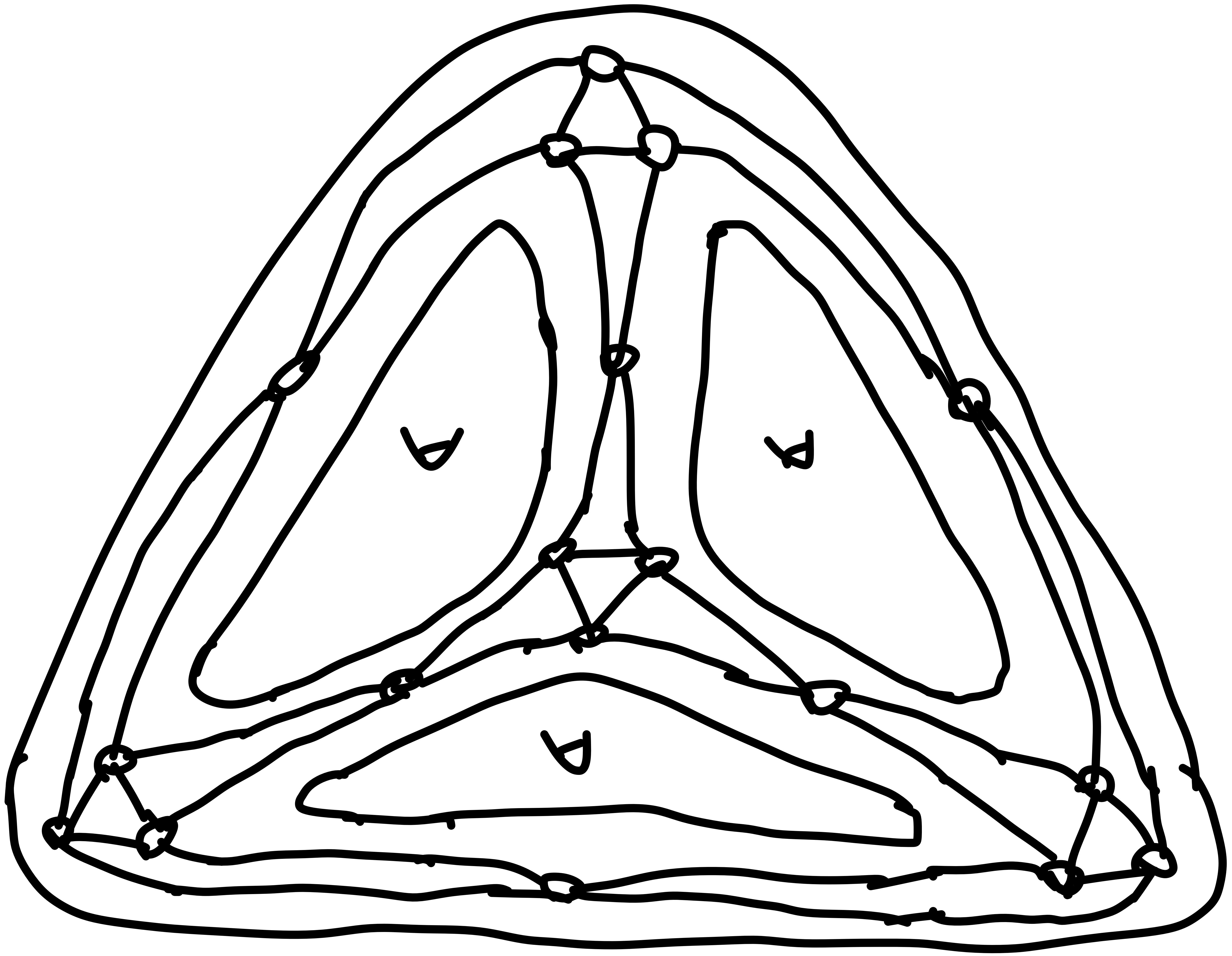


Matchgate circuits

Dimer model / #PM

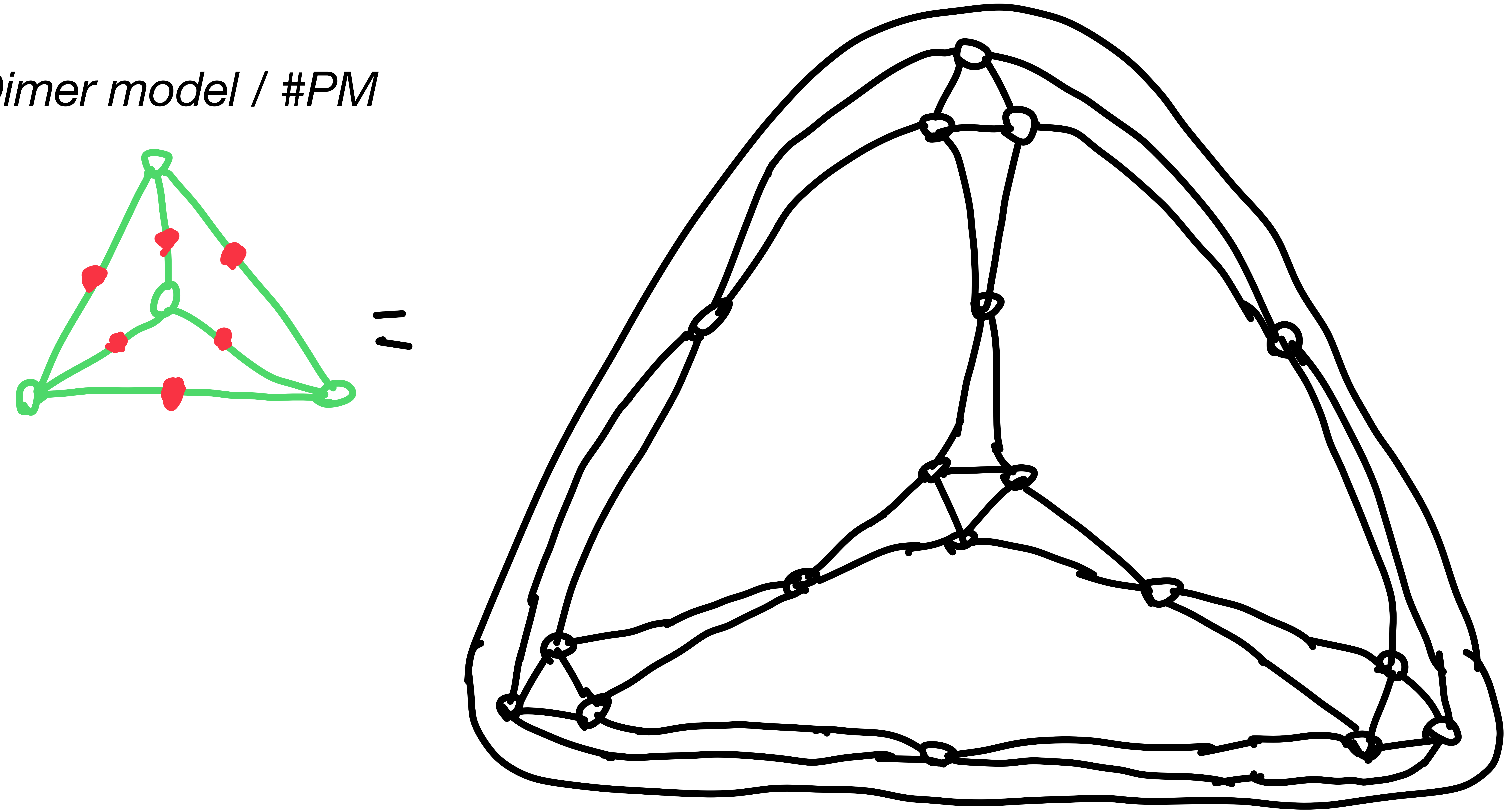


=



Matchgate circuits

Dimer model / #PM



Matchgate circuits

Characterization of matchgate TN in diagram

Definition (matchgate)*: tensor network / quantum circuit composed of the above two kinds of tensors on* ***planar graph***

Valiant, Leslie G. "Expressiveness of matchgates." *Theoretical Computer Science* 289.1 (2002): 457-471.

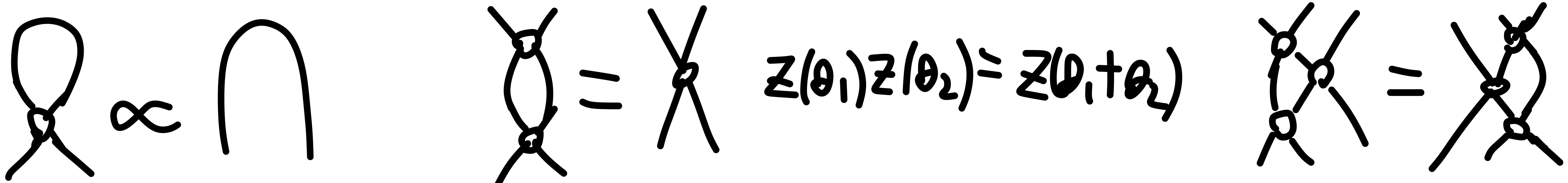
Pictorial characterization*:*

1. the string on the boundary does not braid with other strings

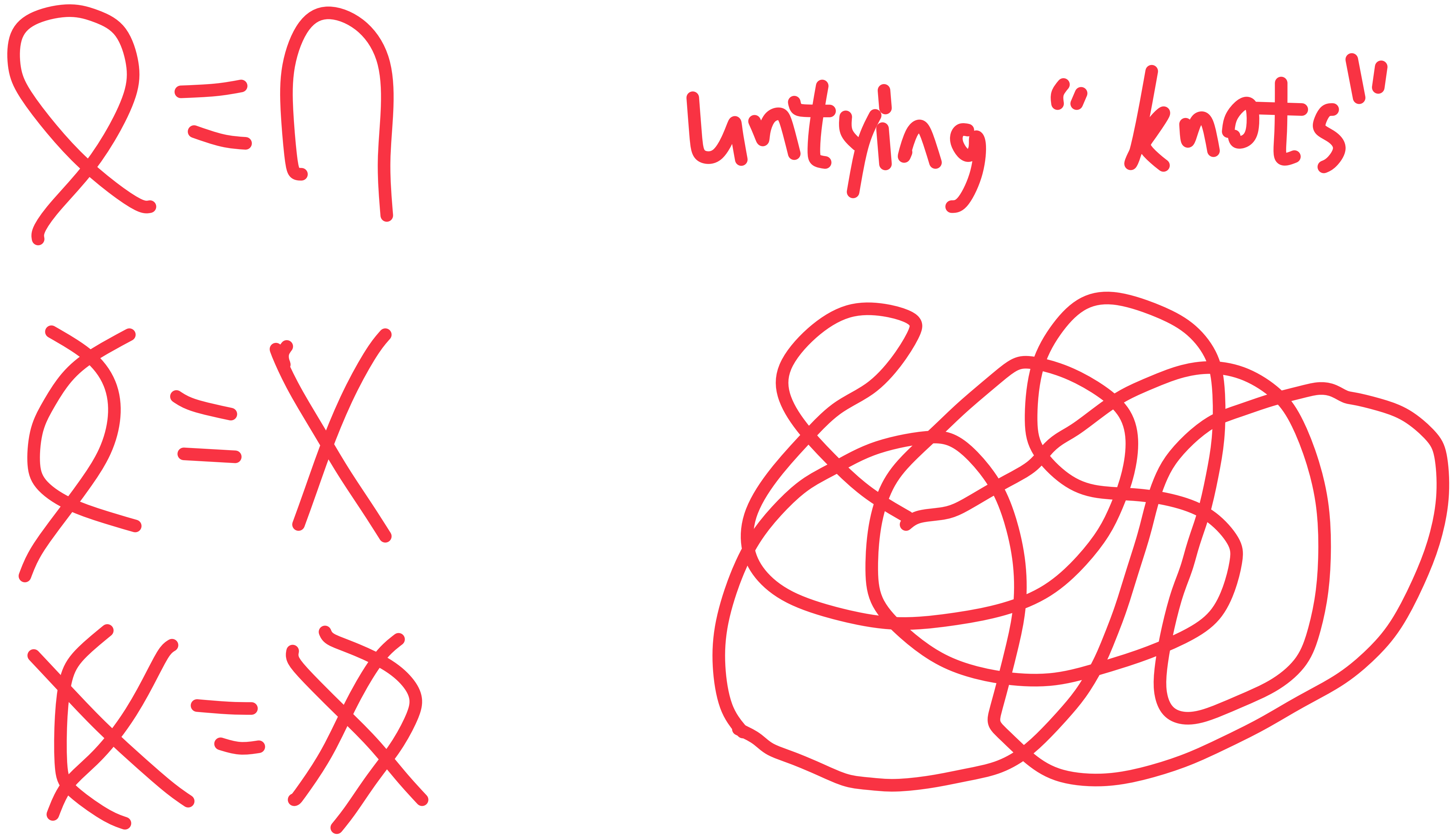
2. no genus inside the bulk

Matchgate circuits

Evaluating matchgate by untying "knots"



Modified Redemeister moves



braidings −1

YB has singular points s.t.

can not apply (but zero-measure)

Clifford circuits

***Definition (Clifford)**: tensor network / quantum circuit composed of H, S, Pauli, copy tensors (copy + H => CZ)*

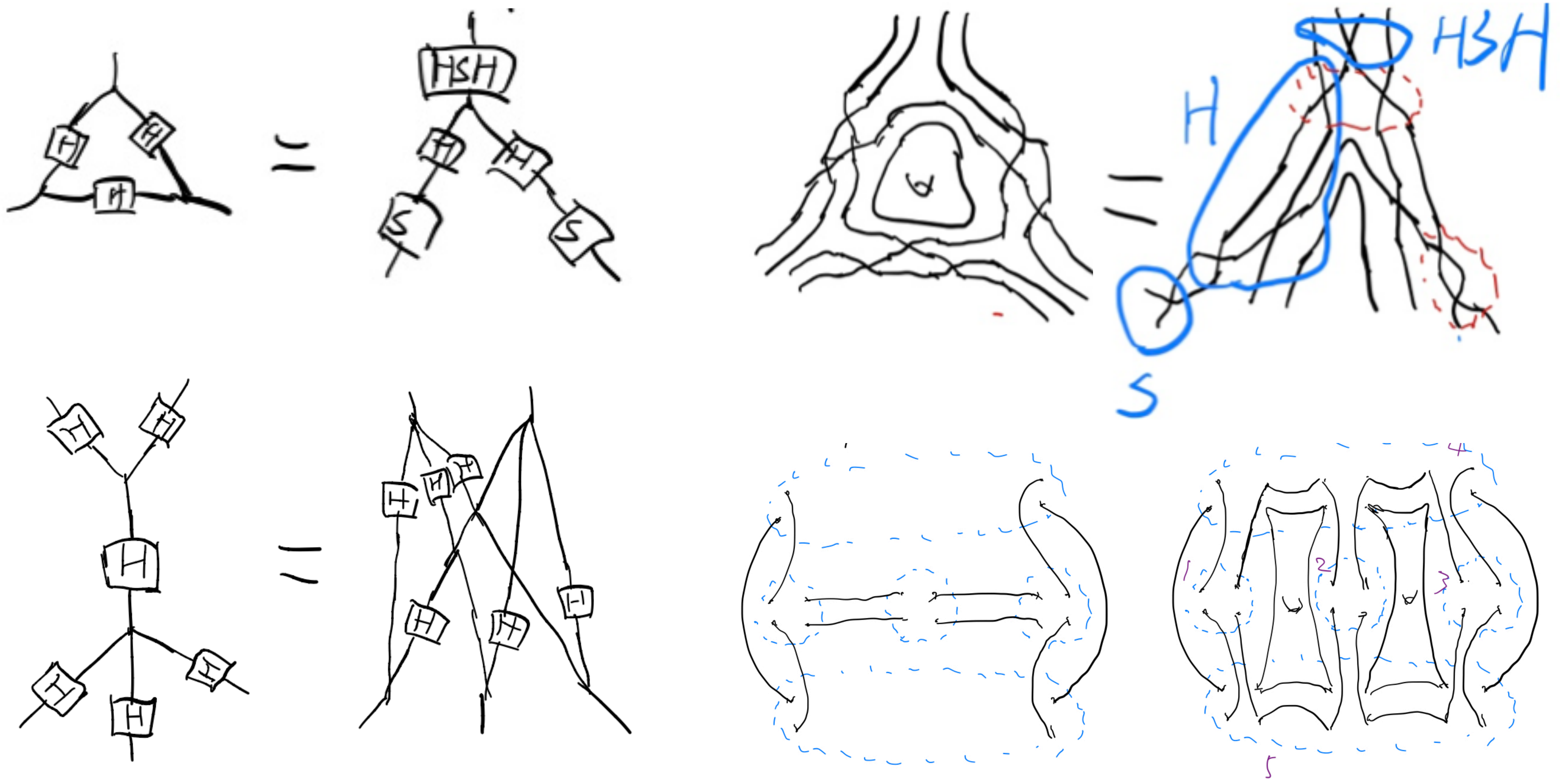
Gottesman, Daniel. "The Heisenberg representation of quantum computers." *arXiv preprint quant-ph/9807006* (1998).

***Pictorial characterization**:*

no parameterized crossing

Clifford circuits

Sketch of proof



deriving Graph States of Local Complements

Van den Nest M, Dehaene J, De Moor B. Graphical description of the action of local Clifford transformations on graph states. Physical Review A. 2004

Duncan R, Perdrix S. Graph states and the necessity of Euler decomposition. InConference on Computability in Europe 2009

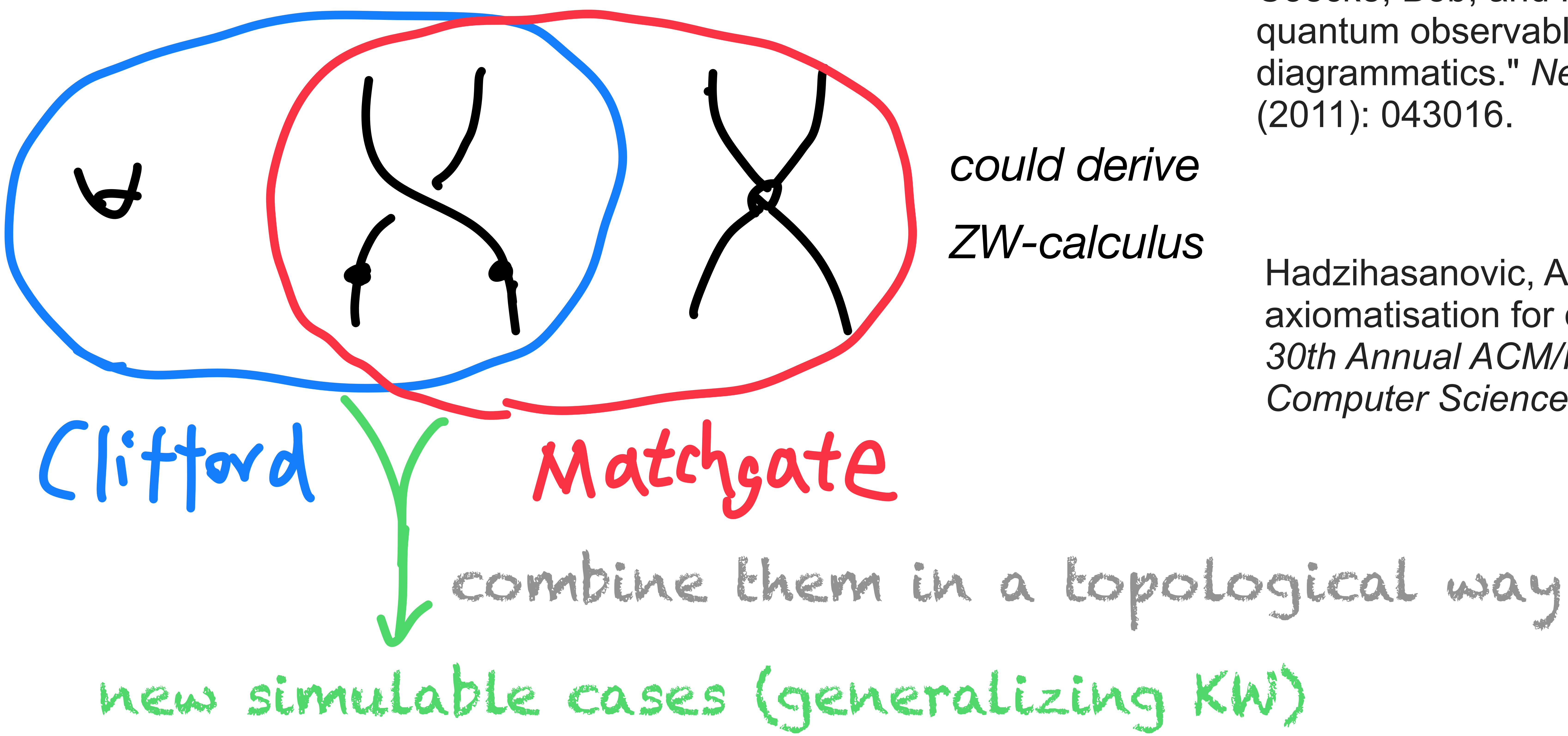
Backens M. The ZX-calculus is complete for stabilizer quantum mechanics. New Journal of Physics. 2014

A unified framework

Classically simulable quantum circuits

could derive

ZX-calculus



could derive

ZW-calculus

Coecke, Bob, and Ross Duncan. "Interacting quantum observables: categorical algebra and diagrammatics." *New Journal of Physics* 13.4 (2011): 043016.

Hadzihasanovic, Amar. "A diagrammatic axiomatisation for qubit entanglement." *2015 30th Annual ACM/IEEE Symposium on Logic in Computer Science*. IEEE, 2015.

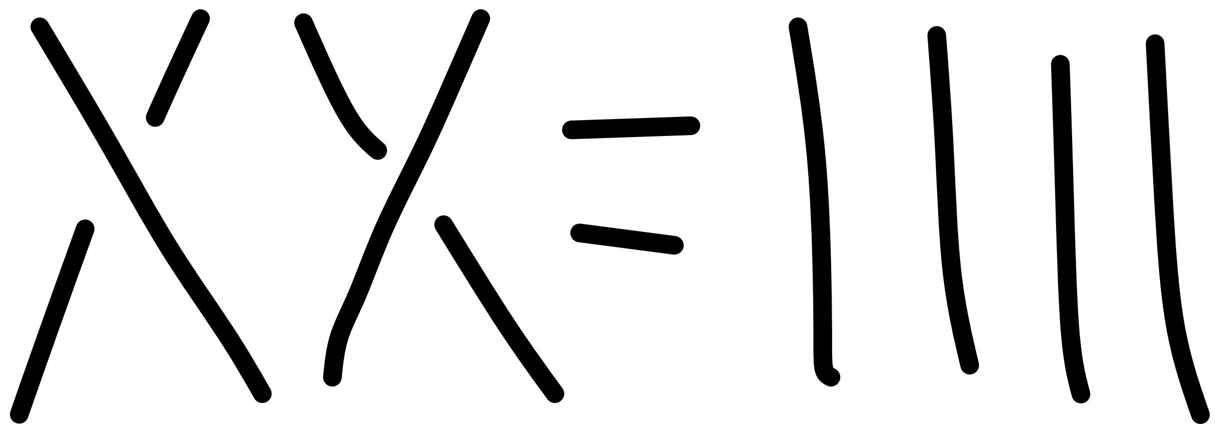
Nested string genus relations as a subclass of Clifford circuits

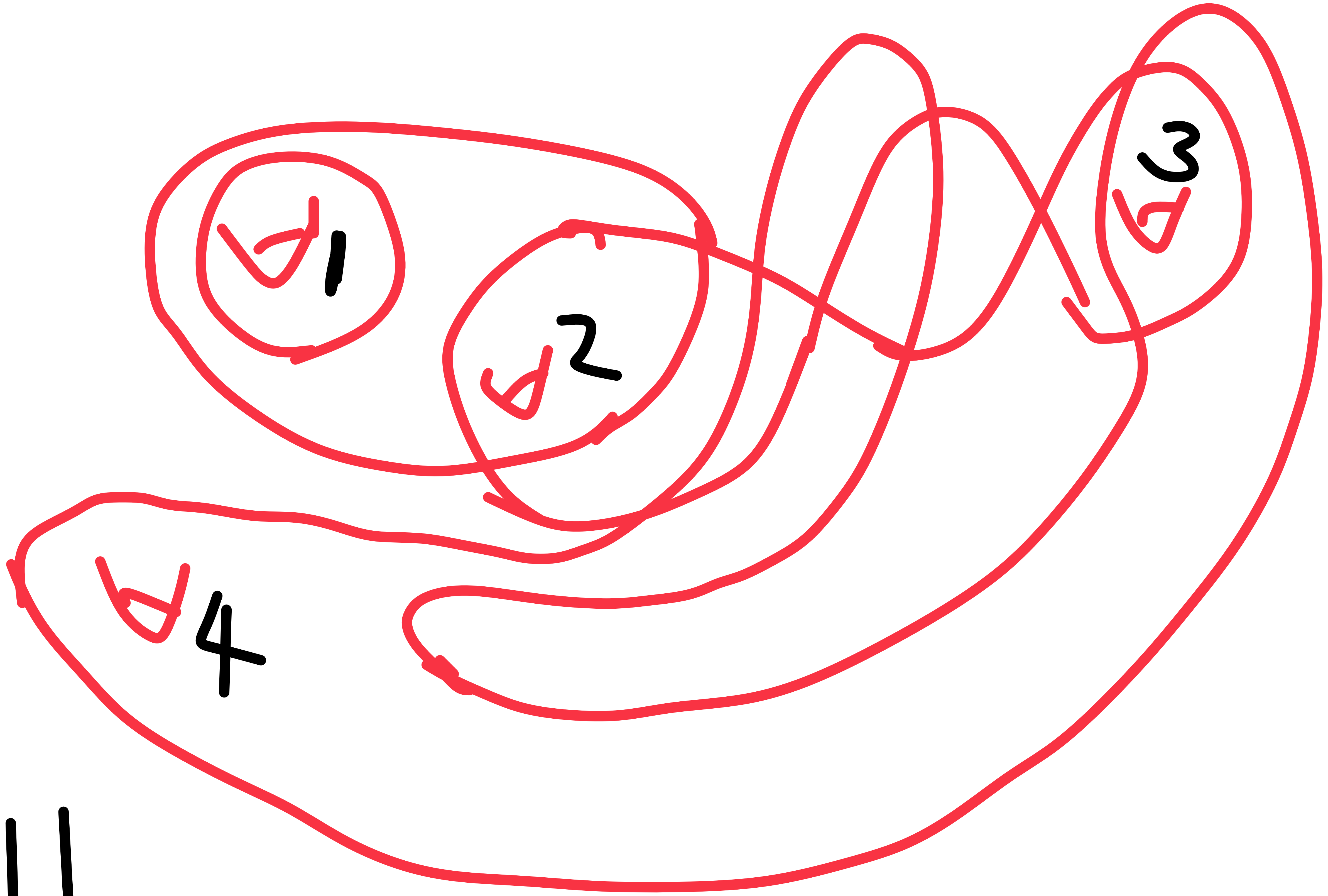
nested string-genus in 3D

special Clifford:

those could be evaluated

without using

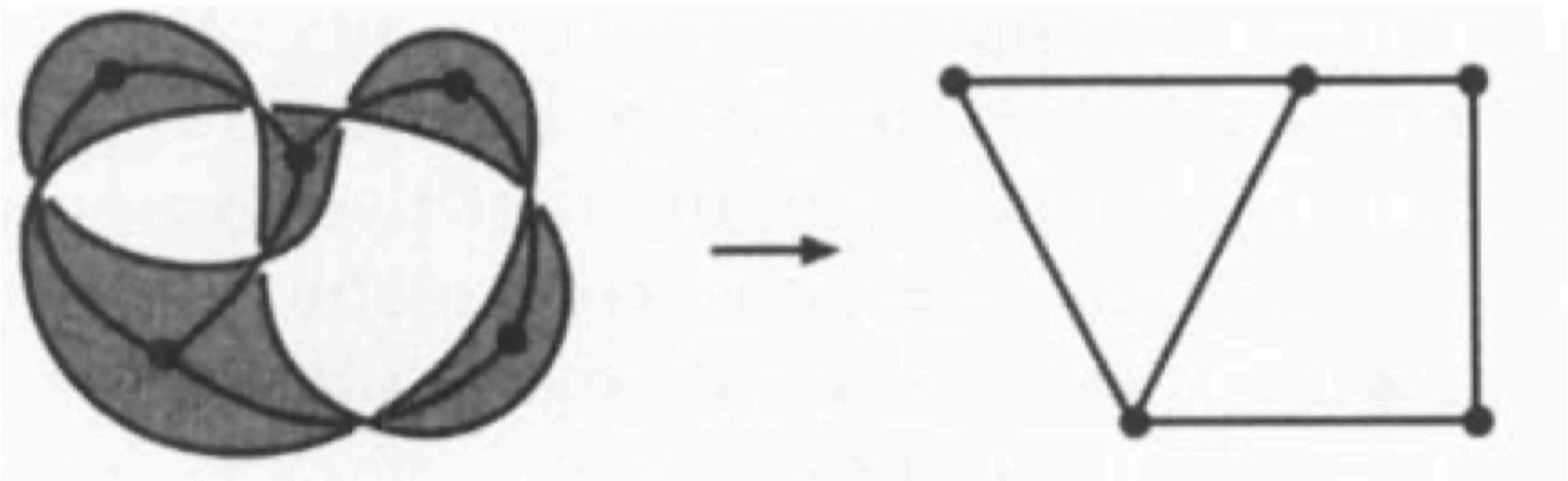


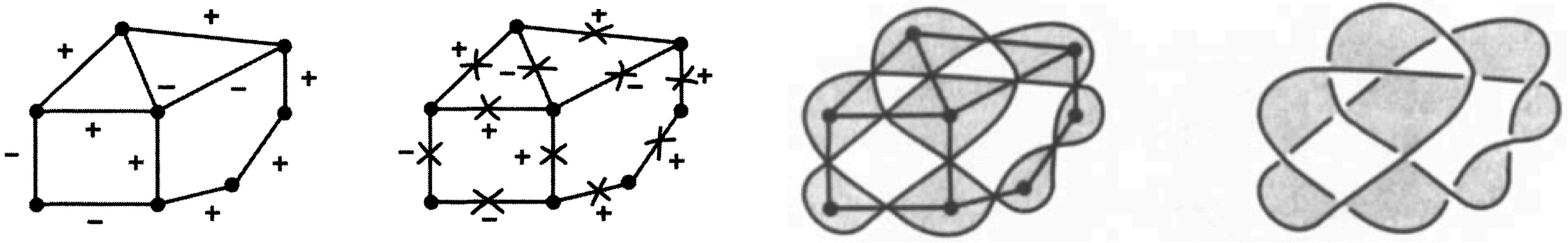


Representation for coding

Medial graph

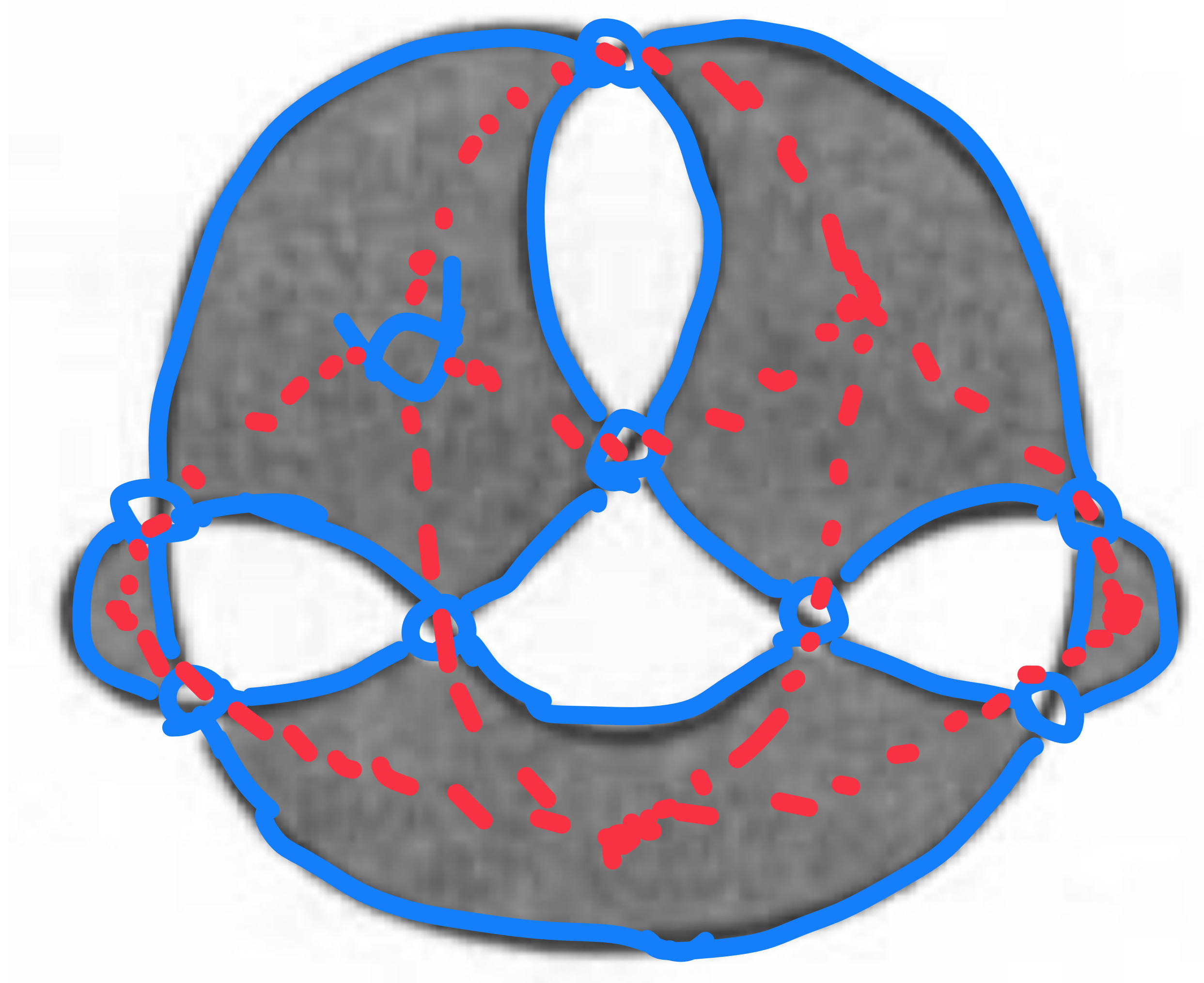
shaded link projection

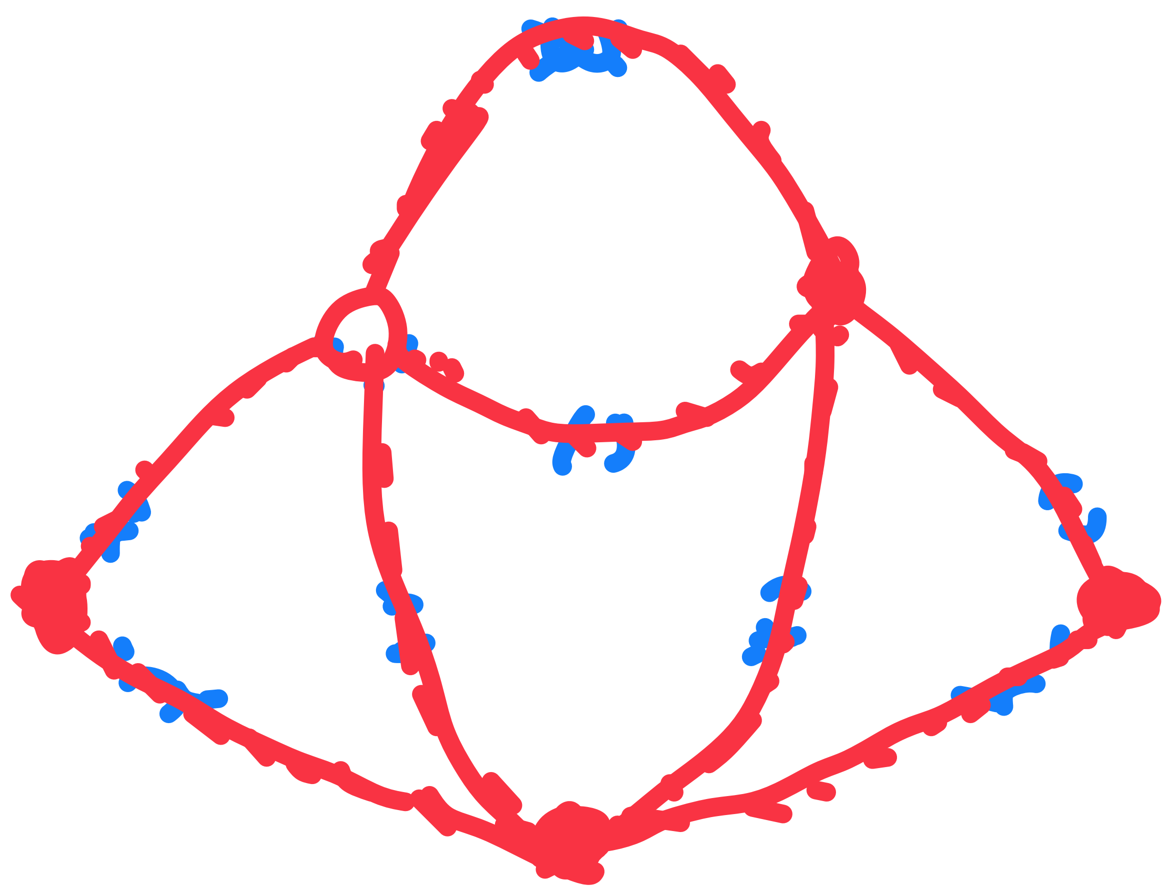




Representation for coding

Two types of vertices and parameterized edges





2 types of vertices:
genus vs. non-genus

rules represented in planar graph,
e.g., YB and parameterized edges

Summary

- Ising model and KW, JW, YB, and locally-removable string-genus relation
- New solvable classes
  - Generalizing Kramers-Wannier duality in a topological way: free fermion+nested string genus relations
  - Interpretation: Ising model with magnetic field / interacting fermion operator
- Examples of other applications
  - A unified framework of Clifford and matchgate circuits, a new class of classically-simulable quantum circuit, evaluation by “untying knots”
  - Medial graph representation

Outlook

- Application to quantum circuit complier for simplifying circuits
- Finding new exactly solvable models with physical implications
- Polynomial algorithm for some new combinatorial problems
- The new solvable classes as variational anzats
  enhance fermion gaussian state (free fermion) by adding non-Gaussian terms (interacting) but still efficiently computable
  potential applications, e.g., quantum chemistry
- Picture language from some encodings of other types of anyons
  (e.g., Quon on Z_d for sparse encoding of parafermion)

Acknowledgement

- Yunxiang Ren, Shengtao Wang, Zhengwei Liu, Arthur Jaffe
- Soonwon Choi, Fernando Pastawski