Symmetry as a shadow topological order

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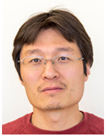
2020/12/08

Kong-Lan-Wen-Zhang-Zheng arXiv:2003.08898; arXiv:2005.14178




Simons Collaboration on
Ultra-Quantum Matter

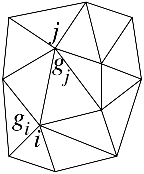
Symmetry in quantum systems

- What is a **quantum system**?

- Total Hilbert space $\mathcal{V} = \bigotimes_i \mathcal{V}_i$ (*$\mathcal{V}_i$ finite Hilbert space on vertex-i*)
- A local Hamiltonian (a hermitian operator) of form $H = \sum O_i + O_{ij}$

- What is a **symmetry** in quantum system?

- A symmetry is a set of **linear constraints** on local operators → **local symmetric operators**
- Sum of local symmetric operators → **symmetric Hamiltonian**.



Symmetry in quantum systems

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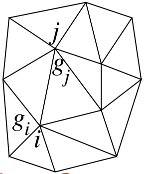
- What is a **symmetry** in quantum system?

- A symmetry is a set of **linear constraints** on local operators $\to$ **local symmetric operators**
- Sum of local symmetric operators $\to$ **symmetric Hamiltonian**.

- **A example**: A lattice model on a triangulation of a n-dimensional space M^n. Local Hilbert space $\mathcal{V}_i$ on each vertex (or site) is spanned by $|g_i\rangle, g_i \in G$.



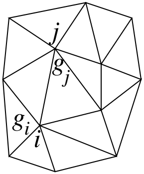
- The **on-site** (gaugable, anomaly-free) symm. (right-action)

$$W_h = \bigotimes_i W_h(i), \quad W_h|\cdots g_i, g_j \cdots\rangle = |\cdots g_i h, g_j h \cdots\rangle, \ h \in G.$$

- The set of **local symmetric operators** (which form an algebra)

$$\mathcal{A} = \{O_{ij} \, \big| W_h O_{ij} = O_{ij} W_h, \ \forall h \in G\}$$

A quantum system with 0-symmetry G



- A **symmetric Hamiltonian** (in the basis $|\cdots g_i, g_j \cdots\rangle$)

$$H_{\text{site}} = -J \sum_i \delta(g_i g_j^{-1}) - B \sum_i \sum_{h \in G} T_h(i)$$

$$\delta(g_i g_j^{-1}), \ \sum_{h \in G} T_h(i) \in \mathcal{A} = \{O_{ij} \,|W_h O_{ij} = O_{ij} W_h\}$$

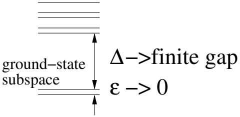
$$T_h(i)|\cdots, g_i, g_j, \cdots\rangle = |\cdots, hg_i, g_j, \cdots\rangle, \ \text{left-action.}$$

- When $B = 0, J > 0$, there are $|G|$ degenerate ground states:

$$\{|\cdots, g, g, g, \cdots\rangle = \bigotimes |g\rangle_i \,\big|\, g \in G\}$$

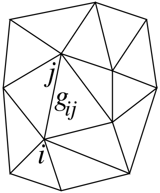
but a unique **symmetric** ground state $\sum_{g \in G} |\cdots, g, g, g, \cdots\rangle$

- The degenerate ground states $\to$ spontaneous symmetry breaking.
- In the ground state subspace, the product states are not symmetric, and the symmetric state is not product states $\to$ **spontaneous symm. breaking**.



- When $J = 0, B > 0$, there is a unique ground state $\otimes_i \sum_{g_i \in G} |g_i\rangle_i$, which is a symmetric product state $\to$ **no symmetry breaking**

A non-Abelian duality



- We consider duality trasnformation of H_{site}. The dual model has local Hilbert space $\mathcal{V}_{ij}$ for each link ij, where $\mathcal{V}_{ij}$ is spanned by $|g_{ij}\rangle, g_{ij} \in G$, with $g_{ij} = g_{ji}^{-1}$.
- The duality map between local operators:

$$g_i g_j^{-1} \to g_{ij}, \quad T_h(i) \to \tilde{T}_h(i) : |\cdot, g_{ij}, g_{ik}, g_{jk}, \cdot\rangle \to |\cdot, hg_{ij}, hg_{ik}, g_{jk}, \cdot\rangle$$

$$H_{\text{link}} = -J \sum_{ij} \delta(g_{ij}) - B \sum_{i} \sum_{h \in G} \tilde{T}_h(i) - U_\infty \sum_{ijk} \delta(g_{ij} g_{jk} g_{ki}),$$

When $J = 0$, H_{link} is a lattice gauge theory.

\- $U_\infty = \infty$ makes $g_{ij} g_{jk} g_{ki} = 1 \to$ flat connection. In this case, there is a $|G|$-to-1 correspondence ($*$ base point)

$$g_i h (g_j h)^{-1} \to g_{ij}, \quad g_i h (g_* h)^{-1} \leftarrow g_{ij} g_{jk} \cdots g_{l*}, \quad h \in G.$$

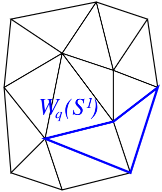
- H_{site} *within the symmetric sub Hilbert space* has identical eigenvalues as H_{link} below U_∞ *if the space M^n satisfies* $\pi_n(M^n) = 0$

Algebraic higher symmetry

Ji-Wen arXiv:1912.13492, Kong-Lan-Wen-Zhang-Zheng arXiv:2005.14178

- $H_{\text{link}} = -J\sum_{ij}\delta(g_{ij}) - B\sum_i\sum_{h\in G}\tilde{T}_h(i) - U_\infty\sum_{ijk}\delta(g_{ij}g_{jk}g_{ik}^{-1})$ has an **Algebraic $(n-1)$-symmetry** $\tilde{G}^{(n-1)}$, generated by Wilson loop operators (for **any loops** S^1) $W_q(S^1) = \mathrm{Tr}\prod_{\langle ij\rangle\in S^1} R_q(g_{ij})$.

  R_q is an irreducible representation of G labeled by q.



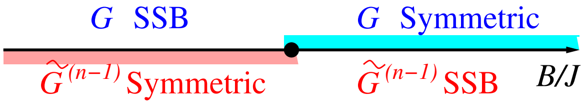
- The **local symmetric operators**
  $$\mathcal{A} = \{O_{ij}\ |W_q(S^1)O_{ij} = O_{ij}W_q(S^1),\ \forall q \text{ and } \forall S^1\}$$
  and H_{link} is a sum of local symmetric operators.

- The symmetry acts on all codimension-$(n-1)$ closed subspace S^1.
- The symmetry satisfies $W_{q_1}(S^1)W_{q_2}(S^1) = \sum_{q_3} N^{q_3}_{q_1q_2}W_{q_3}(S^1)$, which do not form a group for non-Abelian G.
  $\to$ **algebraic $(n-1)$-symmetry (algebraic higher symmetry)**
- If G is Abelian, the $(n-1)$-symmetry (**higher symmetry**) $\tilde{G}^{(n-1)}$ is described by a higher group. Gaiotto-Kapustin-Seiberg-Willett arXiv:1412.5148

Spontaneous breaking of algebraic higher symmetry

- G symmetry has a phenomenon: spontaneous symmetry breaking
$$H_{\text{site}} = -J\sum_i \delta(g_i g_j^{-1}) - B\sum_i \sum_{h\in G} T_h(i)$$

$|G|^{\pi_0(M^n)}$ degenerate ground states — Unique ground state on any M^n



Unique ground states on any M^n — Degenerate ground states on some M^n

$|\text{grnd}\rangle = \bigotimes_{ij} |g_{ij} = 1\rangle$ — $|\text{grnd}_\alpha\rangle = \sum_{g_{ij}g_{jk}g_{ki}=1} \bigotimes_{ij} |g_{ij}\rangle$

- The algebraic $(n-1)$-symm. also has spontaneous symm. breaking
$$H_{\text{link}} = -J\sum_{ij} \delta(g_{ij}) - B\sum_i \sum_{h\in G} \tilde{T}_h(i) - U_\infty \sum_{ijk} \delta(g_{ij}g_{jk}g_{ki})$$
- Detect (define) **spontaneous symmetry breaking (SSB)**:
1) **Degenerate ground states** on some M^n. 2) Some symmetry transformations are **not identity** in the groundstate subspace
- The critical point has both the symm G and the dual algebraic $(n-1)$-symm $\tilde{G}^{(n-1)}$, *ie* has the **categorical symm** $G \vee \tilde{G}^{(n-1)}$

SSB of (algebraic) higher symm. and topo. order

- The ground state degeneracy from SSB of a 0-symmetry is robust against any local **symmetry preserving perturbations**.

- *The ground state degeneracy is a property of the symmetry.*

- The ground state degeneracy from SSB of a finite **(algebraic) higher symmetry** is robust against any **local perturbations** that can break the symmetry.

- *The ground state degeneracy is actually not a property of the (algebraic) higher symmetry.* Hastings-Wen cond-mat/0503554

- SSB of $U(1)$ 1-symmetry gives rise to gapless $U(1)$-gauge bosons, which is robust against any local perturbations.

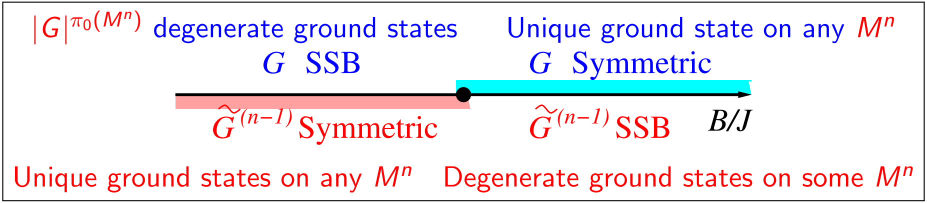
- SSB of finite (algebraic) higher symmetries $\to$ topological orders
- Some topological orders $\to$ (1) emergence of an (algebraic) higher symmetry, (2) which is spontaneously broken. Wen arXiv:1812.02517

- Emergent (algebraic) higher symmetry is present in such topological orders and their continuous transition to the neighboring phases.

The dual-equivalence of two symm. $G \overset{\text{dual}}{\sim} \tilde{G}^{(n-1)}$

The following two Hamiltonians are **dual-equivalent**

$H_{\text{site}} = -J\sum_i \delta(g_i g_j^{-1}) - B\sum_i \sum_{h\in G} T_h(i)$

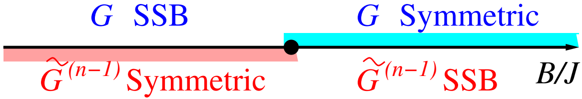


$$H_{\text{link}} = -J\sum_{ij}\delta(g_{ij}) - B\sum_i \sum_{h\in G} \tilde{T}_h(i) - U_\infty \sum_{ijk}\delta(g_{ij}g_{jk}g_{ik}^{-1})$$

- More generally, an arbitary H_{site} with 0-symmetry $G \overset{\text{dual}}{\longleftrightarrow}$ a $H'_{\text{link}} - U_\infty \sum_{ijk}\delta(g_{ij}g_{jk}g_{ik}^{-1})$ with the algebraic $(n-1)$-symmetry, *ie the two Hamiltonians are equivalent, same spectrum etc*
- The 0-symmetry G and the algebraic $(n-1)$-symmetry $\tilde{G}^{(n-1)}$ represent the equivalent constraints that select the "same" class of Hamiltonians. We call them **dual-equivalent** symmetries.

G or $\tilde{G}^{(n-1)} \to$ full categorical symm $G \vee \tilde{G}^{(n-1)}$

$|G|^{\pi_0(M^n)}$ degenerate ground states | Unique ground state on any M^n



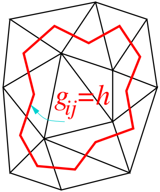
Unique ground states on any M^n | Degenerate ground states on some M^n

- A **Hamiltonian** with symmetry G also has the dual symmetry $\tilde{G}^{(n-1)}$, and also has the categorical symmetry $G \vee \tilde{G}^{(n-1)}$
- A **Hamiltonian** with algebraic higher symmetry $\tilde{G}^{(n-1)}$ also has the dual symmetry G, and also has the categorical symm $G \vee \tilde{G}^{(n-1)}$
- The **gapped ground state** must spontaneous break part of the categorical symmetry $G \vee \tilde{G}^{(n-1)}$, such as G, $\tilde{G}^{(n-1)}$, or some of their combination.
- The ground state with the full categorical symmetry $G \vee \tilde{G}^{(n-1)}$ must be gapless.

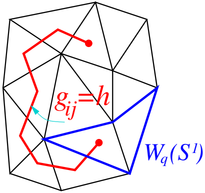
Levin arXiv:1903.09028; Ji-Wen arXiv:1912.13492

Algebraic higher symm $\tilde{G}^{(n-1)}$ & its charge object

- $H_{\text{link}} = -J\sum_{ij}\delta(g_{ij}) - B\sum_i\sum_{h\in G}\tilde{T}_h(i) - U_\infty\sum_{ijk}\delta(g_{ij}g_{jk}g_{ik}^{-1})$



- $H_{\text{link}}\big|_{B=0}$ has an **unique symmetric ground state** $|\text{grnd}\rangle = \bigotimes_{ij}|g_{ij}=1\rangle$ on close space M^n of any topology.
- A $(n-1)$-dimensional excitation h on top of the ground state: chang $g_{ij}=1$ to $g_{ij}=h$ on a $(n-1)$-dimensional **closed subspace** = a **charge object** of the algebraic $(n-1)$-symmetry
- *charge object for* 0*-symmetry* = **charge-anti-charge pair** *on* S^0.



- A **charged object** = changing $g_{ij}=1$ to $g_{ij}=h$ on a $(n-1)$-dimensional **subspace with boundary**. *The boundary = the gauge flux*
- Measure the $(n-1)$-**charge** of a charged object: $W_q(S^1)|h\rangle = \mathrm{Tr}R_q(h)|h>$.
- h and $h' = ghg^{-1}$ carry the same $(n-1)$-charge.

A unified theory of symmetry, higher symmetry, and algebraic higher symmetry

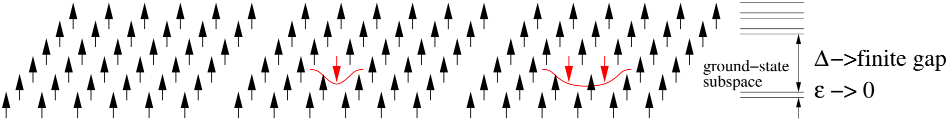
- 0-**symmetry** is described by **group** G.
  Higher symmetry is described by **higher group**.
  Gaiotto-Kapustin-Seiberg-Willett arXiv:1412.5148
  What mathematics describes **algebraic higher symmetry**?
- We plan to use charged excitations to describe symmetry, higher symmetry, and algebraic higher symmetry in a unified way.

- 0-**symmetry**:
  charge object = charge-anti-charge pair S^0.
  charged object = single point (part of S^0).
- 1-**symmetry**:
  charge object = loop excitations S^1.
  charged object = open-string excitations (part of S^1).

- **Algebraic higher symmetry**: charged objects = point-like, string-like, membrane like excitations → **higher fusion category**.

Categorical view of symmetry Lan-Kong-Wen arXiv:1602.05946

- A **symmetric quantum system**: **Hamiltonian** $H = -\sum_i Z_i$ (X_i, Y_i, Z_i Pauli operator) plus **local symmetric operators** $\{\delta H\}$.



- **An excitation** = something can be trapped by $\delta H_{\text{trap}} = 2Z_{i_0}$ or $\delta H_{\text{trap}} = 2(\cos\theta Z_{i_0} + \sin\theta X_{i_0})$. H has a gap
  $H + \delta H_{\text{trap}}$ also has a gap → trapped states: $|\downarrow\rangle$, $|\downarrow\downarrow\rangle$, ...

- **Type**: *(determined by H & local symmetric operators $\{\delta H\}$)*

- Without symmetry *($\{\delta H\}$ are formed by any local operators)*
  $|\downarrow\rangle$, $|\downarrow\downarrow\rangle$ can be deformed into $|0\rangle$ without closing the gap.
  $|\downarrow\rangle \sim |\downarrow\downarrow\rangle \sim |0\rangle$ the same trivial type $\mathbf{1}$.
- With $\mathbb{Z}_2$-symm. $U = \prod_i Z_i$ *(symm. operator $\{\delta H \mid \delta H U = U \delta H\}$)*
  $|\downarrow\downarrow\rangle$ and $|0\rangle$ are of the same trivial type $\mathbf{1}$ *($\mathbb{Z}_2$-charge-0)*.
  $|\downarrow\rangle$ has a different type e *($\mathbb{Z}_2$-charge-1)*.

2+1D charged excitations form a fusion 2-category

- In 2+1D model with the $\mathbb{Z}_2$-symmetry, the excitation fusion rule:
  $\mathbf{1} \otimes \mathbf{1} = \mathbf{1}, \quad \mathbf{1} \otimes e = e \otimes \mathbf{1} = e, \qquad \mathbf{1} =$ the fusion unit
  $e \otimes e = \mathbf{1}, \quad e = \mathbb{Z}_2$-charge (-represenation) with mod-2 conservation

- $\{\mathbf{1}, e\}$ generate a **fusion 2-category** $2\mathcal{R}\mathrm{ep}_{\mathbb{Z}_2}$ in 2+1D:

- string-like excitations ($\{\mathbf{1}_s, e_s\}$ *trivial, descendent*) → objects.
  e_s = e-condensed & spontaneous $\mathbb{Z}_2$-symm. breaking string.
  e_s is called **descendant excitation**

Kong-Wen arXiv:1405.5858
Gaiotto & Johnson-Freyd arXiv:1905.09566

- point-like excitations $\{\mathbf{1}, e\}$ **elementray** → 1-morphisms $\mathbf{1}_s \to \mathbf{1}_s$
  domain-wall excitations $\mathbf{1}_s \to e_s$ → 1-morphisms $\mathbf{1}_s \to e_s$.
- instantons in spacetime connecting point-like excitations →
  2-morphisms = top morphisms = local symmetric operators $\{\delta H\}$.

- 2 layers of morphisms → a fusion 2-category $2\mathcal{R}\mathrm{ep}_{\mathbb{Z}_2}$.

1+1D charged excitations form a fusion 1-category

- 1+1D $\mathbb{Z}_2$-symmetric model: the point-like excitations $\{\mathbf{1}, e\}$ have the same fusion rule $e \otimes e = \mathbf{1}$. $e = \mathbb{Z}_2$ charge (reppresentation).
- $\{\mathbf{1}, e\}$ generate a **fusion 1-category** $\mathcal{R}\mathrm{ep}_{\mathbb{Z}_2}$ in 1+1D.

- bosonic point-like excitations $\{\mathbf{1}, e\}$ $\to$ objects.
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- **Tannaka duality**: The above fusion 1-category is a **symmetric fusion category** $\mathcal{E} \leftrightarrow$ finite group G.
  Objects = R_q representations of G. We denote $\mathcal{E} = \mathcal{R}\text{ep}(G)$.

- In $(n+1)$D, charged excitations described by fusion n-category $n\mathcal{R}\text{ep}(G) \leftrightarrow (n+1)$D 0-symmetry G

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- **Categrical description of symmetry**: an $n+1$D (algebraic higher) symmetry is describe by a fusion n-category

  Fusion rule $\leftrightarrow$ conservation law $\leftrightarrow$ symmetry

- **Examples**: (1) $n+1$D 0-symmetry $G \leftrightarrow n\mathcal{R}\mathrm{ep}_G$.
  (2) $n+1$D algebraic $(n-1)$-symmetry $\tilde{G}^{(n-1)} \leftrightarrow n\mathcal{V}\mathrm{ec}_G$.

Fusion higher category for excitations in topo order

The point-like, string-like, *etc* excitations in a topological order are also described by a fusion higher category $\mathcal{C}$.

- Excitations in 2+1D $\mathbb{Z}_2$ **topological order** ($\mathbb{Z}_2$ guage theory with charges, or toric code) $\to$ a fusion 2-category denoted as $\mathcal{G}^2_{\mathbb{Z}_2}$:

- 2-**morphisms** = top morphisms = local operators $\{\delta H\}$.
- 1-**morphisms** = point-like excitations, domain walls.
  3 bosons: $\mathbf{1}$, e, m; 1 fermion: f (**elementary excitations**).
  $e \otimes e = \mathbf{1} \to$ mod-2 conservation of e-particles $\to \mathbb{Z}_2^e$ symmetry.
  $m \otimes m = \mathbf{1} \to$ mod-2 conservation of m-particles $\to \mathbb{Z}_2^m$ symmetry.
  $f \otimes f = \mathbf{1} \to$ mod-2 conservation of f-particles $\to \mathbb{Z}_2^f$ symmetry.
- **Objects** = string-like excitations = $\{\mathbf{1}_s, e_s, m_s, f_s\}$ (**descendent**)
  e_s-string: e-condasation, $\mathbb{Z}_2^e$ SSB.
  m_s-string: m-condasation, $\mathbb{Z}_2^m$ SSB.
  f_s-string: f-condasation, $\mathbb{Z}_2^f$ SSB. Kitaev cond-mat/0010440
  In 1+1D, fermion can condensed $\to$ p-wave topo. superconductor

Which fusion higher catgeories describe symmetry

- Fusion 2-catgeory $\mathcal{G}^2_{\mathbb{Z}_2}$ describes a 2+1D topological order with no symmetry
- Fusion 2-catgeory $2\mathcal{R}\mathrm{ep}_{\mathbb{Z}_2}$ describes a 2+1D product state (trivial topological order) with $\mathbb{Z}_2$ symmetry

Which fusion higher catgeories describe symmetry

- Fusion 2-catgeory $\mathcal{G}^2_{\mathbb{Z}_2}$ describes a 2+1D topological order with no symmetry → **cannot be mapped to** $2\mathcal{V}\mathrm{ec}$
- Fusion 2-catgeory $2\mathcal{R}\mathrm{ep}_{\mathbb{Z}_2}$ describes a 2+1D product state (trivial topological order) with $\mathbb{Z}_2$ symmetry → **can be mapped to** $2\mathcal{V}\mathrm{ec}$
- For S_3 symmetry, $2\mathcal{R}\mathrm{ep}_{S_3}$ has 3 types of particles: *(3 types of S_3-charge, or 3 S_3 irreducible representations)* (1) $\mathbf{1}$ trivial rep. (2) a 1-dim. rep. (3) b 2-dim. rep.
- $2\mathcal{V}\mathrm{ec}$ has 1 type of particles: (1) $\mathbf{1}$ trivial
- A map β for $2\mathcal{R}\mathrm{ep}_{S_3}$ to $2\mathcal{V}\mathrm{ec}$ (particles to particles at 1-morphism level): $\mathbf{1} \to \mathbf{1}, \quad a \to \mathbf{1}, \quad b \to \mathbf{1} \oplus \mathbf{1}$ (accidental degeneracy)

- A map β for $2\mathcal{R}\mathrm{ep}_{S_3}$ to $2\mathcal{V}\mathrm{ec}$ (operators to operators at 2-morphism level): local symmetric operators → local operators

- The Z_2 topological order $\mathcal{G}^2_{\mathbb{Z}_2}$ conatins fermions and cannot be mapped to trivial topological order $2\mathcal{V}\mathrm{ec}$ with only bosons.

A theory for the most general symmetry

0-symmetry is classified by group G.
Higher symmetry is classified by higher group.

Kong-Lan-Wen-Zhang-Zheng arXiv:2005.14178

- **Algebraic higher symmetry in $n+1$D is classified by local fusion n-category $\mathcal{R}$**, which is fusion n-category that has a top-faithful functor to trivial fusion n-category $n\mathcal{V}\mathrm{ec}$:

$$\beta : \mathcal{R} \overset{\text{top}}{\hookrightarrow} n\mathcal{V}\mathrm{ec}.$$

This includs symmetry and higher symmetry.

\- **top-faithful** means the map β is injective at the top morphisms level (*ie* the map local symmetric operators $\to$ local operators is injective).

- Physically, the functor β means "ingore the symmetry" or "explicitly break the symmetry". The charged excitations $\mathcal{R}$ of a symmetry map to the excitations $n\mathcal{V}\mathrm{ec}$ (with possible accidental degeneracy) in trivial product state, (such as spin-$1/2 \to \mathbf{1} \oplus \mathbf{1}$)

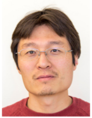
A second theory for algebraic higher symmetry

- Fusion 2-catgeory $\mathcal{G}^2_{\mathbb{Z}_2}$ describes a 2+1D topological order with no symmetry
- Fusion 2-catgeory $2\mathcal{R}\text{ep}_{\mathbb{Z}_2}$ describes a 2+1D product state (trivial topological order) with $\mathbb{Z}_2$ symmetry

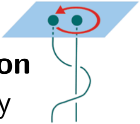
A second theory for algebraic higher symmetry

- Fusion 2-catgeory $\mathcal{G}^2_{\mathbb{Z}_2}$ describes a 2+1D topological order with no symmetry → **non-degenerate**
- Fusion 2-catgeory $2\mathcal{R}\text{ep}_{\mathbb{Z}_2}$ describes a 2+1D product state (trivial topological order) with $\mathbb{Z}_2$ symmetry → **degenerate**

Levin arXiv:1301.7355, Kong-Wen arXiv:1405.5858

- **Non-degenerate = remotely detectable**: Every non-trivial **elementray** excitation is **remotely detectable** by at least one excitation via remote operations (such as braiding) ↔ the topological order is realizable by **lattice model in the same dimension without symmetry** ↔ the topological order is realizable by a **boundary of trivial topological order** (product state) in one higher dimension **without symmetry** = **realizable w/o symm.**



- Non-degenerate = realizable = trivial bulk → no symmetry
- Degenerate = not realizable = bulk topo. order → symmetry

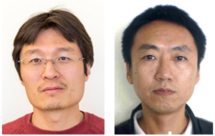
Symmetry as a shadow of topological order

A fusion higher category $\mathcal{C}$ describes excitations in a topological order with or without symmetry. *How to see the symmetry?*

Holographic principle of topological order

- **A conjecture**: A fusion n-category $\mathcal{C}$ can always be realized by the excitations on a certain boundary of a qubit system in one higher dimension (*ie* in $(n+1)$-dimensional space).

- **Another conjecture**: A boundary fusion n-category $\mathcal{C}$ uniquely determines a bulk topological order $\mathcal{M} = Z_1(\mathcal{C})$.



Kong-Wen arXiv:1405.5858; Kong-Wen-Zheng arXiv:1702.00673

- $\mathcal{M}$ is the braided fusion n-category describing codimension-2 and higher excitations in the bulk topological order.
- Z_1 is the E_1 center (Drinfeld center for $n=1$).

- **Symmetry = topological order in one higher dimension**
The symmetry in fusion higher category $\mathcal{C}$ is given by $\mathcal{M} = Z_1(\mathcal{C})$.
- $\mathcal{M} = n\mathcal{V}\mathrm{ec}$ means no symmetry.

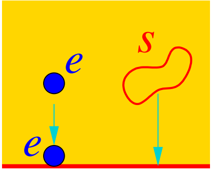
2+1D $\mathbb{Z}_2$ symm as a shadow of 3+1D topo order

2+1D $\mathbb{Z}_2$-symmetry = Fusion 2-category $2\mathcal{R}\mathrm{ep}_{\mathbb{Z}_2}$

Under the holographic point of view:
2+1D $\mathbb{Z}_2$-symm = 3+1D topo. order $Z_1[2\mathcal{R}\mathrm{ep}_{\mathbb{Z}_2}] = \mathcal{G}^3_{\mathbb{Z}_2}$
which is a **3+1D $\mathbb{Z}_2$ gauge theory**

- The **elementray** excitations in 3+1D $\mathbb{Z}_2$-gauge theory point-like excitations e (the bosonic $\mathbb{Z}_2$ charge) and string-like excitations s (the bosonic $\mathbb{Z}_2$-flux string)
- Bulk fusion rule: $e \otimes e = \mathbf{1}$, $\quad s \otimes s = \mathbf{1}_s$ (trivial string)

- $2\mathcal{R}\mathrm{ep}_{\mathbb{Z}_2}$ is a boundary of $\mathcal{G}^3_{\mathbb{Z}_2}$, induced by the $\mathbb{Z}_2$-flux loop condensation, *ie* the boundary excitations are described by $\{\mathbf{1}, e\} = 2\mathcal{R}\mathrm{ep}_{\mathbb{Z}_2}$.



- Meaning of 2+1D symmetry = 3+1D topo order
The class of $2+1$D $\mathbb{Z}_2$-symm. Hamiltonians = The class of boundary Hamiltonians of $3+1$D $\mathbb{Z}_2$ gauge theory (∞ bulk gap)

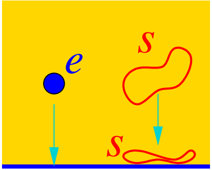
2+1D $\mathbb{Z}_2^{(1)}$ symm as a shadow of 3+1D topo order

$2 + 1$D $\mathbb{Z}_2^{(1)}$-symmetry = Fusion 2-category $2\mathcal{V}\mathrm{ec}_{\mathbb{Z}_2}$

- The objects in $2\mathcal{V}\mathrm{ec}_{\mathbb{Z}_2}$ = the string-like excitations, the charge-object of $\mathbb{Z}_2^{(1)}$ 1-symm, labeled by the elements in group $\mathbb{Z}_2$.

- Under the holographic point of view:
  2+1D $\mathbb{Z}_2(1)$-symm = 3+1D topo. order $Z_1[2\mathcal{V}\mathrm{ec}_{\mathbb{Z}_2}] = \mathcal{G}^3_{\mathbb{Z}_2}$
  which is a 3+1D $\mathbb{Z}_2$ gauge theory

- $2\mathcal{V}\mathrm{ec}_{\mathbb{Z}_2}$ is a boundary of $\mathcal{G}^3_{\mathbb{Z}_2}$, induced by the $\mathbb{Z}_2$-charge condensation, *ie* the boundary excitations are described by $\{\mathbf{1}, s\} = 2\mathcal{V}\mathrm{ec}_{\mathbb{Z}_2}$.



2+1D $\mathbb{Z}_2^{(1)}$ symm as a shadow of 3+1D topo order

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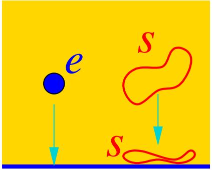
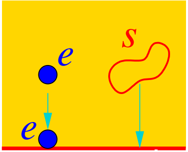


- $2\mathcal{V}\mathrm{ec}_{\mathbb{Z}_2}$ is a boundary of $\mathcal{G}^3_{\mathbb{Z}_2}$, induced by the $\mathbb{Z}_2$-charge condensation, *ie* the boundary excitations are described by $\{\mathbf{1}, s\} = 2\mathcal{V}\mathrm{ec}_{\mathbb{Z}_2}$.
- In $n+1$D, the symmetry G and its dual algebraic $(n-1)$-symmetry $\tilde{G}^{(n-1)}$ are described by the same topological order in one higher dimension $\mathcal{G}^n_G$ – the $n+2$D G-gauge theory.
- **Topo order in one higher dimension = Categorical symmetry**
  G and $\tilde{G}^{(n-1)}$ have the same categorical symm: $\mathcal{G}^n_G = G \vee \tilde{G}^{(n-1)}$, and are **dual-equivalent**.

Boundary symmetry from bulk topological order

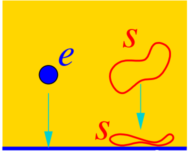
- The mod 2 conservation of the bulk particle $e \to$
  The $\mathbb{Z}_2$ 0-symmetry in the bulk.
  The mod 2 conservation of the bulk string $s \to$
  The $\mathbb{Z}_2^{(1)}$ 1-symmetry in the bulk



- The $\mathbb{Z}_2$ and $\mathbb{Z}_2^{(1)}$ symmetry in the bulk becomes the $\mathbb{Z}_2$ and $\mathbb{Z}_2^{(1)}$ symmetry on the boundary. $\{\mathbf{1}, e\} = 2\mathcal{R}\text{ep}_{\mathbb{Z}_2}$

\- Such a larger symmetry $\mathbb{Z}_2 \vee \mathbb{Z}_2^{(1)}$ = **Categorical symmetry**.

- String-s condensed boundary: $\mathbb{Z}_2$ 0-symmetry is not broken, $\mathbb{Z}_2^{(1)}$ 1-symmetry is spontaneously broken.
  Particle-e condensed boundary: $\mathbb{Z}_2$ 0-symmetry is spontaneously broken, $\mathbb{Z}_2^{(1)}$ 1-symmetry is not broken.

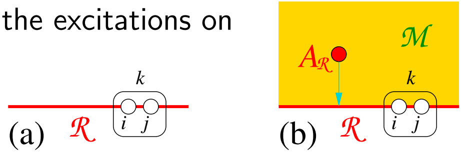


- The boundary with no String-s condensation, nor particle-e condensation is gapless and has the full categorical symmetry $\mathbb{Z}_2 \vee \mathbb{Z}_2^{(1)} = \mathcal{G}^3_{\mathbb{Z}_2}$ ($= 3+1$D $\mathbb{Z}_2$-gauge theory with charges).

Boundary symmetry from bulk topological order

- A general algebraic higher symmetry is described by fusion role in a local fusion higher category $\mathcal{R}$. It is a shadow topological order in one higher dimension given by $\mathcal{M} = Z_1(\mathcal{R})$.

- **Physical meaning**: $\mathcal{R}$ describes the excitations on a boundary of bulk topological order $\mathcal{M} = Z_1(\mathcal{R})$.



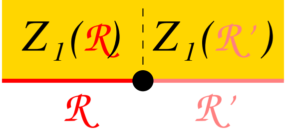
- Algebraic higher symmetry $\mathcal{R}$ implies a "larger" categorical symmetry $\mathcal{M} = Z_1(\mathcal{R})$. *The fusion rule of the excitations in $\mathcal{M}$ gives to the categorical symmetry.*

- The $\mathcal{R}$ boundary of the bulk $\mathcal{M}$ is obtained by condensing a set of excitations $A_{\mathcal{R}}$. The uncondened part is the symmetry $\mathcal{R}$. Thus $\mathcal{R} = \mathcal{M}/A_{\mathcal{R}}$.

- If nothing condense at the boundary, we have a gapless boundary with the full categrical symmetry $\mathcal{M}$. Ji-Wen arXiv:1912.13492

  Proposal: gapless states are fully described by categrical symmetry

Properties of algebraic higher symmetries



- **Two algebraic higher symmetries $\mathcal{R}$ and $\mathcal{R}'$ are dual equivalent if they have the same categorical symmetry (*ie* bulk topo. order): $Z_1(\mathcal{R}) = Z_1(\mathcal{R}')$.**

Kong-Lan-Wen-Zhang-Zheng arXiv:2005.14178

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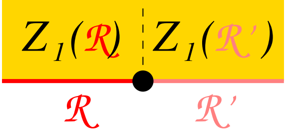
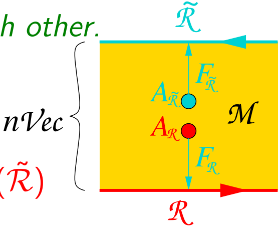


Kong-Lan-Wen-Zhang-Zheng arXiv:2005.14178

We have seen that G and $\tilde{G}^{(n-1)}$ are dual to each other. Or $n\mathcal{R}\mathrm{ep}_G$ and $n\mathcal{V}\mathrm{ec}_G$ are dual to each other.

- **Two algebraic higher symmetries $\mathcal{R}$ and $\tilde{\mathcal{R}}$ are dual to each other if $Z_1(\mathcal{R}) = Z_1(\tilde{\mathcal{R}})$ and their stacking through $\mathcal{M} = Z_1(\mathcal{R}) = Z_1(\tilde{\mathcal{R}})$ is a trivial topological order.**



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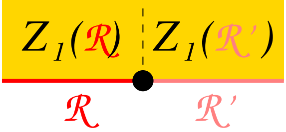

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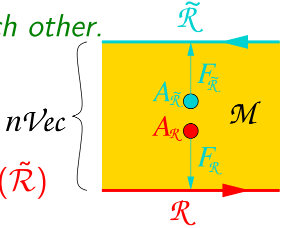

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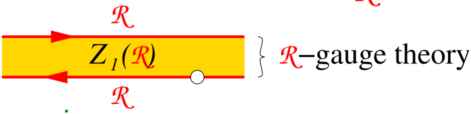
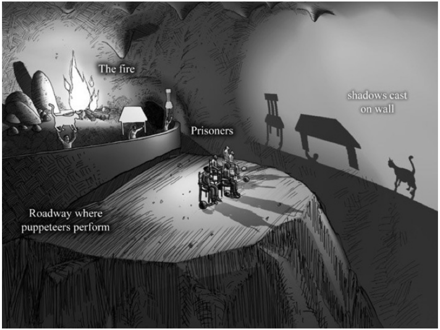
We can also "gauge" an algebraic higher symmetry $\mathcal{R}$ to obtain a topological order of the same dimension.



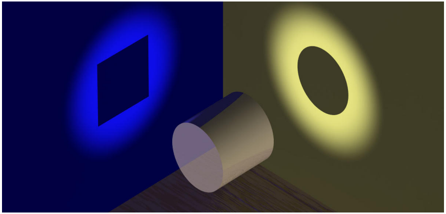
- **The topological order from the gauging algebraic higher symmetry $\mathcal{R}$ is given by stacking two $\mathcal{R}$ through $\mathcal{M} = Z_1(\mathcal{R})$**

The essence of a symmetry

A symmetry is the shadow of a topological order in one higher dimension (*ie* categorical symmetry)



Categorical symmetry
$\rightarrow$ symmetry and dual symmetry



The same topological order (in one higher dimensions) can have different shadows $\rightarrow$ **dual-equivalent** symmetries.

Alg-higher-symm, Cat-symm, and Bulk-topo-order

- A **gapless state** is very special, and has a lot of emergent symmetries. The full (?) emergent symmetry is the **maximal categorical symmetry** Ji-Wen arXiv:1912.13492

- A **categorical symmetry** is a **topological order** in one higher dimension.
- **Maximal categorical symmetry** (*ie* **topogical order** in one higher dimension) may completely (?) determines a **gapless state**.
- We can classify all **gapped liquid phases** in systems with a **categorical symmetry**. Such a classification includes

- SETs with algebraic higher symmetry
- SPTs with algebraic higher symmetry

- Gauge the algebraic higher symmetry
- Anomalous algebraic higher symmetry

Kong-Lan-Wen-Zhang-Zheng arXiv:2005.14178