Invariants of Dirac operators and scalar curvature

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Scalar curvature

Let M be an n-dimensional manifold with a Riemannian metric.

$$\frac{Vol(B_M(p,r))}{Vol(B_{R^n}(r))} = 1 - \frac{Sc(p)}{6(n+1)} r^2 + o(r^4).$$

$Sc(p)$ is the scalar curvature of M at the point $p \in M$.

Scalar curvature: existence

If p is a homogeneous polynomial with degree at most n, then the hypersurface

$$\{[z_0, z_1, \cdots, z_n] \in \mathbb{CP}^n : p(z_0, z_1, \cdots, z_n) = 0\}$$

carries a metric with positive scalar curvature.

(Yau, solution of Calabi-Yau conjecture)

Scalar curvature: non-existence

The Kummer surface $\{[z_0, z_1, z_2, z_3] \in \mathbb{CP}^3 : z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\}$ does not carry any metric with positive scalar curvature (Atiyah-Singer index theory);

T^n does not carry any metric with positive scalar curvature (Schoen-Yau, minimal surface; Gromov-Lawson, index theory);

If M is a closed aspherical method, then M doesn't carry any metric with positive scalar curvature (Connes-Moscovici, noncommutative geometry);

If M is a closed aspherical manifold with dimension 4 or 5, then M doesn't carry any metric with positive scalar curvature (Chodosh-Li, Gromov, minimal surface).

Alexandrov geometry: $K \leq 0$

Dimension 2: $K \geq 0$

Scalar curvature and dihedral angles

Let P be a convex polyhedron in $\mathbb{R}^n$, let g_0 be the Euclidean metric on P.

Dihedral conjecture (Gromov)

Let g be a Riemannian metric on P. If
(1) $H_g(F_i) \geq 0$ for each face F_i of P ($H_g(F_i)$ is the mean curvature of F_i);

(2) $Sc(g) \geq 0$;

(3) $\theta_{ij}(g) \leq \theta_{ij}(g_0)$ (θ_{ij} is the dihedral angle);

then $H_g(F_i) = 0$, $Sc(g) = 0$, and $\theta_{ij}(g) \geq 0$.

Let X be a (possibly singular) space.

Definition

The space (X, g) is said to have non-positive scalar curvature if there exists no convex polyhedron P in X such that

(1) $H_g(F_i) > 0$ for each face F_i of P;

(2) $\theta_{ij}(g) < \theta_{ij}(g_0)$ (θ_{ij} is the dihedral angle).

Applications to the Positive Mass Problem

Theorem (Schoen-Yau)

If (X, g) is a complete asymptotically Euclidean manifold of dimension n 3 such that its scalar curvature is non-negative, then the ADM mass of each end of X is non-negative

By a reduction result of Lohkamp, if the ADM mass of an end of X is negative, then one can reduce to the case where X has only an end and there exists another complete Riemannian metric g' on X with $Sc(g') \geq 0$ and $Sc(g')_x > 0$ for some point $x \in X$ such that there is a compact set $K \subset X$ with $(X - K, g')$ being isometric to the standard Euclidean space minus a ball.

Known cases for the dihedral conjecture

Gromov: cubes.

Chao Li: simplices.

Proved by using Schoen-Yau's minimal surface method.

$$D = c_1 \frac{\partial}{\partial x_1} + \cdots + c_n \frac{\partial}{\partial x_n},$$

where $c_i^2 = -1, \quad c_i c_j + c_j c_i = 0$ when $i \neq j$.
For example, when $n = 2$,

$$c_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$c_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

We have

$$D^2 = -\frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_n^2}.$$

Dirac operator on a manifold

$$D = c_1\nabla_1 + \cdots + c_n\nabla_n.$$

$$D^2 = \nabla^*\nabla + \mathit{curvature}.$$

Proof of the dihedral conjecture (Wang-Xie-Yu)

Key estimate

Let D be the Dirac operator on $S \otimes S$ (S is the spinor bundle). We have

$$\int_P |D\varphi|^2 \geq \int_P |\nabla\varphi|^2 + \int_P \frac{Sc}{4}|\varphi|^2 + \int_{\partial P} \langle D^{\partial}\varphi, \varphi\rangle + \int_{\partial P} \frac{H}{2}|\varphi|^2$$

$$-\frac{1}{2}\sum_{i,j}\int_{F_{ij}} \theta_{ij}(g)|\varphi|^2 + \frac{1}{2}\sum_{i,j}\int_{F_{ij}} \theta_{ij}(g_0)|\varphi|^2$$

for all smooth sections φ of $S \otimes S$.

Outline of the proof for the dihedral conjecture

We identify $S \otimes S$ with $\wedge^* P$. We impose the following local boundary condition: the forms are tangential on the boundary. For any section φ satisfying the boundary condition, we have $\int_{\partial P} \langle D^{\partial} \varphi, \varphi \rangle = 0$. The Fredholm index of the Dirac operator D relative to this boundary condition equals to the Euler number. Since the Euler number is non-zero, there exists a nontrivial section φ such that $D\varphi = 0$.
By the previous estimate, we have

$$\nabla \varphi = 0, \int_P \frac{Sc}{4} |\varphi|^2 = 0, \int_{\partial P} \frac{H}{2} |\varphi|^2 = 0,$$

$$\sum_{i,j} \int_{F_{ij}} (\theta_{ij}(g) - \theta_{ij}(g_0)) |\varphi|^2 = 0.$$

Hence

$$Sc = 0, H = 0, \theta_{ij}(g) - \theta_{ij}(g_0) = 0.$$

A general dihedral theorem

Theorem (WXY)

Let $(N, \overline{g})$ and (M, g) be compact oriented Riemannian manifolds with corners. Suppose the curvature operator of g is non-negative and each codimension one face F_i of M is convex, that is, the second fundamental form of F_i is non-negative. Let $f\colon (N, \overline{g}) \to (M, g)$ be a corner map of manifolds with corners. Suppose f is a spin map and

(1) $Sc(g)_x \geq \| \wedge^2 df \| \cdot Sc(g)_{f(x)}$ for all $x \in N$,

(2) $H_{\overline{g}}(\overline{F}_i)_y \geq \|df\| \cdot H_g(F_i)_{f(y)}$ for all codimension one faces $\overline{F}_i$ of N, $y \in \overline{F}_i$,

(3) $\theta_{ij}(\overline{g})_z \leq \theta_{ij}(g)_{f(z)}$ for all $\overline{F}_i, \overline{F}_j$ and all $z \in \overline{F}_i \cap \overline{F}_j$,

(4) M has nonzero Euler characteristic,

(5) the $\widehat{A}$-degree $\deg_{\widehat{A}}(f)$ of f is nonzero,

Then we have

(a) $Sc(\overline{g})_x = \| \wedge^2 df \| \cdot Sc(g)_{f(x)}$ for all $x \in N$,

(b) $H_{\overline{g}}(\overline{F}_i)_y = \|df\| \cdot H_g(F_i)_{f(y)}$ for all $y \in \overline{F}_i$,

(c) $\theta_{ij}(\overline{g})_z = \theta_{ij}(g)_{f(z)}$ for all $\overline{F}_i, \overline{F}_j$ and all $z \in \overline{F}_i \cap \overline{F}_j$.

The cube inequality

The cube inequality conjecture (Gromov)

Let g be a Riemannian metric on the cube $I^n = [0,1]^n$. If the scalar curvature of g satisfies $k(p) \geq k > 0$ for all point p in the cube, then

$$\sum_i^n \frac{1}{d_i} \geq \frac{kn}{4\pi^2(n-1)},$$

where $d_i = dist(\partial_{i-}, \partial_{i+})$ is the g-distance between the i-th pair of opposite faces ∂_{i-} and ∂_{i+} of the cube. Consequently, we have

$$min_{1\leq i\leq n} \;\; dist(\partial_{i-}, \partial_{i+}) \leq 2\pi.$$

Gromov proves this conjecture in dimension $n \leq 8$ using the minimal surface method of Schoen-Yau.

The cube inequality

Theorem (Gromov: $n \leq 8$ and WXY in general)

A strengthened version of the cube conjectue is true for all dimension, i.e. if the scalar curvature of g satisfies $k(p) \geq k > 0$, then

$$\sum_{i}^{n} \frac{1}{d_i} > \frac{kn}{4\pi^2(n-1)},$$

where $d_i = dist(\partial_{i-}, \partial_{i+})$ is the g-distance between the i-th pair of opposite faces ∂_{i-} and ∂_{i+} of the cube. Consequently, we have

$$min_{1 \leq i \leq n} \;\; dist(\partial_{i-}, \partial_{i+}) < 2\pi.$$

Gromov's proof works only for $\geq$ even in dimension ≤ 8.

An outline of the proof for the cube inequality

We first extend the Riemmanian metric to $\mathbb{R}^n$ such that it is a multiple of the Euclidean metric outside a compact set. Let D be the Dirac operator.

We have $Index(B) = 1$.

Let $B = D + \sum_{i=1}^n \phi_i c_i$. If the cube inequality is not true, then we can construct ϕ such that (1) $\phi_i(x)$ is asymptotically a non-zero scalar multiple of x_i; (2) $B^2 > \delta > 0$ for some positive constant δ.

Finally B is Fredholm and is homotopic to $B' = D + \sum_{i=1}^n x_i c_i$. We have $Index(B) = Index(B') = 1$. This is a contradiction.

Theorem (Jinmin Wang, Zhizhang Xie, G. Yu)

Let X be an n-dimensional compact connected spin manifold with boundary. Suppose $f\colon X \to [-1,1]^m$ is a continuous map, which sends the boundary of X to the boundary of $[-1,1]^m$. Let $\partial_{i\pm}$, $i = 1, \ldots, m$, be the pullbacks of the pairs of the opposite faces of the cube $[-1,1]^m$. Suppose $Y_\pitchfork$ is an $(n-m)$-dimensional closed submanifold (without boundary) in X that satisfies the following conditions:

(1) $\pi_1(Y_\pitchfork) \to \pi_1(X)$ is injective;

(2) $Y_\pitchfork$ is the transversal intersection of m orientable hypersurfaces $\{Y_i\}_{i=1}^m$ of X, where each Y_i separates ∂_{i-} from ∂_{i+};

(3) the higher index $\mathrm{Ind}_\Gamma(D_{Y_\pitchfork})$ does not vanish in $KO_{n-m}(C^*_{\max}(\Gamma;\mathbb{R}))$, where $\Gamma = \pi_1(Y_\pitchfork)$ and $C^*_{\max}(\Gamma;\mathbb{R})$ is the maximal group C^*-algebra of Γ with real coefficients.

If $Sc(X) \geq k > 0$, then the distances $d_i = dist(\partial_{i-}, \partial_{i+})$ satisfy the following inequality:

$$\sum_{i=1}^m \frac{1}{d_i^2} > \frac{kn}{4\pi^2(n-1)}. \quad (1)$$

References

1. A. Connes, Noncommutative Geometry, Academic Press, 1990.
2. M. Gromov, Dirac and Plateau billiards in domains with corners, Cent. Eur. J. Math. 12 (2014), no. 8, 11091156.
3. M. Gromov, Four Lectures on Scalar Curvature, Perspective on scalar curvature, 2019.
4. R. Schoen and S. T. Yau, On the proof of the positive mass conjecture in general relativity. Communications in Mathematical Physics. 65 (1): 4576, 1979.
5. S. T. Yau, On the Ricci curvature of a compact Khler manifold and the complex MongeAmpre equation. I", Communications on Pure and Applied Mathematics, 31 (3): 339411, 1978.
6. C. Li, A polyhedron comparison theorem for three-manifolds with positive scalar curvature, Invent. Math. 219, 1-37 (2020).
7. J. Wang, Z. Xie and G. Yu, On Gromov's dihedral extremality and rigidity conjectures, arXiv:2112.01510.
8. J. Wang, Z. Xie and G. Yu, A proof of Gromov's cube inequality on scalar curvature, arXiv:2105.12054.

Thank you!