Continuous symmetry breaking on the Nishimori Line

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Main Results and Background

New proof of phase transitions for 3D disordered spin systems with continuous symmetry. Reflection positivity is not used.

Classical spins, s_j, $j \in \mathbb{Z}^d$ are group valued: Z_2, $O(2)$, $SU(2)$

Ising, XY, Heisenberg-like

Nishimori disorder is a special temperature dependent, gauge invariant, quenched randomness.

Main Message: Nishimori disorder can make it easier to prove Long Range Order (LRO) - spin alginment at long distances.

Background of proof:

Abbe, Massoulié, Montanari, Sly, and Srivastava. (2018)
Group synchronization on grids. Mathematical Statistics and Learning

H. Nishimori: *Internal energy, specific heat and correlation function of the bond-random Ising model*. Progress of Theoretical Physics, (1981).

Georges, Hansel, Le Doussal, and Bouchaud. (1985)
Exact properties of spin glasses. II. Nishimori's line ; Journal de Physique.

H. Nishimori. *Statistical physics of spin glasses and information processing: an introduction.* Clarendon Press, (2001)

Synchronization and noisy signal recovery

A. Singer; *Angular Synchronization by eigenvectors and semidefinite programming* (2011).

Given a graph with n vertices j and edges j,j' with phases θ_j .
The angular synchronization problem: Determine relative phases $\{\theta_j\}$ from m noisy differences $\theta_j - \theta_{j'}$, $\mod 2\pi$

If noise is too large or m is too small, one cannot get useful information. Where is the threshold? How does it depend on the graph? Find algorithm to recover $\{\theta_j\}$.

O(n+1) Symmetry Breaking - no disorder

Let $s_j \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$, $j \in \Lambda \subset \mathbb{Z}^d$ and define

$$\langle s_0 s_x \rangle_\Lambda(\beta) = Z_\Lambda(\beta)^{-1} \int s_0 s_x \, e^{\beta \sum_{j,j'} s_j \cdot s_{j'}} \prod_{j \in \Lambda} d\mu(s_j)$$

Sum ranges over *adjacent* $j, j' \in \Lambda$ and $d\mu$ is the Haar measure.

For XY, $s_j \in \mathbb{S}^1, s_j = (\cos\theta_j, \sin\theta_j)$, Ising $s_j = \pm 1$, Heisenberg $s_j \in \mathbb{S}^2$

Theorem (Mermin-Wagner '66) In **2** dimensional models with continuous symmetry, $(n \geq 1)$, there is **no** continuous symmetry breaking.

$$\langle s_0 s_x \rangle(\beta) \to 0, \quad as \quad |x| \to \infty.$$

Theorem (Fröhlich-Simon-Sp, '82, Balaban '95) In $d \geq 3$ dimensions, there is long range order (LRO):

$$\langle s_0 s_x \rangle(\beta) \geq M^2(\beta) > 0 \qquad \beta \geq \beta^*$$

FSS rely on Reflection Positivity and translation invariance.

Balaban develops a robust multi-scale spinwave analysis

For the 3D XY case and 4D U(1) gauge theories there are other methods by A. Guth, Fröhlich-Sp, Kennedy-King using Fourier Transform and duality.

Phase disordered XY model

Let $\omega_{j,j'} \in [0, 2\pi]$ be random phases, j, j' **adjacent**. Define

$$\langle \cos(\theta_0 - \theta_x) \rangle_\Lambda(\omega, \beta)$$

$$= Z_\Lambda^{-1}(\omega, \beta) \int \cos(\theta_0 - \theta_x) e^{\boldsymbol{\beta} \sum_{j,j'} \cos(\theta_j - \theta_{j'} + \omega_{j,j'})} \prod^{\Lambda} d\theta_j$$

Distribution of independent $\omega_{j,j'}$: $\quad \prod_{jj'} \frac{e^{\bar{\boldsymbol{\beta}} \cos(\omega_{j,j'})}}{z(\bar{\beta})}$

Nishimori Line: $\boldsymbol{\beta} = \bar{\boldsymbol{\beta}}$

Remarks: When β is large and $\bar{\beta}$ small - expect spinglass in 3D.

When $\bar{\beta} \to \infty, \quad \omega = 0$ - No disorder.

Theorem (Garban-Sp) In 3D there is finite β^* so that for $\beta = \bar{\beta} \geq \beta^*$ there is long range order:

$$\mathbf{[}\langle\cos(\theta_0 - \theta_x)\rangle(\omega, \beta)\mathbf{]} \geq M^2 \approx 1 - O(\frac{\ln\beta}{\beta^{1/2}})$$

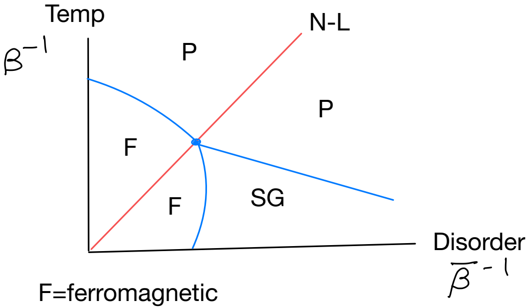
Here $\boldsymbol{[A(\omega)]}$ denotes the average over disorder $\boldsymbol{\omega}$.

Remark (J. Fröhlich) Apply inequality of Messager et al. : $\langle\cos(\theta_0 - \theta_x)\rangle(\beta, \omega = 0) \geq \langle\cos(\theta_0 - \theta_x)\rangle(\beta, \omega)$, *for all* ω.

Nishimori: $[\langle\cos(\theta_0 - \theta_x)\rangle(\omega, \beta)] = [\langle\cos(\theta_0 - \theta_x)\rangle^2(\omega, \beta)]$.

Holds in all dimensions when $\beta = \bar{\beta}$
Implies that NL does not enter spin glass phase.

Conjectured 3D Phase diagram



F=ferromagnetic
P=paramagnetic
SG=spin glass

● -multi-critical point

Remark: Diagram for 3D random phase XY is taken from Alba-Vicari (Monte-Carlo)

Remark: Le Doussal and Harris ('88) predict muliticritical point M is on Nishimori line and calculate critical exponents at M in $6-\epsilon$ expansion.

Random phase SU(2)

Let $U_j \in SU(2)$ and let $\Omega_{jj'} \in SU(2)$ denote random phases. Consider the Gibbs weight

$$e^{Re\,\boldsymbol{\beta}\sum_{jj'} Tr(U_j^* \Omega_{jj'} U_{j'})} \prod_j d\mu(U_j)$$

Invariant under $U_j \to U_j V, \;\; V \in SU(2)$

Disorder distribution: $\Omega_{jj'}$: $\quad \prod_{j,j'} e^{Re\bar{\boldsymbol{\beta}}\, Tr(\Omega_{jj'})} d\mu(\Omega_{jj'})$.

Nishimori Line: $\boldsymbol{\beta} = \bar{\boldsymbol{\beta}}$

Theorem(Garban-Sp) If $d \geq 3$, long range order holds for $\beta = \bar{\beta}$ large.

Advantages and Drawbacks

Advantages:

- Translation invariance not needed: $\beta_{j,j'} = \bar{\beta}_{j,j'}$ can vary.
- Reflection positivity not used. Next to nearest neighbor OK
- For XY we can add a random field $\sum_j h_j \cos(\theta_j - \psi_j)$ where distribution of ϕ_j is $\propto e^{\sum_j h_j \cos \psi_j}$

Drawbacks:

- Works only for $d \geq 3$, (For d=2 cannot get LRO of Ising or KT for XY)
- For SU(2) LRO proved only with Nishimori disorder.
- Does not apply to Heisenberg. Spins must be group valued.

Nishimori gauge invariance - XY case

Let $f_{j,j'}(u)$ be periodic smooth functions eg. $f(u) = e^{iu}$.

Theorem For all $\beta = \bar{\beta}$

$$[\langle \prod_{j,j'} f_{j,j'}(\theta_j - \theta_{j'} + \omega_{j,j'})\rangle(\omega, \beta)]_\beta \ = \prod_{j,j'} [f_{j,j'}(\omega_{j,j'})]_\beta$$

Proof:

$$[\langle \prod_{j,j'} f_{j,j'}(\theta_j - \theta_{j'} + \omega_{j,j'})\rangle]$$

$$= \int \prod_{j,j'} d\omega_{j,j'} \frac{e^{\beta \cos \omega_{j,j'}}}{z(\beta)} \frac{\int \prod_{j,j'} f_{j.j'} \, e^{\beta \cos(\theta_j - \theta_{j'} + \omega_{j,j'})} \prod d\theta_j}{Z(\omega, \beta)}$$

Change of variables: $\theta_j \to \theta_j - \phi_j, \quad \omega_{j,j'} \to \omega_{j,j'} + \phi_j - \phi_{j'}$

Key Observation

Gibbs weight is invariant but the disorder is shifted: $\omega_{j,j'} + \phi_j - \phi_{j'}$.

Since the integral is independent of ϕ_j we integrate over ϕ_j.

This integral exactly cancels $Z(\omega, \beta)$.

Now the remaining integrals factorize and can be computed exactly. $\square$.

Corollary:

$$\frac{d}{d\beta} |\Lambda|^{-1} [\ln Z_\Lambda(\omega, \beta)] = [cos(\omega)]_\beta$$

is analyic on the Nishimori line.

Nevertheless there is a phase transition on NL in 3D.

Sketch of proof of LRO for XY following **Abbe et al.**

Consider family of paths $p : (0,0,0)\, to\ \vec{n} = (n,n,n)$ formed from sums of vectors: $e_1 = (0,0,1)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$.
Each path has 3n edges.

Theorem (Benjamini, Pemantle, Peres '98) In 3D there exists a measure $\boldsymbol{M}(p)$ on such paths with the property that the probabilty two independent paths p_1, p_2 intersect at K edges

$$Prob_{\boldsymbol{M}}\{|p_1 \cap p_2| = K\} \leq Ce^{-c_0 K}, \ \ c_0 > 0\ .$$

In 3D, uniform measure on such paths does not work.

These paths are more unpredictable than random walk.

Result cannot hold in 2D.

Proof of LRO : For paths p: $0 \to \vec{n}$:

$$Y_p \equiv \prod_{jj' \in p} e^{i(\theta_j - \theta_{j'} + \omega_{j,j'})} = e^{i(\theta_0 - \theta_{\vec{n}})} \prod_{jj' \in p} e^{i\omega_{j,j'}} \equiv e^{i(\theta_0 - \theta_{\vec{n}})} R_p(\omega).$$

By the Nishimori gauge property, for every p,

$$[\langle Y_p \rangle] = \prod_{j,j' \in p} [e^{i\omega_{j,j'}}] \equiv \lambda(\beta)^{3n}$$

$$[\langle e^{i(\theta_0 - \theta_x)} \rangle(\omega, \beta) \hat{R}_p(\omega)] = 1, \quad \textit{where} \quad \hat{R}_p(\omega) = \lambda(\beta)^{-3n} R_p(\omega)$$

$$[\langle e^{i(\theta_0 - \theta_x)} \rangle(\omega, \beta)] = 1 - [\langle e^{i(\theta_0 - \theta_x)} \rangle(\omega, \beta) \, \mathbb{E}_{\boldsymbol{M}} (1 - \hat{R}_p(\omega))]$$

Thus it suffices to show:

$$[|\mathbb{E}_{\boldsymbol{M}}(1 - \hat{R}_p(\omega))|^2]^{1/2} = [|\mathbb{E}_{\boldsymbol{M}} \hat{R}_p(\omega)|^2 - 1]^{1/2} \le O(\frac{\ln(\beta)}{\beta^{1/2}})$$

Recall $\lambda^{-1}(\beta)[e^{i\omega_{jj'}}] = 1$ and $\lambda \approx e^{-1/2\beta}$.

$$\mathbb{E}_{\boldsymbol{M}\times\boldsymbol{M}}[\hat{R}_{p_1}\hat{R}^*_{p_2}] = \mathbb{E}_{\boldsymbol{M}\times\boldsymbol{M}}[\{\lambda^{-3n}\prod_{jj'\in p_1} e^{+i\omega_{jj'}}\}\{\lambda^{-3n}\prod_{jj'\in p_2} e^{-i\omega_{jj'}}\}]$$

For every common edge we get a factor of $\lambda(\beta)^{-2} = e^{1/\beta}$.

Let P_K be the probability p_1, p_2 have **K** common edges wrt $\boldsymbol{M}$.

$$[|\mathbb{E}_{\boldsymbol{M}}\hat{R}_p(\omega)|^2] = \sum_K P_K\lambda^{-2K} = 1 + \sum_K P_K(e^{K/\beta} - 1)$$

$$
\begin{aligned}
&= 1 + \sum_{K \leq ln^2 \beta} P_K(e^{K/\beta} - 1) + \sum_{K > ln^2 \beta} P_K(e^{k/\beta} - 1) \\
&= 1 + \frac{ln^2 \beta}{\beta} + C \sum_{K > ln^2 \beta} e^{-c_0 K + K/\beta} \\
&= 1 + \frac{ln^2 \beta}{\beta} + O(\beta^{-1}) \quad for \ \beta \gg 1 \quad \square
\end{aligned}
$$

Similar proof works for SU(N) case.

Define paths p as before and

$$Y_p \equiv \prod_{j,j' \in p} U_j^* \, \Omega_{jj'} \, U_{j'} = U_0^* \{ \prod_{jj' \in p} \Omega_{jj'} \} U_{\vec{n}}$$

where the product is ordered along the path.

Concluding Remarks

According to Nishimori, one referee of his fundamental paper wrote: “the model is too artificial and thin to warrant publication.”

There is interesting work on dynamics along the Nishimori line (Ozeki '95) He shows that there is no ageing (growth of time correlations). This is another indication that NL does not enter the SG phase.

For the Ising case, one can add small correlated (gauge invariant) disorder and still keep analyticy along NL. (Nishimori'22)

Nishimori Line and Bayesian statistics + error correcting codes.
P. Ruján '93 Earlier work by N Sourlas '89, Review Y. Iba; J Phys. A (1999).

2D Ising bond disorder - SUSY (Gruzberg, Ludwig, Read '01), Network model (Merz-Chalker '02)