Symmetry as a shadow topological order

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Symmetry in quantum systems

- What is a **quantum system**?
  - Total Hilbert space $\mathcal{V} = \bigotimes_i \mathcal{V}_i$ ($\mathcal{V}_i$ finite Hilbert space on vertex-$i$)
  - A local Hamiltonian (a hermitian operator) of form $H = \sum O_i + O_{ij}$

- What is a **symmetry** in quantum system?
  - A symmetry is a set of **linear constraints** on local operators $\rightarrow$ **local symmetric operators**
  - Sum of local symmetric operators $\rightarrow$ **symmetric Hamiltonian**.

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Example: A lattice model on a triangulation of an $n$-dimensional space $M_n$. Local Hilbert space $\mathcal{V}_i$ on each vertex (or site) is spanned by $|g_i\rangle$, $g_i \in G$.

- The on-site (gaugable, anomaly-free) symmetry $W_h = \bigotimes_i W_h(i)$, $W_h |\cdots g_i, g_j \cdots\rangle = |\cdots g_i h, g_j h \cdots\rangle$, $h \in G$.

- The set of local symmetric operators (which form an algebra) $A = \{ O_{ij} |\ W_h O_{ij} = O_{ij} W_h, \forall h \in G \}$.
Symmetry in quantum systems

• What is a **quantum system**?
  - Total Hilbert space \( \mathcal{V} = \bigotimes_i \mathcal{V}_i \) (**\( \mathcal{V}_i \) finite Hilbert space on vertex-\( i \)**)
  - A local Hamiltonian (a hermitian operator) of form \( H = \sum O_i + O_{ij} \)

• What is a **symmetry** in quantum system?
  - A symmetry is a set of **linear constraints** on local operators \( \rightarrow \) **local symmetric operators**
  - Sum of local symmetric operators \( \rightarrow \) **symmetric Hamiltonian**.

• **A example**: A lattice model on a triangulation of a \( n \)-dimensional space \( M^n \). Local Hilbert space \( \mathcal{V}_i \) on each vertex (or site) is spanned by \( |g_i\rangle \), \( g_i \in G \).
  - The **on-site** (gaugable, anomaly-free) symm. (right-action)
    \[
    W_h = \bigotimes_i W_h(i), \quad W_h | \cdots g_i, g_j \cdots \rangle = | \cdots g_i h, g_j h \cdots \rangle, \quad h \in G.
    \]
  - The set of **local symmetric operators** (which form an algebra)
    \[
    \mathcal{A} = \{ O_{ij} | W_h O_{ij} = O_{ij} W_h, \quad \forall h \in G \}\]
A quantum system with 0-symmetry $G$

- A symmetric Hamiltonian (in the basis $|\cdots g_i, g_j \cdots\rangle$)

$$H_{\text{site}} = -J \sum_i \delta(g_i g_j^{-1}) - B \sum_i \sum_{h \in G} T_h(i)$$

$$\delta(g_i g_j^{-1}), \sum_{h \in G} T_h(i) \in A = \{ O_{ij} \mid W_h O_{ij} = O_{ij} W_h \}$$

$$T_h(i)|\cdots, g_i, g_j, \cdots\rangle = |\cdots, hg_i, g_j, \cdots\rangle,$$ left-action.

- When $B = 0, J > 0$, there are $|G|$ degenerate ground states:

$$\{ |\cdots, g, g, g, \cdots\rangle = \bigotimes_i |g_i\rangle \mid g \in G \}$$

but a unique symmetric ground state $\sum_{g \in G} |\cdots, g, g, g, \cdots\rangle$

- The degenerate ground states $\rightarrow$ spontaneous symmetry breaking.

- In the ground state subspace, the product states are not symmetric, and the symmetric state is not product states $\rightarrow$ spontaneous symm. breaking.

- When $J = 0, B > 0$, there is a unique ground state $\bigotimes_i \sum_{g_i \in G} |g_i\rangle i$, which is a symmetric product state $\rightarrow$ no symmetry breaking
A non-Abelian duality

- We consider duality transformation of $H_{\text{site}}$. The dual model has local Hilbert space $\mathcal{V}_{ij}$ for each link $ij$, where $\mathcal{V}_{ij}$ is spanned by $|g_{ij}\rangle$, $g_{ij} \in G$, with $g_{ij} = g_{ji}^{-1}$.

- The duality map between local operators:

$$g_ig_j^{-1} \rightarrow g_{ij}, \quad T_h(i) \rightarrow \tilde{T}_h(i) : |\cdot, g_{ij}, g_{ik}, g_{jk}, \cdot\rangle \rightarrow |\cdot, hg_{ij}, hg_{ik}, g_{jk}, \cdot\rangle$$

$$H_{\text{link}} = -J \sum_{ij} \delta(g_{ij}) - B \sum_i \sum_h \sum_{h \in G} \tilde{T}_h(i) - U_\infty \sum_{ijk} \delta(g_{ij}g_{jk}g_{ki}),$$

When $J = 0$, $H_{\text{link}}$ is a lattice gauge theory.

- $U_\infty = \infty$ makes $g_{ij}g_{jk}g_{ki} = 1 \rightarrow$ flat connection. In this case, there is a $|G|$-to-1 correspondence ($^\ast$ base point)

$$g_i h(g_j h)^{-1} \rightarrow g_{ij}, \quad g_i h(g^\ast h)^{-1} \leftarrow g_{ij}g_{jk} \cdots g_{l^\ast}, \quad h \in G.$$ 

- $H_{\text{site}}$ within the symmetric sub Hilbert space has identical eigenvalues as $H_{\text{link}}$ below $U_\infty$ if the space $M^n$ satisfies $\pi_n(M^n) = 0$. 

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Algebraic higher symmetry


• \( H_{\text{link}} = -J \sum_{ij} \delta(g_{ij}) - B \sum_i \sum_{h \in G} \tilde{T}_h(i) - U_\infty \sum_{ijk} \delta(g_{ij}g_{jk}g_{ik}^{-1}) \)

has an **Algebraic \((n-1)\)-symmetry** \( \tilde{G}^{(n-1)} \), generated by Wilson loop operators (for **any loops** \( S^1 \))

\[ W_q(S^1) = \text{Tr} \prod_{\langle ij \rangle \in S^1} R_q(g_{ij}). \]

\( R_q \) is an irreducible representation of \( G \) labeled by \( q \).

• The **local symmetric operators**

\[ \mathcal{A} = \{ O_{ij} \mid W_q(S^1)O_{ij} = O_{ij}W_q(S^1), \ \forall q \text{ and } \forall S^1 \} \]

and \( H_{\text{link}} \) is a sum of local symmetric operators.

- The symmetry acts on all codimension-\((n-1)\) closed subspace \( S^1 \).
- The symmetry satisfies \( W_{q_1}(S^1)W_{q_2}(S^1) = \sum_{q_3} N_{q_1q_2}^{q_3} W_{q_3}(S^1) \),

which do not form a group for non-Abelian \( G \).

\[ \rightarrow \text{ algebraic \((n-1)\)-symmetry (algebraic higher symmetry)} \]

- If \( G \) is Abelian, the \((n-1)\)-symmetry (**higher symmetry**) \( \tilde{G}^{(n-1)} \)

is described by a higher group. Gaiotto-Kapustin-Seiberg-Willett arXiv:1412.5148
Spontaneous breaking of algebraic higher symmetry

- $G$ symmetry has a phenomenon: spontaneous symmetry breaking
  $$H_{\text{site}} = -J \sum_i \delta(g_i g_j^{-1}) - B \sum_i \sum_{h \in G} T_h(i)$$

| $G|^{\pi_0(M^n)}$ degenerate ground states | Unique ground state on any $M^n$
| $G$ SSB | $G$ Symmetric

| $\tilde{G}^{(n-1)}$ Symmetric | $\tilde{G}^{(n-1)}$ SSB $B/J$

Unique ground states on any $M^n$
$$|\text{grnd}\rangle = \bigotimes_{ij} |g_{ij} = 1\rangle$$

Degenerate ground states on some $M^n$
$$|\text{grnd}_\alpha\rangle = \sum_{g_{ij} g_{jk} g_{ki} = 1} \bigotimes_{ij} |g_{ij}\rangle$$

- The algebraic $(n-1)$-symm. also has spontaneous symm. breaking
  $$H_{\text{link}} = -J \sum_{ij} \delta(g_{ij}) - B \sum_i \sum_{h \in G} \tilde{T}_h(i) - U_{\infty} \sum_{ijk} \delta(g_{ij} g_{jk} g_{ki})$$

- Detect (define) spontaneous symmetry breaking (SSB):
  1) Degenerate ground states on some $M^n$. 2) Some symmetry transformations are not identity in the groundstate subspace

- The critical point has both the symm $G$ and the dual algebraic $(n-1)$-symm $\tilde{G}^{(n-1)}$, ie has the categorical symm $G \vee \tilde{G}^{(n-1)}$
SSB of (algebraic) higher symm. and topo. order

- The ground state degeneracy from SSB of a 0-symmetry is robust against any local symmetry preserving perturbations.
  - *The ground state degeneracy is a property of the symmetry.*

- The ground state degeneracy from SSB of a finite (algebraic) higher symmetry is robust against any local perturbations that can break the symmetry.
  - *The ground state degeneracy is actually not a property of the (algebraic) higher symmetry.*

- SSB of $U(1)$ 1-symmetry gives rise to gapless $U(1)$-gauge bosons, which is robust against any local perturbations.

- SSB of finite (algebraic) higher symmetries $\rightarrow$ topological orders

- Some topological orders $\rightarrow$ (1) emergence of an (algebraic) higher symmetry, (2) which is spontaneously broken.

- Emergent (algebraic) higher symmetry is present in such topological orders and their continuous transition to the neighboring phases.
The dual-equivalence of two symm. $G \xrightarrow{\text{dual}} \tilde{G}(n-1)$

The following two Hamiltonians are dual-equivalent:

\[ H_{\text{site}} = -J \sum_i \delta(g_i g_j^{-1}) - B \sum_i \sum_{h \in G} T_h(i) \]

- $|G|^{\pi_0(M^n)}$ degenerate ground states
- $G$ SSB
- $\tilde{G}^{(n-1)}$ Symmetric

- Unique ground state on any $M^n$
- $G$ Symmetric
- $\tilde{G}^{(n-1)}$ SSB
- $B/J$

- Unique ground states on any $M^n$
- Degenerate ground states on some $M^n$

\[ H_{\text{link}} = -J \sum_{ij} \delta(g_{ij}) - B \sum_i \sum_{h \in G} \tilde{T}_h(i) - U_\infty \sum_{ijk} \delta(g_{ij} g_{jk} g_{ik}^{-1}) \]

- More generally, an arbitrary $H_{\text{site}}$ with 0-symmetry $G \overset{\text{dual}}{\leftrightarrow}$ a $H'_{\text{link}} - U_\infty \sum_{ijk} \delta(g_{ij} g_{jk} g_{ik}^{-1})$ with the algebraic $(n-1)$-symmetry, i.e. the two Hamiltonians are equivalent, same spectrum etc.

- The 0-symmetry $G$ and the algebraic $(n-1)$-symmetry $\tilde{G}^{(n-1)}$ represent the equivalent constraints that select the “same” class of Hamiltonians. We call them dual-equivalent symmetries.
$G$ or $\tilde{G}^{(n-1)} \rightarrow$ full categorical symm $G \lor \tilde{G}^{(n-1)}$

| $G|_{\pi_0(M^n)}$ degenerate ground states | Unique ground state on any $M^n$ |
|---------------------------------------------|----------------------------------|
| $G$ SSB                                    | $G$ Symmetric                     |
| $\tilde{G}^{(n-1)}$ Symmetric              | $\tilde{G}^{(n-1)}$ SSB B/J      |

Unique ground states on any $M^n$  Degenerate ground states on some $M^n$

- A **Hamiltonian** with symmetry $G$ also has the dual symmetry $\tilde{G}^{(n-1)}$, and also has the categorical symmetry $G \lor \tilde{G}^{(n-1)}$
- A **Hamiltonian** with algebraic higher symmetry $\tilde{G}^{(n-1)}$ also has the dual symmetry $G$, and also has the categorical symm $G \lor \tilde{G}^{(n-1)}$
- The **gapped ground state** must spontaneous break part of the categorical symmetry $G \lor \tilde{G}^{(n-1)}$, such as $G$, $\tilde{G}^{(n-1)}$, or some of their combination.
- The ground state with the full categorical symmetry $G \lor \tilde{G}^{(n-1)}$ must be gapless.

Algebraic higher symm $\tilde{G}^{(n-1)}$ & its charge object

- $H_{\text{link}} = -J \sum_{ij} \delta(g_{ij}) - B \sum_i \sum_{h \in G} \tilde{T}_h(i) - U_\infty \sum_{ijk} \delta(g_{ij}g_{jk}g_{ik}^{-1})$

- $H_{\text{link}}|_{B=0}$ has an **unique symmetric ground state** $|\text{grnd}\rangle = \bigotimes_{ij} |g_{ij} = 1\rangle$ on close space $M^n$ of any topology.

- A $(n-1)$-dimensional excitation $h$ on top of the ground state: change $g_{ij} = 1$ to $g_{ij} = h$ on a $(n-1)$-dimensional closed subspace = a **charge object** of the algebraic $(n-1)$-symmetry

- **Charge object for 0-symmetry** = **charge-anti-charge pair** on $S^0$.

- A **charged object** = changing $g_{ij} = 1$ to $g_{ij} = h$ on a $(n-1)$-dimensional subspace with boundary. **The boundary = the gauge flux**

- Measure the $(n-1)$-charge of a charged object:
  $W_q(S^1)|h\rangle = \text{Tr} R_q(h)|h\rangle$.

- $h$ and $h' = ghg^{-1}$ carry the same $(n-1)$-charge.
0-symmetry is described by group $G$. Higher symmetry is described by higher group. Gaiotto-Kapustin-Seiberg-Willett arXiv:1412.5148

What mathematics describes algebraic higher symmetry?

- We plan to use charged excitations to describe symmetry, higher symmetry, and algebraic higher symmetry in a unified way.

- 0-symmetry:
  - charge object = charge-anti-charge pair $S^0$.
  - charged object = single point (part of $S^0$).

- 1-symmetry:
  - charge object = loop excitations $S^1$.
  - charged object = open-string excitations (part of $S^1$).

- Algebraic higher symmetry: charged objects = point-like, string-like, membrane like excitations $\rightarrow$ higher fusion category.
• A symmetric quantum system: Hamiltonian $H = - \sum_i Z_i$
($X_i, Y_i, Z_i$ Pauli operator) plus local symmetric operators $\{\delta H\}$.

• An excitation = something can be trapped $H$ has a gap by $\delta H_{\text{trap}} = 2Z_{i_0}$ or $\delta H_{\text{trap}} = 2(\cos \theta Z_{i_0} + \sin \theta X_{i_0})$. $H + \delta H_{\text{trap}}$ also has a gap $\rightarrow$ trapped states: $|\downarrow\rangle, |\downarrow\downarrow\rangle$, ... ...

• Type: (determined by $H$ & local symmetric operators $\{\delta H\}$)
  - Without symmetry ($\{\delta H\}$ are formed by any local operators)
    $|\downarrow\rangle, |\downarrow\downarrow\rangle$ can be deformed into $|0\rangle$ without closing the gap. $|\downarrow\rangle \sim |\downarrow\downarrow\rangle \sim |0\rangle$ the same trivial type $1$.
  - With $\mathbb{Z}_2$-symm. $U = \prod_i Z_i$ (symm. operator $\{\delta H \ | \ \delta H U = U \delta H\}$)
    $|\downarrow\downarrow\rangle$ and $|0\rangle$ are of the same trivial type $1$ ($\mathbb{Z}_2$-charge-0).
    $|\downarrow\rangle$ has a different type $e$ ($\mathbb{Z}_2$-charge-1).
2+1D charged excitations form a fusion 2-category

- In 2+1D model with the $\mathbb{Z}_2$-symmetry, the excitation fusion rule:
  \[ 1 \otimes 1 = 1, \quad 1 \otimes e = e \otimes 1 = e, \quad 1 = \text{the fusion unit} \]
  \[ e \otimes e = 1, \quad e = \mathbb{Z}_2\text{-charge (-representation)} \text{ with mod-2 conservation} \]

- $\{1, e\}$ generate a fusion 2-category $2\text{Rep}_{\mathbb{Z}_2}$ in 2+1D:
  - string-like excitations ($\{1_s, e_s\}$ trivial, descendant) $\rightarrow$ objects.
  - $e_s = e$-condensed & spontaneous $\mathbb{Z}_2$-symm. breaking string.
  - $e_s$ is called descendant excitation

- Kong-Wen arXiv:1405.5858
- Gaiotto & Johnson-Freyd arXiv:1905.09566

- point-like excitations $\{1, e\}$ elementray $\rightarrow$ 1-morphisms $1_s \rightarrow 1_s$
  - domain-wall excitations $1_s \rightarrow e_s \rightarrow 1$-morphisms $1_s \rightarrow e_s$.

- instantons in spacetime connecting point-like excitations $\rightarrow$
  - 2-morphisms $=$ top morphisms $=$ local symmetric operators $\{\delta H\}$.

- 2 layers of morphisms $\rightarrow$ a fusion 2-category $2\text{Rep}_{\mathbb{Z}_2}$. 
1+1D charged excitations form a fusion 1-category

- 1+1D $\mathbb{Z}_2$-symmetric model: the point-like excitations $\{1, e\}$ have the same fusion rule $e \otimes e = 1$. $e = \mathbb{Z}_2$ charge (representation).
- $\{1, e\}$ generate a fusion 1-category $\text{Rep}_{\mathbb{Z}_2}$ in 1+1D.
  - bosonic point-like excitations $\{1, e\} \rightarrow$ objects.
  - instantons in spacetime connecting point-like excitations $\rightarrow$ 1-morphisms $=$ top morphisms $=$ local symmetric operators $\{\delta H\}$. 
1+1D charged excitations form a fusion 1-category

- 1+1D $\mathbb{Z}_2$-symmetric model: the point-like excitations $\{1, e\}$ have the same fusion rule $e \otimes e = 1$. $e = \mathbb{Z}_2$ charge (representation).
- $\{1, e\}$ generate a **fusion 1-category** $\text{Rep}_{\mathbb{Z}_2}$ in 1+1D.
  - bosonic point-like excitations $\{1, e\} \rightarrow$ objects.
  - instantons in spacetime connecting point-like excitations $\rightarrow$ 1-morphisms = top morphisms = local symmetric operators $\{\delta H\}$.
- **Tannaka duality**: The above fusion 1-category is a **symmetric fusion category** $\mathcal{E} \leftrightarrow$ finite group $G$.
  - Objects = $R_q$ representations of $G$. We denote $\mathcal{E} = \text{Rep}(G)$.
- In $(n + 1)$D, charged excitations described by fusion $n$-category $n\text{Rep}(G) \leftrightarrow (n + 1)$D 0-symmetry $G$.
1+1D charged excitations form a fusion 1-category

- 1+1D $\mathbb{Z}_2$-symmetric model: the point-like excitations $\{1, e\}$ have the same fusion rule $e \otimes e = 1$. $e = \mathbb{Z}_2$ charge (representation).
- $\{1, e\}$ generate a **fusion 1-category** $\text{Rep}_{\mathbb{Z}_2}$ in 1+1D.
  - bosonic point-like excitations $\{1, e\}$ → objects.
  - instantons in spacetime connecting point-like excitations → 1-morphisms = top morphisms = local symmetric operators $\{\delta H\}$.
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**Categorical description of symmetry**: an $n+1$D (algebraic higher) symmetry is describe by a fusion $n$-category

  **Fusion rule** $\leftrightarrow$ **conservation law** $\leftrightarrow$ **symmetry**

- **Examples**: (1) $n+1$D 0-symmetry $G \leftrightarrow n\text{Rep}_G$.
  (2) $n+1$D algebraic $(n-1)$-symmetry $\tilde{G}^{(n-1)} \leftrightarrow n\text{Vec}_G$. 

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The point-like, string-like, etc excitations in a topological order are also described by a fusion higher category $\mathcal{C}$.

- **Excitations in 2+1D $\mathbb{Z}_2$ topological order** ($\mathbb{Z}_2$ gauge theory with charges, or toric code) $\rightarrow$ a fusion 2-category denoted as $G^2_{\mathbb{Z}_2}$:
  - **2-morphisms** $= \text{top morphisms} = \text{local operators} \{\delta H\}$.
  - **1-morphisms** $= \text{point-like excitations, domain walls}$.

3 bosons: $1$, $e$, $m$; 1 fermion: $f$ (elementary excitations).

- $e \otimes e = 1 \rightarrow \text{mod-2 conservation of } e\text{-particles} \rightarrow \mathbb{Z}_2^e \text{ symmetry}$.
- $m \otimes m = 1 \rightarrow \text{mod-2 conservation of } m\text{-particles} \rightarrow \mathbb{Z}_2^m \text{ symmetry}$.
- $f \otimes f = 1 \rightarrow \text{mod-2 conservation of } f\text{-particles} \rightarrow \mathbb{Z}_2^f \text{ symmetry}$.

- **Objects** $= \text{string-like excitations} \subseteq \{1_s, e_s, m_s, f_s\}$ (descendent)

  - $e_s$-string: $e$-condensation, $\mathbb{Z}_2^e$ SSB.
  - $m_s$-string: $m$-condensation, $\mathbb{Z}_2^m$ SSB.
  - $f_s$-string: $f$-condensation, $\mathbb{Z}_2^f$ SSB.

  *In 1+1D, fermion can condensed $\rightarrow$ p-wave topo. superconductor*
Which fusion higher categories describe symmetry

- Fusion 2-category $G^2_{\mathbb{Z}_2}$ describes a 2+1D topological order with no symmetry.
- Fusion 2-category $2\text{Rep}_{\mathbb{Z}_2}$ describes a 2+1D product state (trivial topological order) with $\mathbb{Z}_2$ symmetry.
Which fusion higher categories describe symmetry

- Fusion 2-category $G^2_{\mathbb{Z}_2}$ describes a 2+1D topological order with no symmetry → cannot be mapped to $\text{2Vec}$
- Fusion 2-category $\text{2Rep}_{\mathbb{Z}_2}$ describes a 2+1D product state (trivial topological order) with $\mathbb{Z}_2$ symmetry → can be mapped to $\text{2Vec}$
- For $S_3$ symmetry, $\text{2Rep}_{S_3}$ has 3 types of particles:
  (3 types of $S_3$-charge, or 3 $S_3$ irreducible representations)
  (1) $\mathbf{1}$ trivial rep. (2) $a$ 1-dim. rep. (3) $b$ 2-dim. rep.
- $\text{2Vec}$ has 1 type of particles: (1) $\mathbf{1}$ trivial
- A map $\beta$ for $\text{2Rep}_{S_3}$ to $\text{2Vec}$ (particles to particles at 1-morphism level): $\mathbf{1} \rightarrow \mathbf{1}, \quad a \rightarrow \mathbf{1}, \quad b \rightarrow \mathbf{1} \oplus \mathbf{1}$ (accidental degeneracy)
- A map $\beta$ for $\text{2Rep}_{S_3}$ to $\text{2Vec}$ (operators to operators at 2-morphism level): local symmetric operators → local operators
- The $\mathbb{Z}_2$ topological order $G^2_{\mathbb{Z}_2}$ contains fermions and cannot be mapped to trivial topological order $\text{2Vec}$ with only bosons.
A theory for the most general symmetry

0-symmetry is classified by group $G$.
Higher symmetry is classified by higher group.

• **Algebraic higher symmetry** in $n + 1$D is classified by local fusion $n$-category $\mathcal{R}$, which is fusion $n$-category that has a top-faithful functor to trivial fusion $n$-category $n\text{Vec}$:

$$\beta : \mathcal{R} \overset{\text{top}}{\rightarrow} n\text{Vec}.$$ 

*This includes symmetry and higher symmetry.*

- **top-faithful** means the map $\beta$ is injective at the top morphisms level (*ie* the map local symmetric operators $\rightarrow$ local operators is injective).

• Physically, the functor $\beta$ means “ignore the symmetry” or “explicitly break the symmetry”. The charged excitations $\mathcal{R}$ of a symmetry map to the excitations $n\text{Vec}$ (with possible accidental degeneracy) in trivial product state, (such as spin-$1/2 \rightarrow 1 \oplus 1$)
A second theory for algebraic higher symmetry

- Fusion 2-category $G^2_{\mathbb{Z}_2}$ describes a 2+1D topological order with no symmetry
- Fusion 2-category $2\text{Rep}_{\mathbb{Z}_2}$ describes a 2+1D product state (trivial topological order) with $\mathbb{Z}_2$ symmetry

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Symmetry as a shadow topological order
A second theory for algebraic higher symmetry

- Fusion $\mathcal{G}^2_{\mathbb{Z}_2}$ describes a 2+1D topological order with no symmetry $\rightarrow$ **non-degenerate**

- Fusion 2-category $\mathcal{2Rep}_{\mathbb{Z}_2}$ describes a 2+1D product state (trivial topological order) with $\mathbb{Z}_2$ symmetry $\rightarrow$ **degenerate**


- **Non-degenerate = remotely detectable:**
  Every non-trivial *elementary* excitation is remotely detectable by at least one excitation via remote operations (such as braiding) $\leftrightarrow$ the topological order is realizable by lattice model in the same dimension without symmetry $\leftrightarrow$ the topological order is realizable by a boundary of trivial topological order (product state) in one higher dimension without symmetry $= \text{realizable w/o symm.}$

- Non-degenerate = realizable = trivial bulk $\rightarrow$ no symmetry

- Degenerate = not realizable = bulk topo. order $\rightarrow$ symmetry
A fusion higher category $C$ describes excitations in a topological order with or without symmetry. *How to see the symmetry?*

**Holographic principle of topological order**

- **A conjecture**: A fusion $n$-category $C$ can always be realized by the excitations on a certain boundary of a qubit system in one higher dimension (i.e., in $(n+1)$-dimensional space).

- **Another conjecture**: A boundary fusion $n$-category $C$ uniquely determines a bulk topological order $\mathcal{M} = Z_1(C)$.  
  - $\mathcal{M}$ is the braided fusion $n$-category describing codimension-2 and higher excitations in the bulk topological order.
  - $Z_1$ is the $E_1$ center (Drinfeld center for $n = 1$).

- **Symmetry = topological order in one higher dimension**
  - The symmetry in fusion higher category $C$ is given by $\mathcal{M} = Z_1(C)$.
  - $\mathcal{M} = n\text{Vec}$ means no symmetry.
2+1D $\mathbb{Z}_2$ symm as a shadow of 3+1D topo order

2+1D $\mathbb{Z}_2$-symmetry = Fusion 2-category $2\text{Rep}_{\mathbb{Z}_2}$

Under the holographic point of view:
2+1D $\mathbb{Z}_2$-symm $\equiv$ 3+1D topo. order $\mathbb{Z}_1[2\text{Rep}_{\mathbb{Z}_2}] = G_3^{\mathbb{Z}_2}$
which is a 3+1D $\mathbb{Z}_2$ gauge theory

- The elementray excitations in 3+1D $\mathbb{Z}_2$-gauge theory
  point-like excitations $e$ (the bosonic $\mathbb{Z}_2$ charge) and
  string-like excitations $s$ (the bosonic $\mathbb{Z}_2$-flux string)

- Bulk fusion rule: $e \otimes e = 1, \quad s \otimes s = 1_s$ (trivial string)

- $2\text{Rep}_{\mathbb{Z}_2}$ is a boundary of $G_3^{\mathbb{Z}_2}$, induced by the
  $\mathbb{Z}_2$-flux loop condensation, i.e the boundary
  excitations are described by $\{1, e\} = 2\text{Rep}_{\mathbb{Z}_2}$.

- Meaning of 2+1D symmetry $\equiv$ 3+1D topo order
  The class of 2 + 1D $\mathbb{Z}_2$-symm. Hamiltonians = The class of
  boundary Hamiltonians of 3 + 1D $\mathbb{Z}_2$ gauge theory ($\infty$ bulk gap)
2+1D \( \mathbb{Z}_2^{(1)} \) symm as a shadow of 3+1D topo order

2 + 1D \( \mathbb{Z}_2^{(1)} \)-symmetry = Fusion 2-category \( 2\text{Vec}_{\mathbb{Z}_2} \)

- The objects in \( 2\text{Vec}_{\mathbb{Z}_2} = \) the string-like excitations, the charge-object of \( \mathbb{Z}_2^{(1)} \) 1-symm, labeled by the elements in group \( \mathbb{Z}_2 \).

- Under the holographic point of view:
  \( 2+1\text{D } \mathbb{Z}_2^{(1)}\)-symm = 3+1D topo. order \( \mathbb{Z}_1[2\text{Vec}_{\mathbb{Z}_2}] = G^3_{\mathbb{Z}_2} \) which is a 3+1D \( \mathbb{Z}_2 \) gauge theory

- \( 2\text{Vec}_{\mathbb{Z}_2} \) is a boundary of \( G^3_{\mathbb{Z}_2} \), induced by the \( \mathbb{Z}_2 \)-charge condensation, ie the boundary excitations are described by \( \{1, s\} = 2\text{Vec}_{\mathbb{Z}_2} \).
2+1D $\mathbb{Z}_2^{(1)}$ symm as a shadow of 3+1D topo order

2+1D $\mathbb{Z}_2^{(1)}$-symmetry = Fusion 2-category $2\mathcal{V}ec_{\mathbb{Z}_2}$

- The objects in $2\mathcal{V}ec_{\mathbb{Z}_2}$ = the string-like excitations, the charge-object of $\mathbb{Z}_2^{(1)}$ 1-symm, labeled by the elements in group $\mathbb{Z}_2$.

- Under the holographic point of view:
  2+1D $\mathbb{Z}_2(1)$-symm = 3+1D topo. order $\mathbb{Z}_1[2\mathcal{V}ec_{\mathbb{Z}_2}] = G^3_{\mathbb{Z}_2}$

which is a 3+1D $\mathbb{Z}_2$ gauge theory

- $2\mathcal{V}ec_{\mathbb{Z}_2}$ is a boundary of $G^3_{\mathbb{Z}_2}$, induced by the $\mathbb{Z}_2$-charge condensation, ie the boundary excitations are described by $\{1, s\} = 2\mathcal{V}ec_{\mathbb{Z}_2}$.

- In $n + 1$D, the symmetry $G$ and its dual algebraic $(n - 1)$-symmetry $\tilde{G}^{(n-1)}$ are described by the same topological order in one higher dimension $G^n_G$ – the $n + 2$D $G$-gauge theory.

- Topo order in one higher dimension = Categorical symmetry $G$ and $\tilde{G}^{(n-1)}$ have the same categorical symm: $G^n_G = G \lor \tilde{G}^{(n-1)}$, and are dual-equivalent.
Boundary symmetry from bulk topological order

- The mod 2 conservation of the bulk particle $e \rightarrow$ The $\mathbb{Z}_2$ 0-symmetry in the bulk.
  The mod 2 conservation of the bulk string $s \rightarrow$ The $\mathbb{Z}_2^{(1)}$ 1-symmetry in the bulk
- The $\mathbb{Z}_2$ and $\mathbb{Z}_2^{(1)}$ symmetry in the bulk becomes the $\mathbb{Z}_2$ and $\mathbb{Z}_2^{(1)}$ symmetry on the boundary.
  Such a larger symmetry $\mathbb{Z}_2 \vee \mathbb{Z}_2^{(1)} = \text{Categorical symmetry}$.

- String-$s$ condensed boundary: $\mathbb{Z}_2$ 0-symmetry is not broken, $\mathbb{Z}_2^{(1)}$ 1-symmetry is spontaneously broken.
  Particle-$e$ condensed boundary: $\mathbb{Z}_2$ 0-symmetry is spontaneously broken, $\mathbb{Z}_2^{(1)}$ 1-symmetry is not broken.

- The boundary with no String-$s$ condensation, nor particle-$e$ condensation is gapless and has the full categorical symmetry $\mathbb{Z}_2 \vee \mathbb{Z}_2^{(1)} = \mathcal{G}_Z^3 (= 3 + 1D \mathbb{Z}_2$-gauge theory with charges).
Boundary symmetry from bulk topological order

- A general algebraic higher symmetry is described by fusion role in a local fusion higher category $\mathcal{R}$. It is a shadow topological order in one higher dimension given by $\mathcal{M} = Z_1(\mathcal{R})$.

- **Physical meaning**: $\mathcal{R}$ describes the excitations on a boundary of bulk topological order $\mathcal{M} = Z_1(\mathcal{R})$.

- Algebraic higher symmetry $\mathcal{R}$ implies a “larger” categorical symmetry $\mathcal{M} = Z_1(\mathcal{R})$. *The fusion rule of the excitations in $\mathcal{M}$ gives to the categorical symmetry.*

- The $\mathcal{R}$ boundary of the bulk $\mathcal{M}$ is obtained by condensing a set of excitations $A_{\mathcal{R}}$. The uncondensed part is the symmetry $\mathcal{R}$. Thus $\mathcal{R} = \mathcal{M} / A_{\mathcal{R}}$.

- If nothing condense at the boundary, we have a gapless boundary with the full categorical symmetry $\mathcal{M}$. *Ji-Wen arXiv:1912.13492*

**Proposal**: gapless states are fully described by categorical symmetry
Two algebraic higher symmetries $\mathcal{R}$ and $\mathcal{R}'$ are dual equivalent if they have the same categorical symmetry (i.e., bulk topological order): $Z_1(\mathcal{R}) = Z_1(\mathcal{R}')$.

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We have seen that $G$ and $\tilde{G}^{(n-1)}$ are dual to each other. Or $n\text{Rep}_G$ and $n\text{Vec}_G$ are dual to each other.

Two algebraic higher symmetries $\mathcal{R}$ and $\tilde{\mathcal{R}}$ are dual to each other if $Z_1(\mathcal{R}) = Z_1(\tilde{\mathcal{R}})$ and their stacking through $\mathcal{M} = Z_1(\mathcal{R}) = Z_1(\tilde{\mathcal{R}})$ is a trivial topological order.
Properties of algebraic higher symmetries

• Two algebraic higher symmetries $\mathcal{R}$ and $\mathcal{R}'$ are dual equivalent if they have the same categorical symmetry (i.e., bulk topo. order): $Z_1(\mathcal{R}) = Z_1(\mathcal{R}')$.


We have seen that $G$ and $\tilde{G}^{(n-1)}$ are dual to each other. Or $n\text{Rep}_G$ and $n\text{Vec}_G$ are dual to each other.

• Two algebraic higher symmetries $\mathcal{R}$ and $\tilde{\mathcal{R}}$ are dual to each other if $Z_1(\mathcal{R}) = Z_1(\tilde{\mathcal{R}})$ and their stacking through $\mathcal{M} = Z_1(\mathcal{R}) = Z_1(\tilde{\mathcal{R}})$ is a trivial topological order.

We can also “gauge” an algebraic higher symmetry $\mathcal{R}$ to obtain a topological order of the same dimension.

• The topological order from the gauging algebraic higher symmetry $\mathcal{R}$ is given by stacking two $\mathcal{R}$ through $\mathcal{M} = Z_1(\mathcal{R})$.
The essence of a symmetry

A symmetry is the shadow of a topological order in one higher dimension (\textit{ie} categorical symmetry)

The same topological order (in one higher dimensions) can have different shadows $\rightarrow$ \textbf{dual-equivalent} symmetries.
• A **gapless state** is very special, and has a lot of emergent symmetries. The full (?) emergent symmetry is the **maximal categorical symmetry**

  Ji-Wen arXiv:1912.13492

• A **categorical symmetry** is a **topological order** in one higher dimension.

• **Maximal categorical symmetry** (*i.e.* topological order in one higher dimension) may completely (?) determines a **gapless state**.

• We can classify all **gapped liquid phases** in systems with a **categorical symmetry**. Such a classification includes
  - SETs with algebraic higher symmetry
  - SPTs with algebraic higher symmetry

• Gauge the algebraic higher symmetry

• Anomalous algebraic higher symmetry