Recent progress on random field Ising model

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Based on joint works with
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Mathematical Picture Language Seminar
(Lenz–)Ising model: a model for magnetization

- $\mathbb{Z}^d$: $d$-dimensional lattice (nearest neighbor graph);
- $\Lambda_N$: box with side length $2N$ centered at origin $o$;
- configuration $\sigma \in \Omega = \{-1, 1\}^{\Lambda_N}$.
- $E(A, B)$ for $A, B \subset \mathbb{Z}^d$: edges between $A$ and $B$;

For temperature $T \geq 0$, define the Ising measure by

$$\mu^\pm T, \Lambda_N(\sigma) \propto \exp(-\frac{1}{T}H^\pm(\sigma, \Lambda_N))$$

Observation: Ising measure favors configurations with more agreeing neighboring pairs.
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**Hamiltonian** with plus (resp. minus) boundary condition:

$$H^\pm(\sigma, \Lambda_N) = -\left( \sum_{(u, v) \in E(\Lambda_N, \Lambda_N)} \sigma_u \sigma_v \pm \sum_{(u, v) \in E(\Lambda_N, \Lambda_N^c)} \sigma_u \right).$$
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Long range order for Ising model

• boundary influence = \langle \sigma_o \rangle \mu + T, \Lambda N - \langle \sigma_o \rangle \mu - T, \Lambda N \\
  by symmetry.

• Ising model is monotone, so the plus (minus) boundary condition is the maximum (minimum) boundary condition.

We say long range order exists if boundary influence stays above a positive constant as \( N \to \infty \).

• Ising 1925: for \( d = 1 \), no long range order for any \( T > 0 \).

• Peierls 1936: for \( d \geq 2 \), long range order exists at low temperatures but not at high temperatures.

• Much more on Ising model has been understood, but not our focus today.
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**Question:** do local spin interactions lead to long range order?
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Boundary influence: \[
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At low temperatures long range order exists for $d \geq 2$, i.e.,

$$\frac{1}{2}(\langle \sigma_o \rangle \mu + T, \Lambda N - \langle \sigma_o \rangle \mu - T, \Lambda N) = \langle \sigma_o \rangle \mu + T, \Lambda N \geq \text{const}.$$
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- **Construction** (of flip mapping): if spin at origin disagrees with boundary condition, flip all spins enclosed by its sign component.

Out boundary (in the dual graph) for the sign component of the origin

Flip all spins enclosed by the out boundary

Conclusion (summing over $\ell$): at low temperature, the origin agrees with the boundary condition with good probability.
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Random field Ising model

Disorder

\( h \): \( v \in \mathbb{Z}^d \)

The RFIM Hamiltonian on \( \Lambda_N \) with external field \( \epsilon h \) and plus (resp. minus) boundary condition:

\[
H_{\pm}(\sigma, \Lambda_N, \epsilon h) = H_{\pm}(\sigma, \Lambda_N) - \sum_{u \in \Lambda_N} \epsilon h_u \sigma_u.
\]

For temperature \( T \geq 0 \), define the RFIM measure by

\[
\mu_{\pm T, \Lambda_N, \epsilon h}(\sigma) \propto \exp\left( -\frac{1}{T} H_{\pm}(\sigma, \Lambda_N, \epsilon h) \right).
\]

At \( T = 0 \), \( \mu_{\pm T, \Lambda_N, \epsilon h} \) is supported on the ground state \( \sigma_{\pm \Lambda_N, \epsilon h} \) (unique a.s. since Gaussian distribution is continuous).

Boundary influence on spin magnetization

\[
m_{T, \Lambda_N, \epsilon} = \frac{1}{2} E(\langle \sigma_o \rangle_{\pm T, \Lambda_N, \epsilon h} - \langle \sigma_o \rangle_{- T, \Lambda_N, \epsilon h}),
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where \( \langle \cdot \rangle_{\pm T, \Lambda_N, \epsilon h} \) denotes expectation with respect to \( \mu_{\pm T, \Lambda_N, \epsilon h} \).

Main question today: how does the random field affects the long range order? I.e., what is the limiting behavior of \( m_{T, \Lambda_N, \epsilon} \)?
Random field Ising model

Disorder \( \{ h_v : v \in \mathbb{Z}^d \} \): independent standard Gaussian variables. The RFIM Hamiltonian on \( \Lambda_N \) with external field \( \epsilon h \) and plus (resp. minus) boundary condition:

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Boundary influence on spin magnetization

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Main question today: how does the random field affects the long range order? I.e., what is the limiting behavior of \( m_{T, \Lambda_N, \epsilon} \)?
Can we still apply Peierls argument?

- No, since the random field has influence on the probability change for the flip mapping.
- Such influence depends on $\sigma$ and thus a uniform bound is not possible.

For large $\epsilon$, exponential decay for boundary influence in any dimension was proved in Berretti 85, Fröhlich–Imbrie 84, von Dreifus–Klein–Perez 95 and Camia–Jiang–Newman 18, Aizenman–Peled 18.

For small $\epsilon$, it is much more delicate and challenging (the focus for the rest of the talk).
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RFIM with weak disorder: Imry–Ma prediction

Imry–Ma prediction:
- For small $\epsilon$, long range order exists for $d \geq 3$ at low temperatures;
- No long range order for $d = 2$.

Underlying intuition of Imry–Ma:
- Gaussian volume (i.e., the sum of the Gaussian disorder) of $\Lambda_n$ is $\approx n^{d/2}$;
- Boundary effect from $\partial \Lambda_n$ is $\approx n^{d-1}$.

Difficulties for proving Imry–Ma prediction:
- For $d = 3$ there exists a random connected set $S$ such that $P_{v \in S} \epsilon_h v \gg |\partial S|$.
  For instance, let $S$ be $\Lambda_n$ with $n^2$ vertices of least field values removed.
  This prevents a straightforward application of Peierls argument.
- For $d = 2$ with small $\epsilon$, we need the collective influence from disorder on a large set to fight against the boundary effect.
  But why should they collaborate?
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Difficulties for proving Imry–Ma prediction:

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- For $d = 2$ with small $\epsilon$, we need the collective influence from disorder on a large set to fight against the boundary effect.
RFIM with weak disorder: Imry–Ma prediction

Imry–Ma 75: predicted that for small $\epsilon$
- long range order exists for $d \geq 3$ at low temperatures;
- no long range order for $d = 2$.

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RFIM with weak disorder: two dimensions

Aizenman–Wehr 90: boundary influence

\[ T, \Lambda N, \epsilon \rightarrow \Lambda N \rightarrow \infty \]

Work with free energy difference \( \Delta F = F(\Lambda N, \epsilon h) - F(\Lambda N, \epsilon h) \) where \( F_{\pm}(\Lambda N, \epsilon h) = -T \log X_{\sigma} \in \{-1, 1\} \Lambda N \exp(-T H_{\pm}(\sigma, \Lambda N, \epsilon h)) \).

• Deterministically \(|\Delta F| \leq 2|\partial \Lambda N| = 16 N\) (via direct comparison of Hamiltonians with plus and minus boundary conditions for the same configuration).

• Suffices to show that if \( m T, \Lambda N, \epsilon \geq \text{const} \), then \( \Delta F > 16 N \) with positive probability.

\( \Delta F \) has variance \( \approx \epsilon^2 N^2 \) (by Cacoullos 82).

† Partial derivatives of \( \Delta F \) are given by boundary influences.

⋄ the key is to show a central limit theorem for \( \Delta F \).

Debate among physicists on decay rate: polynomial decay in some regime (Berezinskii–Kosterlitz–Thouless transition) vs. exponential decay for all \( \epsilon \)?
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Correlation length for 2D RFIM

Question: as $\epsilon \to 0$, what is the minimal size of a box to see influence from the random field? I.e., what is the scaling of the correlation length $\psi(T, \epsilon, m) = \min\{N: mT, \Lambda N, \epsilon \leq m\}$ for $0 < m < 1$ (say, $m = 1/2$)?

Physics predictions: many studies but no consensus even at $T = 0$. A common belief was $\psi(T, \epsilon, m) = e^{\epsilon - 2}$, and some recent work supported $\psi(T, \epsilon, m) = e^{\epsilon - 1}$.

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RFIM with weak disorder: three dimensions and above

Chalker 83, Fisher–Fröhlich–Spencer 84: with positive probability

$|\sum_{v \in S} \epsilon_h v| < |\partial S|$ (i.e., Gaussian volume is smaller than the boundary size) for all simply connected set $S \ni o$.

Assume that for any simply connected set $S \ni o$,

$\sum_{v \in S} \epsilon_h v \leq |\partial S|$.

Not possible as a ground state

Still possible as a ground state

Imbrie 85 (T = 0) and Bricmont–Kupiainen 88 (small $T > 0$): long range order exists.

• Control sign clusters within sign clusters (i.e., holes) by an involved renormalization group theoretic argument.

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An overview of our proof (for 3D RFIM)

Our key insight is that Peierls argument can be extended.

- Recall the obstacle for Peierls argument with external field is the challenge in keeping track of spin interactions with disorder after flipping spins.
- Solution: we flip the external field as well.

A one-sentence summary: instead of fixing the disorder and applying Peierls argument on spin configurations, we consider the joint space of disorder and spin configurations and apply Peierls argument in this larger space.

Define the joint measure \( Q_{\pm}(h \in A, \sigma \in B) = R_A \mu_{\pm T, \Lambda_N, h}(B) dP(h) \).

Goal: show \( Q^+(\sigma_o = -1) \ll 1 \) for small \( \epsilon, T \) and \( d \geq 3 \).
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• Recall the obstacle for Peierls argument with external field is the challenge in keeping track of spin interactions with disorder after flipping spins.
• Solution: we flip the external field as well.

A one-sentence summary: instead of fixing the disorder and applying Peierls argument on spin configurations, we consider the joint space of disorder and spin configurations and apply Peierls argument in this larger space.
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Define joint measure \( Q^\pm(h \in A, \sigma \in B) = \int_A \mu^\pm_{T, \Lambda_N, h}(B) d\mathbb{P}(h). \)
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Goal: show $\mathbb{Q}^+(\sigma_o = -1) \ll 1$ for small $\epsilon, T$ and $d \geq 3$. 
A sketch of the new Peierls argument
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Consider the simply connected component enclosed by sign component at $o$ and flip signs of **spins and disorder** inside.

Out boundary (in the dual graph) for the sign component of the origin

Flip all **spins and disorder** enclosed by the out boundary

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Analysis: two competing effects for sign component with outmost boundary of size $\ell$.

- Flipping gains a factor of $e^{\ell/T}$ in probability;
- Multiplicity of the mapping is $e^{O(\ell)}$.

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Conclusion: (summing over $\ell$): at low temperature, the origin agrees with the boundary condition with good probability.

A caveat: the partition function for the Ising model is changed since the external field is changed!
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- Plus disorder
- Minus disorder
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The density for $Q^+$ on $(h, \sigma)$ is

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\nu^+(h, \sigma) = \prod_v \frac{1}{\sqrt{2\pi}} e^{-\frac{h_v^2}{2\epsilon^2}} \prod_v e^{-\frac{1}{T} H_{\Lambda_N, h}^+(\sigma)} \frac{Z_{T, \Lambda_N, h}^+}{Z_{T, \Lambda_N, h}}
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(recall $Z_{T, \Lambda_N, h}^+ = \sum_{\sigma} e^{-\frac{1}{T} H_{\Lambda_N, h}^+(\sigma)}$ is the partition function).
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**Solution:** show that the change of the free energy $-\frac{1}{T} \log \mathcal{Z}_{T,\Lambda_N, h}^+$ is bounded by $\ell$ after flipping any component with boundary size $\ell$. 
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- Adapting the proof of Fisher–Fröhlich–Spencer 84 verbatim: with high probability, the change of free energy after flipping the sign of disorder in any simply connected component of boundary size $\ell$ is bounded by $\ell$. 
A closer look at three dimensions

Recall: in 3D phase transition persists with weak disorder.

Question \( \ast \): does disorder strictly decrease the critical temperature?

Question \( \ast \ast \): does disorder at least not increase the critical temperature? I.e., when \( T > T_c \) (critical temperature without disorder), always exponential decay?

- Camia–Jiang–Newman 18: yes when \( T > T_d > T_c \).
- D.–Song–Sun 21: yes when \( T > T_c \), as a corollary of \( \Box \) the boundary influence is maximized at zero external field.

† The above inequality has many other applications.

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- This is hard since our understanding for critical 3D Ising without disorder remains limited.
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Future direction: away from monotone models

Aizenman–Wehr (the qualitative result) applies to a wide class of models including Potts model, XY model, spin glasses, etc. All aforementioned quantitative bounds with weak random field only apply to Ising model. Dario–Harel–Peled 21: some quantitative bounds on non-monotone models.

Interesting challenges next:
• Exponential decay for 2D random field Potts model?
• Correlation length for 2D random field Potts model?

Remark. D.–Zhuang 21: for 3D random field Potts model with weak disorder, long range order exists at low temperatures.
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Question: what about scaling limits for interfaces of random field Ising model?