Uncertainty Principles on Quantum Symmetries

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Uncertainty Principles on Groups

Classical Fourier Duality

In the early 1800's, Joseph Fourier introduced his transformation to solve differential equations describing heat.
The Fourier transform $\mathcal{F}$ on measurable functions f on $\mathbb{R}$ is

$$\mathcal{F}(f)(x) = \int_{-\infty}^{\infty} f(t)e^{-2\pi itx}dt \,.$$

Convolution for such functions is:

$$(f_1 * f_2)(s) = \int_{-\infty}^{\infty} f_1(t)f_2(s-t)dt \,,$$

yielding the Fourier duality

$$\mathcal{F}(f_1 * f_2) = \mathcal{F}(f_1)\mathcal{F}(f_2) \,. \quad (1)$$

The Fourier transform switches the position operator x and the momentum operator $p = -i\frac{d}{dx}$.

Take

$$\|f\|_p = \left(\int_{-\infty}^{\infty} |f(t)|^p dt \right)^{1/p}, \ 0 < p < \infty.$$

For $p \geqslant 1$, $\| \cdot \|_p$ is the p-norm of measurable functions and $\|f\|_\infty$ is the essential maximum of f.

Plancherel formula (1910):

$$\|\mathcal{F}f\|_2 = \|f\|_2.$$

Interpolating with the elementary inequality $\|\mathcal{F}(f)\|_\infty \leqslant \|f\|_1$, one obtains the Hausdorff-Young inequality,

$$\|\mathcal{F}(f)\|_q \leqslant \|f\|_p \, , \quad 1 \leqslant p \leqslant 2, \quad 1/p + 1/q = 1 \, . \tag{2}$$

Heisenberg Uncertainty Principles

In 1927, Heisenberg showed in quantum theory that position and momentum cannot simultaneously be precisely measured (the Heisenberg's uncertainty principle). This phenomenon has been mathematically formulated by Kennard and by Weyl:

$$\sigma_x \sigma_p \geqslant \frac{\hbar}{2},$$

where $\hbar$ is Planck's constant, and σ is the standard deviation.
More precisely,

$$\|xf\|_2 \ \|pf\|_2 \geqslant (4\pi)^{-1} \ , \quad \|f\|_2 = 1 \ .$$

Equivalently

$$\|xf\|_2 \ \|x\mathcal{F}f\|_2 \geqslant (4\pi)^{-1} \ , \quad \|f\|_2 = 1 \ .$$

Entropic Uncertainty Principles on $\mathbb{R}$

In 1957, Hirschman studied the Shannon entropy $H(|f|^2) = -\int_{-\infty}^{\infty} |f(x)|^2 \log |f(x)|^2 dx$ of $|f|^2$, proving

$$H(|f|^2) + H(|\mathcal{F}(f)|^2) \geqslant 0 \,, \quad \|f\|_2 = 1 \,.$$

Everett conjectured the lower bound is $\log \frac{e}{2}$, which follows from Beckner's sharp Hausdorff-Young inequality (1975 Ann. Math.):

$$\|\mathcal{F}(f)\|_q \leqslant A_p \|f\|_p \,.$$

$A_p = p^{1/2p} q^{-1/2q}$, and $1/p + 1/q = 1$, $1 \leqslant p \leqslant 2$,
A Physical interpretation was given by Bialynicki and Birula (1975 CMP).
The extremizers of these inequalities are Gaussian functions.
The Hirschman-Beckner uncertainty principle ensures Heisenberg's uncertainty principle.

Donoho-Stark Uncertainty Principles

In 1989, Donoho and Stark established an uncertainty principle for functions f on cyclic groups G in terms of the cardinality of their support:

$$|\mathcal{S}(f)|\,|\mathcal{S}(\mathcal{F}(f))| \geqslant |G|\,,$$

here f is a function on G, $\mathcal{S}(f) = \{x : f(x) \neq 0\}$, and $|A|$ is the cardinality of the set A.

The Donoho-Stark uncertainty principle was generalized by

- Smith, on finite abelian groups (1990 SIAM),
- Meshulam, on finite groups in (1992 EJC),
- Alagic-Russell, on compact groups (2008 IJM).

This type of uncertainty principle has been applied by Candes-Romberg-Tao and by Donoho in compressed sensing in 2006.

Meshulam's uncertainty principle

The Meshulam's uncertainty principle for a (non-abelian) finite group G is generalization of the Donoho-Stark uncertainty principle. More precisely, take $\mathcal{A} := L^{\infty}(G)$, functions on G, with the discrete measure d. Let π be the left regular representation of G, and $\mathcal{B} := \mathcal{L}G$, the group algebra of G acting on $L^2(G)$ with the (unnormalized) trace τ. The Fourier transform $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is defined as

$$\mathcal{F}(x) := |G|^{-1/2} \sum_{g} x(g)\pi(g), \ \forall\, x \in \mathcal{A}.$$

Then

$$\mathcal{S}(x)\mathcal{S}(\mathcal{F}(x)) \geqslant |G|, \ \forall\, x \neq 0,$$

where $\mathcal{S}(\mathcal{F}(x))$ is the rank of $\mathcal{F}(x)$.

Uncertainty Principles on Quantum Symmetries

Quantum Donoho-Stark Uncertainty Principles

The Donoho-Stark type of uncertainty principles on quantum symmetries have been formulated and proved by:

- Jiang-Liu-Wu, on subfactors (2016 JFA),
- Liu-Wu, on Kac algebras (2017 JMP),
- Jiang-Liu-Wu, on locally compact quantum groups (2017 JFA).

Indeed, quantum symmetries are closely related to quantum theory and they are beyond classical group symmetries. For example, from a subfactor (with finiteness conditions), one obtains 2+1 Turaev-Viro Topological Quantum Field Theory with reflection positivity.

Subfactors

A factor is a von Neumann algebra with trivial center.
In this talk, we only consider a type II_1 factor, which is infinite dimensional with a (unique) trace.
(The trace plays the role of a non-commutative measure.)
Example: $\mathcal{R} := \overline{\bigotimes_{k=1}^{\infty} M_2(\mathbb{C})}$ under the GNS construction w.r.t. the unique tracial state.

A subfactor is an inclusion of factors $\mathcal{N} \subseteq \mathcal{M}$.

A subfactor produces a pair of von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$ with traces d and τ respectively, as well as a quantum Fourier transform $\mathcal{F} : \mathcal{A} \to \mathcal{B}$.
Example: The subfactor $\mathcal{R} \subset \mathcal{R} \rtimes G$, for a finite group G, produces $\mathcal{A} = L^{\infty}(G)$ and $\mathcal{B} = \mathcal{L}(G)$ with the corresponding measure d, τ and the Fourier transform $\mathcal{F}$ as mentioned above (Page 8).

Jones index

In general, a subfactor can be regarded as $\mathcal{R} \subseteq \mathcal{R} \rtimes G$, for the action of a "quantum group" G, whose "order" is called the Jones index.
Remarkably, the Jones index μ of a subfactor (1983 Inv. Math.) takes values in

$$\left\{ 4\cos^2 \frac{\pi}{2+\ell}, \ell = 1, 2, \ldots, \right\} \cup [4, \infty].$$

Quantum Uncertainty Principles on Subfactors

For the quintuple $(\mathcal{A}, \mathcal{B}, d, \tau, \mathcal{F})$ from an (irreducible) subfactor with Jones index μ, we have the following quantum uncertainty principles:

Theorem (Jiang-L-Wu 2016 JFA)

For any non-zero $x \in \mathcal{A}$, we have

$$\mathcal{S}(\mathfrak{F}(x))\mathcal{S}(x) \geqslant \mu.$$

where $\mathcal{S}(x)$ is the trace of the range projection of x.
Moreover, " $=$ " holds if and only if x is a bi-shift of a biprojection.

Remark: When G is a finite group, $\mathcal{S}(x)$ is the rank of the operator x. We recover Meshulam's uncertainty principle, and Donoho-Stark uncertainty principle. Moreover, bi-shifts of biprojections are modulations and translations of indicator functions on subgroups.
Remark: In general, μ could be non-integers! Bisch proved a bijection between biprojections and intermediate subfactors (1994 PJM).

Quantum Uncertainty Principles on Subfactors

Theorem (Jiang-L-Wu 2016 JFA)

For any non-zero $x \in \mathcal{A}$, we have

$$H(|x|^2) + H(|\mathcal{F}(x)|^2) \geqslant \|x\|_2^2 \log \mu - 2\|x\|_2^2 \log \|x\|_2^2.$$

where $H(\cdot)$ is the von Neumann entropy, $H(|x|^2) = -d(|x|^2 \log |x|^2)$. Moreover, " $=$ " holds if and only if x is a bi-shift of a biprojection.

Remark: For the reals, the minimizers are Gaussian functions. Bi-shifts of biprojections play the role of Gaussian functions in various ways.

Quantum Uncertainty Principles on Subfactors

Hardy's uncertainty principle (1933 JLMS) said that for a Gaussian function g and a Schwartz function f, if there is a constant $c > 0$, such that

$$|f(x)| \leqslant c|g(x)|, \quad |\mathcal{F}f(\xi)| \leqslant c|\mathcal{F}g(\xi)|, \; \forall\, x, \xi \in \mathbb{R},$$

then f is a multiple of g.

Theorem (Jiang-L-Wu 2016 JFA)

For a bi-shift of a biprojection g and an operator w in $\mathcal{A}$, if there is a constant $c > 0$, such that

$$|w| \leqslant c|g|, \quad |\mathcal{F}w| \leqslant c|\mathcal{F}g|,$$

then w is a multiple of g.

The Norm of the Fourier Transform

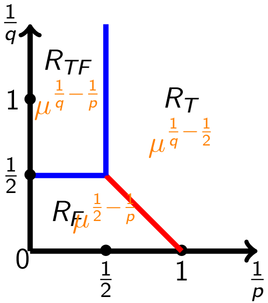
Theorem (L-Wu 2020 Sci. China Math)

For any $x \in \mathcal{A}$, and any $p, q > 0$, we have

$$K(1/q,1/p)^{-1}\|x\|_p \leqslant \|\mathcal{F}(x)\|_q \leqslant K(1/p,1/q)\|x\|_p.$$

Hausdorff-Young inequality: $1/p + 1/q = 1,\ 1/2 \leqslant 1/p \leqslant 1.$

K(1/p,1/q):=



Extremizers

Analysis	Algebra
Regions	Extremizers
$1/p + 1/q > 1, 1/p > 1/2$	trace-one projections
$1/p + 1/q = 1, 1/2 < 1/p < 1$	bishifts of biprojections
$1/p = 1, 1/q = 0$	extremal elements
$1/p = 1/2, 1/q = 1/2$	$\mathcal{A}$
$1/p + 1/q < 1, 0 < 1/q < 1/2$	Fourier transform of trace-one projections
$1/q = 0, 0 \leqslant 1/p < 1$	extremal unitary elements
$1/q = 1/2, 0 \leqslant 1/p < 1/2$	unitary elements
$1/q > 1/2, 1/p = 1/2$	Fourier transform of unitary elements
$1/q > 1/2, 1/p < 1/2$	biunitary elements if exist

Table: Table for extremizers

Rényi entropic uncertainty principles

For $p \in (0,1) \cup (1,\infty)$, the Rényi entropy of order p of x in $\mathcal{A}$ is defined as

$$h_p(x) = \frac{p}{1-p} \log \|x\|_p.$$

$$h_1(x) = H(x) = d(-|x| \log |x|), \text{ when } \|x\|_1 = 1.$$

Theorem (L-Wu 2020 Sci. China Math)

Let $x \in \mathcal{A}$ be such that $\|x\|_2 = 1$. Then for any $p, q > 0$,

$$(\frac{1}{p} - \frac{1}{2})h_{p/2}(|x|^2) + (\frac{1}{2} - \frac{1}{q})h_{q/2}(|\mathfrak{F}(x)|^2) \geqslant -\log K(1/p, 1/q);$$

and

$$h_{p/2}(|x|^2) + h_{q/2}(|\mathfrak{F}(x)|^2) \geqslant (-1 + \frac{2}{2-p} + \frac{2}{2-q}) \log K(1/p, 1/q).$$

Uncertainty Principles

When $1/p, 1/q \to 1/2$, we recover the quantum Hirschman-Beckner uncertainty principle:

Theorem (Jiang-L-Wu 16)

For any $x \in \mathcal{A}$, we have

$$H(|x|^2) + H(|\mathcal{F}(x)|^2) \geqslant \|x\|_2^2 \log \mu - 2\|x\|_2^2 \log \|x\|_2^2.$$

When $1/p, 1/q \to \infty$, we recover the quantum Donoho-Stark uncertainty principle:

Theorem (Jiang-L-Wu 16)

For any non-zero $x \in \mathcal{A}$, we have

$$\mathcal{S}(\mathfrak{F}(x))\mathcal{S}(x) \geqslant \mu.$$

A unified strategy

Define

$$\|\mathcal{F}\|_{p\to q} := \sup_{x\neq 0} \frac{\|\mathcal{F}x\|_q}{\|x\|_p}.$$

Our strategy of proving the uncertainty principles:

$\|\mathcal{F}\|_{1\to\infty}$ and $\|\mathcal{F}\|_{2\to 2}$

$\Rightarrow \|\mathcal{F}\|_{p\to q}$

$\Rightarrow$ Entropic uncertainty principles

$\Rightarrow$ Donoho-Stark type uncertainty principles

This strategy led to uncertainty principles on

- Subfactor planar algebras (Jiang-L-Wu 2016 JFA)
- Kac algebras (L-Wu 2017 JMP),
- locally compact quantum groups (Jiang-L-Wu 2017 JFA),
- fusion bialgebras (L-Palcoux-Wu 2021 Adv. Math.)

Highlight: For subfactors, the proof of $\|\mathcal{F}\|_{1\to\infty} = \mu^{-1/2}$ is non-trivial. Both definitions and proofs are pictorial in planar algebras!

Quantum Fourier Analysis

We proposed a program *Quantum Fourier Analysis* to investigate analytic aspects of quantum symmetries and their Fourier dualities.
See the paper Quantum Fourier Analysis (PNAS 2020) joint with


Arthur Jaffe
Harvard Univ.


Chunlan Jiang
Hebei Normal Univ.


Yunxiang Ren
Harvard Univ.


Jinsong Wu
BIMSA & HIT

Wigderson-Wigderson Uncertainty Principles

In recent work of A. Wigderson and Y. Wigderson (2021 Bull. AMS), they introduce a k-Hadamard matrix $\mathcal{F}$ as a matrix such that
$\|\mathcal{F}\|_{1\to\infty} \leqslant 1$ and $\|(\mathcal{F}^*\mathcal{F})^{-1}\|_{\infty\to\infty} \leqslant k^{-1}$.
They proved a primary uncertainty principle:

Theorem (Wigderson-Wigderson 2021 Bull. AMS)

For a k-Hadamard matrix $\mathcal{F}$:

$$\frac{\|x\|_1\|\mathcal{F}x\|_1}{\|x\|_\infty\|\mathcal{F}x\|_\infty} \geqslant k.$$

When $\mathcal{F}$ is the Fourier transform matrix, the Wigderson-Wigderson uncertainty principle implies the Donoho-Stark type uncertain principles, as

$$S(x) \geqslant \frac{\|x\|_1}{\|x\|_\infty}.$$

Wigderson-Wigderson Uncertainty Principles on $\mathbb{R}$

For any non-zero Schwartz function $f \in \mathcal{S}(\mathbb{R})$, $q \in (1, \infty]$, define $E_q(f)$ as

$$E_q(f) = \frac{\|f\|_q \|\hat{f}\|_q}{\|f\|_2 \|\hat{f}\|_2} = \frac{\|f\|_q \|\hat{f}\|_q}{\|f\|_2^2}.$$

Wigderson-Wigderson conjectured that the image of E_q is $\mathbb{R}_{>0}$, for any $1 \leqslant q$, $q \neq 2$. (Bull. AMS 2021)

Wigderson-Wigderson Uncertainty Principles on $\mathbb{R}$

For any non-zero Schwartz function $f \in \mathcal{S}(\mathbb{R})$, $q \in (1, \infty]$, define $E_q(f)$ as

$$E_q(f) = \frac{\|f\|_q \|\hat{f}\|_q}{\|f\|_2 \|\hat{f}\|_2} = \frac{\|f\|_q \|\hat{f}\|_q}{\|f\|_2^2}.$$

Wigderson-Wigderson conjectured that the image of E_q is $\mathbb{R}_{>0}$, for any $1 \leqslant q$, $q \neq 2$. (Bull. AMS 2021)

Theorem (Huang-L-Wu arXiv:2107.09057)

1. *When $1 < q < 2$, take $1/p + 1/q = 1$, then*

$$E_q(f) \geqslant [p^{1/p}/q^{1/q}]^{1/2}, \ \forall f \in \mathcal{S}(\mathbb{R}) \setminus \{0\},$$

2. *When $q > 2$, the image of E_q is $\mathbb{R}_{>0}$.*

Question (Huang-L-Wu arXiv:2107.09057)

When $1 < q < 2$, what is the range of $E_q(f)$?

Approximate Support Uncertainty Principles

For a function f on a finite set G with counting measure d, for any $1 \leqslant p \leqslant \infty$ and $\varepsilon > 0$, define

$$supp_\varepsilon^p(f) = \min\{|Q| : Q \subseteq G, \|f|_{Q^c}\|_p \leqslant \varepsilon \|f\|_p\}.$$

Theorem (Wigderson-Wigderson 2021 Bull. AMS)

Suppose $\mathcal{F}$ is a k-Hadamard $n \times m$ matrix and f is a non-zero vector in $\mathbb{C}^n$. For any $\varepsilon, \eta \in [0,1]$, $\varepsilon + \eta \leqslant 1$, we have that

$$supp_\varepsilon^2(f) supp_\eta^2(\mathcal{F}f) \geqslant k(1-\varepsilon-\eta)^2;$$

$$supp_\varepsilon^1(f) supp_\eta^1(\mathcal{F}f) \geqslant k(1-\varepsilon)(1-\eta).$$

When G is a finite Abelian group, $m = n = |G|$, and $\mathcal{F}$ is the Fourier transform, the uncertainty principle for $supp_\varepsilon^2$ was proved by Donoho-Stark (SIAM 1989).

von Neumann k-bi-Algebras

Inspired by the early work on Quantum Fourier analysis and the notion of k-Hadamard matrices, we introduce *von Neumann k-bi-algebras* and their uncertainty principles.

Definition (Huang-L-Wu arXiv:2107.09057)

Suppose $\mathcal{A}$ and $\mathcal{B}$ are two von Neumann algebras with positive normal traces d and τ respectively. For $k > 0$, a k-**transform** $\mathcal{F}$ from $\mathcal{A}$ into $\mathcal{B}$ is a linear map such that $\|\mathcal{F}\|_{1\to\infty} \leqslant 1$ and $\|\mathcal{F}^*\mathcal{F}(x)\|_\infty \geqslant k\|x\|_\infty$ for any $x \in \mathcal{A}$. We call the quintuple $(\mathcal{A}, \mathcal{B}, d, \tau, \mathcal{F})$ a **von Neumann k-bi-algebra**. (Here $\mathcal{F} : L^2(\mathcal{A}) \to L^2(\mathcal{B})$ is assumed to be bounded and $\mathcal{F}^*$ is the adjoint operator.)

We recover the notion of a k-Hadamard matrix $\mathcal{F}$, when $\mathcal{A}$ and $\mathcal{B}$ are finite-dimensional Abelian von Neumann algebras with counting measure.

New Smooth Supports

We introduce a new smooth support, slightly different from the classical approximate support.

Definition (Huang-L-Wu arXiv:2107.09057)

Let $\mathcal{M}$ be a von Neumann algebra with a positive normal trace $\tau_{\mathcal{M}}$. Let $\epsilon \in [0,1]$ and $p \in [1,\infty]$. For any element $x \in \mathcal{M}$, we define the (p,ϵ)-smooth support to be

$$\mathcal{S}_\epsilon^p(x) = \inf\{\tau_{\mathcal{M}}(H\mathcal{R}(x)) : H \in \mathcal{M},\ 0 \leqslant H \leqslant I,\ \|(I-H)x\|_p \leqslant \epsilon\|x\|_p\},$$

where $\mathcal{R}(x)$ is the range projection of x.

Remark: Note that $supp_\epsilon^p(x) \in \mathbb{N}$, so it is discontinuous w.r.t. ε.
In contrast, $\mathcal{S}_\epsilon^p(x)$ is continuous w.r.t. ε.
Moreover, $supp_\epsilon^p(x) \geqslant \mathcal{S}_\epsilon^p(x)$ when comparable.

Quantum Smooth Uncertainty Principles

Theorem (Huang-L-Wu arXiv:2107.09057)

Suppose $(\mathcal{A}, \mathcal{B}, d, \tau, \mathcal{F})$ is a von Neumann k-bi-algebra, and x is non-zero in $\mathcal{A}$. For any $\varepsilon, \eta \in [0,1]$, $\varepsilon + \eta \leqslant 1$, we have that

$$\mathcal{S}_\varepsilon^2(f)\mathcal{S}_\eta^2(\mathcal{F}x) \geqslant k(1-\varepsilon-\eta)^2;$$

$$\mathcal{S}_\varepsilon^1(f)\mathcal{S}_\eta^1(\mathcal{F}x) \geqslant k(1-\varepsilon)(1-\eta).$$

When $(\mathcal{A}, \mathcal{B}, d, \tau, \mathcal{F})$ comes from quantum symmetries, such as subfactors, we obtain quantum uncertainty principles for smooth supports.
When $\mathcal{F}$ is a k-Hardmard matrix, our uncertain principles are stronger than Wigderson-Wigderson approximated support uncertainty principle, as $supp_\epsilon^p(x) \geqslant \mathcal{S}_\epsilon^p(x)$.

Example

Let $\mathcal{A} = \mathcal{B} = \mathbb{C} \oplus \mathbb{C}$ and $d(x) = \tau(x) = x(0) + x(1)$, $x \in \mathbb{C}^2$. Take $x = (1,1) \in \mathbb{C}^2$ and $\epsilon = \eta = 1/3$. Then $|supp_\epsilon^1(x)| = |supp_\epsilon^2(x)| = 2$ while $\mathcal{S}_\epsilon^1(x) = \mathcal{S}_\epsilon^2(x) = 4/3$. Let $\mathcal{F} = I$ be the 1-transform, we have

$$4 = |supp_\epsilon^1(x)||supp_\eta^1(\mathcal{F}(x))| > \mathcal{S}_\epsilon^1(x)\mathcal{S}_\eta^1(\mathcal{F}(x)) = \frac{16}{9},$$
$$4 = |supp_\epsilon^2(x)||supp_\eta^2(\mathcal{F}(x))| > \mathcal{S}_\epsilon^2(x)\mathcal{S}_\eta^2(\mathcal{F}(x)) = \frac{16}{9}.$$

Quantum Smooth Entropic Uncertainty Principles

Definition (Huang-L-Wu arXiv:2107.09057)

Let $\mathcal{M}$ be a finite von Neumann algebra. For any $x \in \mathcal{M}$, $\epsilon \in [0,1]$ and $p \in [1,\infty]$, the (p,ϵ) **smooth entropy** of $|x|^2$ is defined by

$$H_\epsilon^p(|x|^2) := \inf\{H(|y|^2) : \; y \in \mathcal{M}, \; \|x-y\|_p \leqslant \epsilon\},$$

Theorem (Huang-L-Wu arXiv:2107.09057)

Let $(\mathcal{A}, \mathcal{B}, d, \tau, \mathcal{F})$ be a von Neumann k-bi-algebra. Suppose $\mathcal{A}$ and $\mathcal{B}$ are finite dimensional and $\mathcal{F}^*\mathcal{F} = kI$. Then for any $x \in \mathcal{A}$, $\epsilon, \eta \in [0,1]$ and $p, q \in [1,\infty]$, we have

$$\frac{H_\epsilon^p(|x|^2)}{\|x\|_2^2} + \frac{H_\eta^q(|\mathcal{F}(x)|^2)}{\|\mathcal{F}(x)\|_2^2} \geqslant -4\log\|x\|_2 - \frac{C(\|x\|)}{\|x\|_2^2} d(I)^{1-\frac{1}{p}}\epsilon - \frac{C(\|\mathcal{F}x\|)}{\|\mathcal{F}(x)\|_2^2}\tau(I)^{1-\frac{1}{q}}\eta.$$

Thank you!