Uncertainty Principles on Quantum Symmetries

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Uncertainty Principles on Groups
Classical Fourier Duality

In the early 1800’s, Joseph Fourier introduced his transformation to solve differential equations describing heat. The Fourier transform $\mathcal{F}$ on measurable functions $f$ on $\mathbb{R}$ is

$$\mathcal{F}(f)(x) = \int_{-\infty}^{\infty} f(t) e^{-2\pi itx} \, dt.$$  

Convolution for such functions is:

$$(f_1 \ast f_2)(s) = \int_{-\infty}^{\infty} f_1(t) f_2(s - t) \, dt,$$

yielding the Fourier duality

$$\mathcal{F}(f_1 \ast f_2) = \mathcal{F}(f_1) \mathcal{F}(f_2). \quad (1)$$

The Fourier transform switches the position operator $x$ and the momentum operator $p = -i \frac{d}{dx}$. 
Take
\[ \|f\|_p = \left( \int_{-\infty}^{\infty} |f(t)|^p dt \right)^{1/p}, \quad 0 < p < \infty. \]

For \( p \geq 1 \), \( \| \cdot \|_p \) is the \( p \)-norm of measurable functions and \( \|f\|_\infty \) is the essential maximum of \( f \).

Plancherel formula (1910):
\[ \|\mathcal{F}f\|_2 = \|f\|_2. \]

Interpolating with the elementary inequality \( \|\mathcal{F}(f)\|_\infty \leq \|f\|_1 \), one obtains the Hausdorff-Young inequality,
\[ \|\mathcal{F}(f)\|_q \leq \|f\|_p, \quad 1 \leq p \leq 2, \quad 1/p + 1/q = 1. \]  \( (2) \)
In 1927, Heisenberg showed in quantum theory that position and momentum cannot simultaneously be precisely measured (the Heisenberg’s uncertainty principle). This phenomenon has been mathematically formulated by Kennard and by Weyl:

\[ \sigma_x \sigma_p \geq \frac{\hbar}{2}, \]

where \( \hbar \) is Planck’s constant, and \( \sigma \) is the standard deviation. More precisely,

\[ \| xf \|_2 \| pf \|_2 \geq (4\pi)^{-1}, \quad \| f \|_2 = 1. \]

Equivalently

\[ \| xf \|_2 \| xF f \|_2 \geq (4\pi)^{-1}, \quad \| f \|_2 = 1. \]
In 1957, Hirschman studied the Shannon entropy
\[ H(|f|^2) = - \int_{-\infty}^{\infty} |f(x)|^2 \log |f(x)|^2 dx \]
of \( |f|^2 \), proving
\[ H(|f|^2) + H(|\mathcal{F}(f)|^2) \geq 0, \quad \|f\|_2 = 1. \]

Everett conjectured the lower bound is \( \log \frac{e}{2} \), which follows from Beckner’s sharp Hausdorff-Young inequality (1975 Ann. Math.):
\[ \|\mathcal{F}(f)\|_q \leq A_p \|f\|_p. \]

\[ A_p = p^{1/2p}q^{-1/2q}, \quad \text{and} \quad 1/p + 1/q = 1, \ 1 \leq p \leq 2, \]
A Physical interpretation was given by Bialynicki and Birula (1975 CMP).
The extremizers of these inequalities are Gaussian functions.
The Hirschman-Beckner uncertainty principle ensures Heisenberg’s uncertainty principle.
In 1989, Donoho and Stark established an uncertainty principle for functions $f$ on cyclic groups $G$ in terms of the cardinality of their support:

$$|\mathcal{S}(f)| \cdot |\mathcal{S}(\mathcal{F}(f))| \geq |G|,$$

here $f$ is a function on $G$, $\mathcal{S}(f) = \{x : f(x) \neq 0\}$, and $|A|$ is the cardinality of the set $A$.

The Donoho-Stark uncertainty principle was generalized by

- Smith, on finite abelian groups (1990 SIAM),
- Meshulam, on finite groups in (1992 EJC),
- Alagic-Russell, on compact groups (2008 IJM).

This type of uncertainty principle has been applied by Candes-Romberg-Tao and by Donoho in compressed sensing in 2006.
The Meshulam’s uncertainty principle for a (non-abelian) finite group $G$ is a generalization of the Donoho-Stark uncertainty principle. More precisely, take $\mathcal{A} := L^\infty(G)$, functions on $G$, with the discrete measure $d$. Let $\pi$ be the left regular representation of $G$, and $\mathcal{B} := L^2 G$, the group algebra of $G$ acting on $L^2(G)$ with the (unnormalized) trace $\tau$. The Fourier transform $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ is defined as

$$\mathcal{F}(x) := |G|^{-1/2} \sum_g x(g) \pi(g), \forall x \in \mathcal{A}.$$ 

Then

$$S(x) S(\mathcal{F}(x)) \geq |G|, \forall x \neq 0,$$

where $S(\mathcal{F}(x))$ is the rank of $\mathcal{F}(x)$. 

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Uncertainty Principles on Quantum Symmetries
The Donoho-Stark type of uncertainty principles on quantum symmetries have been formulated and proved by:

- Jiang-Liu-Wu, on subfactors (2016 JFA),
- Liu-Wu, on Kac algebras (2017 JMP),
- Jiang-Liu-Wu, on locally compact quantum groups (2017 JFA).

Indeed, quantum symmetries are closely related to quantum theory and they are beyond classical group symmetries. For example, from a subfactor (with finiteness conditions), one obtains 2+1 Turaev-Viro Topological Quantum Field Theory with reflection positivity.
A factor is a von Neumann algebra with trivial center. In this talk, we only consider a type $II_1$ factor, which is infinite dimensional with a (unique) trace. (The trace plays the role of a non-commutative measure.)

Example: $\mathcal{R} := \bigotimes_{k=1}^{\infty} M_2(\mathbb{C})$ under the GNS construction w.r.t. the unique tracial state.

A subfactor is an inclusion of factors $\mathcal{N} \subseteq \mathcal{M}$.

A subfactor produces a pair of von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$ with traces $d$ and $\tau$ respectively, as well as a quantum Fourier transform $\mathcal{F} : \mathcal{A} \to \mathcal{B}$. Example: The subfactor $\mathcal{R} \subset \mathcal{R} \rtimes G$, for a finite group $G$, produces $\mathcal{A} = L^\infty(G)$ and $\mathcal{B} = L(G)$ with the corresponding measure $d$, $\tau$ and the Fourier transform $\mathcal{F}$ as mentioned above (Page 8).
In general, a subfactor can be regarded as $\mathcal{R} \subseteq \mathcal{R} \rtimes G$, for the action of a “quantum group” $G$, whose “order” is called the Jones index. Remarkably, the Jones index $\mu$ of a subfactor (1983 Inv. Math.) takes values in

$$\left\{4 \cos^2 \frac{\pi}{2 + \ell}, \ell = 1, 2, \ldots, \right\} \cup [4, \infty].$$
Quantum Uncertainty Principles on Subfactors

For the quintuple $(\mathcal{A}, \mathcal{B}, d, \tau, \mathcal{F})$ from an (irreducible) subfactor with Jones index $\mu$, we have the following quantum uncertainty principles:

Theorem (Jiang-L-Wu 2016 JFA)

For any non-zero $x \in \mathcal{A}$, we have

$$S(\mathcal{F}(x))S(x) \geq \mu.$$ 

where $S(x)$ is the trace of the range projection of $x$. Moreover, “=” holds if and only if $x$ is a bi-shift of a biprojection.

Remark: When $G$ is a finite group, $S(x)$ is the rank of the operator $x$. We recover Meshulam’s uncertainty principle, and Donoho-Stark uncertainty principle. Moreover, bi-shifts of biprojections are modulations and translations of indicator functions on subgroups.

Remark: In general, $\mu$ could be non-integers! Bisch proved a bijection between biprojections and intermediate subfactors (1994 PJM).
For any non-zero $x \in A$, we have

$$H(|x|^2) + H(|\mathcal{F}(x)|^2) \geq \|x\|_2^2 \log \mu - 2\|x\|_2^2 \log \|x\|_2^2.$$ 

where $H(\cdot)$ is the von Neumann entropy, $H(|x|^2) = -d(|x|^2 \log |x|^2)$. Moreover, “=” holds if and only if $x$ is a bi-shift of a biprojection.

Remark: For the reals, the minimizers are Gaussian functions. Bi-shifts of biprojections play the role of Gaussian functions in various ways.
Hardy’s uncertainty principle (1933 JLMS) said that for a Gaussian function $g$ and a Schwartz function $f$, if there is a constant $c > 0$, such that

$$|f(x)| \leq c|g(x)|, \quad |\mathcal{F}f(\xi)| \leq c|\mathcal{F}g(\xi)|, \quad \forall \ x, \xi \in \mathbb{R},$$

then $f$ is a multiple of $g$.

**Theorem (Jiang-L-Wu 2016 JFA)**

*For a bi-shift of a biprojection $g$ and an operator $w$ in $\mathcal{A}$, if there is a constant $c > 0$, such that

$$|w| \leq c|g|, \quad |\mathcal{F}w| \leq c|\mathcal{F}g|,$$

then $w$ is a multiple of $g$.***
The Norm of the Fourier Transform

Theorem (L-Wu 2020 Sci. China Math)

For any $x \in A$, and any $p, q > 0$, we have

$$K(1/q, 1/p)^{-1} \|x\|_p \leq \|\mathcal{F}(x)\|_q \leq K(1/p, 1/q) \|x\|_p.$$ 

Hausdorff-Young inequality: $1/p + 1/q = 1$, $1/2 \leq 1/p \leq 1$. 

$$K(1/p, 1/q) :=$$ 

Diagram showing the relationships between $p$ and $q$. 

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## Extremizers

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**Table:** Table for extremizers
Rényi entropic uncertainty principles

For $p \in (0, 1) \cup (1, \infty)$, the Rényi entropy of order $p$ of $x$ in $A$ is defined as

$$h_p(x) = \frac{p}{1 - p} \log \|x\|_p.$$ 

$$h_1(x) = H(x) = d(-|x| \log |x|), \text{ when } \|x\|_1 = 1.$$ 

**Theorem (L-Wu 2020 Sci. China Math)**

Let $x \in A$ be such that $\|x\|_2 = 1$. Then for any $p, q > 0$,

$$\left(\frac{1}{p} - \frac{1}{2}\right)h_{p/2}(|x|^2) + \left(\frac{1}{2} - \frac{1}{q}\right)h_{q/2}(|\mathcal{F}(x)|^2) \geq -\log K(1/p, 1/q);$$

and

$$h_{p/2}(|x|^2) + h_{q/2}(|\mathcal{F}(x)|^2) \geq (-1 + \frac{2}{2 - p} + \frac{2}{2 - q}) \log K(1/p, 1/q).$$
Uncertainty Principles

When $1/p, 1/q \to 1/2$, we recover the quantum Hirschman-Beckner uncertainty principle:

**Theorem (Jiang-L-Wu 16)**

For any $x \in A$, we have

$$H(|x|^2) + H(|\mathcal{F}(x)|^2) \geq \|x\|^2_2 \log \mu - 2\|x\|^2_2 \log \|x\|^2_2.$$ 

When $1/p, 1/q \to \infty$, we recover the quantum Donoho-Stark uncertainty principle:

**Theorem (Jiang-L-Wu 16)**

For any non-zero $x \in A$, we have

$$\mathcal{S}(\tilde{f}(x))\mathcal{S}(x) \geq \mu.$$
A unified strategy

Define
\[ \|\mathcal{F}\|_{p \to q} := \sup_{x \neq 0} \frac{\|\mathcal{F}x\|_q}{\|x\|_p}. \]

Our strategy of proving the uncertainty principles:
- \( \|\mathcal{F}\|_{1 \to \infty} \) and \( \|\mathcal{F}\|_{2 \to 2} \)
- \( \Rightarrow \) \( \|\mathcal{F}\|_{p \to q} \)
- \( \Rightarrow \) Entropic uncertainty principles
- \( \Rightarrow \) Donoho-Stark type uncertainty principles

This strategy led to uncertainty principles on
- Subfactor planar algebras (Jiang-L-Wu 2016 JFA)
- Kac algebras (L-Wu 2017 JMP),
- locally compact quantum groups (Jiang-L-Wu 2017 JFA),
- fusion bialgebras (L-Palcoux-Wu 2021 Adv. Math.)

Highlight: For subfactors, the proof of \( \|\mathcal{F}\|_{1 \to \infty} = \mu^{-1/2} \) is non-trivial. Both definitions and proofs are pictorial in planar algebras!
Quantum Fourier Analysis

We proposed a program *Quantum Fourier Analysis* to investigate analytic aspects of quantum symmetries and their Fourier dualities. See the paper Quantum Fourier Analysis (PNAS 2020) joint with

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Harvard Univ.

Chunlan Jiang
Hebei Normal Univ.

Yunxiang Ren
Harvard Univ.

Jinsong Wu
BIMSA & HIT
Wigderson-Wigderson Uncertainty Principles

In recent work of A. Wigderson and Y. Wigderson (2021 Bull. AMS), they introduce a $k$-Hadamard matrix $\mathcal{F}$ as a matrix such that $\|\mathcal{F}\|_{1 \rightarrow \infty} \leq 1$ and $\|\text{adj}(\mathcal{F})\|_{\infty \rightarrow \infty} \leq k^{-1}$.

They proved a primary uncertainty principle:

**Theorem (Wigderson-Wigderson 2021 Bull. AMS)**

For a $k$-Hadamard matrix $\mathcal{F}$:

$$\frac{\|x\|_1 \|\mathcal{F}x\|_1}{\|x\|_{\infty} \|\mathcal{F}x\|_{\infty}} \geq k.$$  

When $\mathcal{F}$ is the Fourier transform matrix, the Wigderson-Wigderson uncertainty principle implies the Donoho-Stark type uncertainty principles, as

$$S(x) \geq \frac{\|x\|_1}{\|x\|_{\infty}}.$$
For any non-zero Schwartz function $f \in S(\mathbb{R})$, $q \in (1, \infty]$, define $E_q(f)$ as

$$E_q(f) = \frac{\|f\|_q \|\hat{f}\|_q}{\|f\|_2 \|\hat{f}\|_2} = \frac{\|f\|_q \|\hat{f}\|_q}{\|f\|^2_2}.$$  

Wigderson-Wigderson conjectured that the image of $E_q$ is $\mathbb{R}_{>0}$, for any $1 \leq q$, $q \neq 2$. (Bull. AMS 2021)
Wigderson-Wigderson Uncertainty Principles on $\mathbb{R}$

For any non-zero Schwartz function $f \in S(\mathbb{R})$, $q \in (1, \infty]$, define $E_q(f)$ as

$$E_q(f) = \frac{\|f\|_q \|\hat{f}\|_q}{\|f\|_2 \|\hat{f}\|_2} = \frac{\|f\|_q \|\hat{f}\|_q}{\|f\|_2^2}.$$ 

Wigderson-Wigderson conjectured that the image of $E_q$ is $\mathbb{R}_{>0}$, for any $1 \leq q$, $q \neq 2$. (Bull. AMS 2021)

**Theorem (Huang-L-Wu arXiv:2107.09057)**

1. When $1 < q < 2$, take $1/p + 1/q = 1$, then

   $$E_q(f) \geq [p^{1/p} / q^{1/q}]^{1/2}, \forall f \in S(\mathbb{R}) \setminus \{0\},$$

2. When $q > 2$, the image of $E_q$ is $\mathbb{R}_{>0}$.

**Question (Huang-L-Wu arXiv:2107.09057)**

*When $1 < q < 2$, what is the range of $E_q(f)$?*
Approximate Support Uncertainty Principles

For a function $f$ on a finite set $G$ with counting measure $d$, for any $1 \leq p \leq \infty$ and $\varepsilon > 0$, define

$$supp_{\varepsilon}^p(f) = \min \{|Q| : Q \subseteq G, \|f|_{Q^c}\|_p \leq \varepsilon\|f\|_p\}.$$

**Theorem (Wigderson-Wigderson 2021 Bull. AMS)**

Suppose $\mathcal{F}$ is a $k$-Hadamard $n \times m$ matrix and $f$ is a non-zero vector in $\mathbb{C}^n$. For any $\varepsilon, \eta \in [0, 1]$, $\varepsilon + \eta \leq 1$, we have that

$$supp_{\varepsilon}^2(f)supp_{\eta}^2(\mathcal{F}f) \geq k(1 - \varepsilon - \eta)^2;$$

$$supp_{\varepsilon}^1(f)supp_{\eta}^1(\mathcal{F}f) \geq k(1 - \varepsilon)(1 - \eta).$$

When $G$ is a finite Abelian group, $m = n = |G|$, and $\mathcal{F}$ is the Fourier transform, the uncertainty principle for $supp_{\varepsilon}^2$ was proved by Donoho-Stark (SIAM 1989).
Inspired by the early work on Quantum Fourier analysis and the notion of $k$-Hadamard matrices, we introduce *von Neumann $k$-bi-algebras* and their uncertainty principles.

**Definition (Huang-L-Wu arXiv:2107.09057)**

Suppose $A$ and $B$ are two von Neumann algebras with positive normal traces $d$ and $\tau$ respectively. For $k > 0$, a *$k$-transform* $\mathcal{F}$ from $A$ into $B$ is a linear map such that $\|\mathcal{F}\|_{1 \to \infty} \leq 1$ and $\|\mathcal{F}^*\mathcal{F}(x)\|_\infty \geq k\|x\|_\infty$ for any $x \in A$. We call the quintuple $(A, B, d, \tau, \mathcal{F})$ a *von Neumann $k$-bi-algebra*. (Here $\mathcal{F} : L^2(A) \to L^2(B)$ is assumed to be bounded and $\mathcal{F}^*$ is the adjoint operator.)

We recover the notion of a $k$-Hadamard matrix $\mathcal{F}$, when $A$ and $B$ are finite-dimensional Abelian von Neumann algebras with counting measure.
New Smooth Supports

We introduce a new smooth support, slightly different from the classical approximate support.

Definition (Huang-L-Wu arXiv:2107.09057)

Let $\mathcal{M}$ be a von Neumann algebra with a positive normal trace $\tau_{\mathcal{M}}$. Let $\epsilon \in [0, 1]$ and $p \in [1, \infty]$. For any element $x \in \mathcal{M}$, we define the $(p, \epsilon)$-smooth support to be

$$S_{\epsilon}^p(x) = \inf \{ \tau_{\mathcal{M}}(HR(x)) : H \in \mathcal{M}, 0 \leq H \leq I, \| (I - H)x \|_p \leq \epsilon \| x \|_p \},$$

where $R(x)$ is the range projection of $x$.

Remark: Note that $supp_{\epsilon}^p(x) \in \mathbb{N}$, so it is discontinuous w.r.t. $\epsilon$. In contrast, $S_{\epsilon}^p(x)$ is continuous w.r.t. $\epsilon$.
Moreover, $supp_{\epsilon}^p(x) \geq S_{\epsilon}^p(x)$ when comparable.
Theorem (Huang-L-Wu arXiv:2107.09057)

Suppose \((\mathcal{A}, \mathcal{B}, d, \tau, \mathcal{F})\) is a von Neumann \(k\)-bi-algebra, and \(x\) is non-zero in \(\mathcal{A}\). For any \(\varepsilon, \eta \in [0, 1]\), \(\varepsilon + \eta \leq 1\), we have that

\[
S^2_\varepsilon(f)S^2_\eta(\mathcal{F}x) \geq k(1 - \varepsilon - \eta)^2;
\]

\[
S^1_\varepsilon(f)S^1_\eta(\mathcal{F}x) \geq k(1 - \varepsilon)(1 - \eta).
\]

When \((\mathcal{A}, \mathcal{B}, d, \tau, \mathcal{F})\) comes from quantum symmetries, such as subfactors, we obtain quantum uncertainty principles for smooth supports. When \(\mathcal{F}\) is a \(k\)-Hardmard matrix, our uncertain principles are stronger than Wigderson-Wigderson approximated support uncertainty principle, as

\[
supp^p_\varepsilon(x) \geq S^p_\varepsilon(x).
\]
Example

Let $\mathcal{A} = \mathcal{B} = \mathbb{C} \oplus \mathbb{C}$ and $d(x) = \tau(x) = x(0) + x(1)$, $x \in \mathbb{C}^2$. Take $x = (1, 1) \in \mathbb{C}^2$ and $\epsilon = \eta = 1/3$. Then $|\text{supp}_\epsilon^1(x)| = |\text{supp}_\epsilon^2(x)| = 2$ while $S_\epsilon^1(x) = S_\epsilon^2(x) = 4/3$. Let $\mathcal{F} = I$ be the 1-transform, we have

$$4 = |\text{supp}_\epsilon^1(x)||\text{supp}_\eta^1(\mathcal{F}(x))| > S_\epsilon^1(x)S_\eta^1(\mathcal{F}(x)) = \frac{16}{9},$$

$$4 = |\text{supp}_\epsilon^2(x)||\text{supp}_\eta^2(\mathcal{F}(x))| > S_\epsilon^2(x)S_\eta^2(\mathcal{F}(x)) = \frac{16}{9}.$$
Quantum Smooth Entropic Uncertainty Principles

**Definition (Huang-L-Wu arXiv:2107.09057)**

Let \( \mathcal{M} \) be a finite von Neumann algebra. For any \( x \in \mathcal{M}, \epsilon \in [0, 1] \) and \( p \in [1, \infty] \), the \((p, \epsilon)\) smooth entropy of \(|x|^2\) is defined by

\[
H_{\epsilon}^p(|x|^2) := \inf \{ H(|y|^2) : y \in \mathcal{M}, \|x - y\|_p \leq \epsilon \},
\]

**Theorem (Huang-L-Wu arXiv:2107.09057)**

Let \((\mathcal{A}, \mathcal{B}, d, \tau, \mathcal{F})\) be a von Neumann \(k\)-bi-algebra. Suppose \(\mathcal{A}\) and \(\mathcal{B}\) are finite dimensional and \(\mathcal{F}^*\mathcal{F} = kI\). Then for any \(x \in \mathcal{A}, \epsilon, \eta \in [0, 1]\) and \(p, q \in [1, \infty]\), we have

\[
\frac{H_{\epsilon}^p(|x|^2)}{\|x\|^2_2} + \frac{H_{\eta}^q(|\mathcal{F}(x)|^2)}{\|\mathcal{F}(x)\|^2_2} \geq -4 \log \|x\|_2 - \frac{C(\|x\|)}{\|x\|^2_2} d(I)^{1 - \frac{1}{p} \epsilon} - \frac{C(\|\mathcal{F}x\|)}{\|\mathcal{F}(x)\|^2_2} \tau(I)^{1 - \frac{1}{q} \eta}.
\]
Thank you!