Continuous symmetry breaking on the Nishimori Line

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Main Results and Background

New proof of phase transitions for 3D disordered spin systems with continuous symmetry. Reflection positivity is not used.

Classical spins, $s_j, j \in \mathbb{Z}^d$ are group valued: $\mathbb{Z}_2$, $O(2)$, $SU(2)$

Ising, XY, Heisenberg-like

Nishimori disorder is a special temperature dependent, gauge invariant, quenched randomness.

**Main Message:** Nishimori disorder can make it easier to prove Long Range Order (LRO) - spin alignment at long distances.
Background of proof:

Group synchronization on grids. Mathematical Statistics and Learning


Georges, Hansel, Le Doussal, and Bouchaud. (1985)
Exact properties of spin glasses. II. Nishimori’s line ; Journal de Physique.

Synchronization and noisy signal recovery

A. Singer; *Angular Synchronization by eigenvectors and semidefinite programming* (2011).

Given a graph with $n$ vertices $j$ and edges $j,j'$ with phases $\theta_j$.

**The angular synchronization problem:** Determine relative phases $\{\theta_j\}$ from $m$ noisy differences $\theta_j - \theta_{j'} \mod 2\pi$.

If noise is too large or $m$ is too small, one cannot get useful information. Where is the threshold? How does it depend on the graph? Find algorithm to recover $\{\theta_j\}$. 
**O(n+1) Symmetry Breaking - no disorder**

Let \( s_j \in S^n \subset \mathbb{R}^{n+1} \), \( j \in \Lambda \subset \mathbb{Z}^d \) and define

\[
\langle s_0 s_x \rangle_{\Lambda}(\beta) = Z_{\Lambda}(\beta)^{-1} \int s_0 s_x e^{\beta \sum_{j,j'} s_j s_{j'}^*} \prod_{j \in \Lambda} d\mu(s_j)
\]

Sum ranges over adjacent \( j, j' \in \Lambda \) and \( d\mu \) is the Haar measure.

For XY, \( s_j \in S^1 \), \( s_j = (\cos \theta_j, \sin \theta_j) \), Ising \( s_j = \pm 1 \), Heisenberg \( s_j \in S^2 \)

**Theorem** (Mermin-Wagner '66) In 2 dimensional models with continuous symmetry, \(( n \geq 1 )\), there is **no** continuous symmetry breaking.

\[
\langle s_0 s_x \rangle(\beta) \to 0, \quad \text{as} \quad |x| \to \infty.
\]
Theorem (Fröhlich-Simon-Sp, '82, Balaban '95) In $d \geq 3$ dimensions, there is long range order (LRO):

$$\langle s_0 s_x \rangle(\beta) \geq M^2(\beta) > 0 \quad \beta \geq \beta^*$$

FSS rely on Reflection Positivity and translation invariance. Balaban develops a robust multi-scale spinwave analysis.

For the 3D XY case and 4D U(1) gauge theories there are other methods by A. Guth, Fröhlich-Sp, Kennedy-King using Fourier Transform and duality.
Phase disordered XY model

Let $\omega_{j,j'} \in [0, 2\pi]$ be random phases, $j, j'$ adjacent. Define

$$\langle \cos(\theta_0 - \theta_x) \rangle^\Lambda(\omega, \beta)$$

$$= Z^-1(\omega, \beta) \int \cos(\theta_0 - \theta_x) e^{\beta \sum_{j,j'} \cos(\theta_j - \theta_{j'} + \omega_{j,j'})} \prod d\theta_j$$

**Distribution** of independent $\omega_{j,j'}$:  

$$\prod_{jj'} \frac{e^{\bar{\beta} \cos(\omega_{j,j'})}}{z(\bar{\beta})}$$

**Nishimori Line:** $\beta = \bar{\beta}$

**Remarks:** When $\beta$ is large and $\bar{\beta}$ small - expect spinglass in 3D.

When $\bar{\beta} \to \infty$, $\omega = 0$ - No disorder.
**Theorem** (Garban-Sp) In 3D there is finite $\beta^*$ so that for $\beta = \bar{\beta} \geq \beta^*$ there is long range order:

$$\left[ \langle \cos(\theta_0 - \theta_x) \rangle(\omega, \beta) \right] \geq M^2 \approx 1 - O\left( \frac{\ln \beta}{\beta^{1/2}} \right)$$

Here $[A(\omega)]$ denotes the average over disorder $\omega$.

**Remark** (J. Fröhlich) Apply inequality of Messager et al.:

$$\langle \cos(\theta_0 - \theta_x) \rangle(\beta, \omega = 0) \geq \langle \cos(\theta_0 - \theta_x) \rangle(\beta, \omega), \text{ for all } \omega.$$  

**Nishimori:** $[\langle \cos(\theta_0 - \theta_x) \rangle(\omega, \beta)] = [\langle \cos(\theta_0 - \theta_x) \rangle^2(\omega, \beta)]$.

Holds in all dimensions when $\beta = \bar{\beta}$

Implies that NL does not enter spin glass phase.
Conjectured 3D Phase diagram

- Conjectured 3D Phase diagram:
  - N-L
  - F
  - SG
  - P
  - F
  - Disorder
  - Temp
  - F=ferromagnetic
  - P=paramagnetic
  - SG=spin glass
  - Multi-critical point
  - B-I
  - B--

- Temp vs Disorder
  - F
  - P
  - SG

- **F=ferromagnetic**
- **P=paramagnetic**
- **SG=spin glass**
- **-multi-critical point**
Remark: Diagram for 3D random phase XY is taken from Alba-Vicari (Monte-Carlo)

Remark: Le Doussal and Harris ('88) predict multicritical point M is on Nishimori line and calculate critical exponents at M in $6 - \epsilon$ expansion.
Random phase SU(2)

Let $U_j \in SU(2)$ and let $\Omega_{jj'} \in SU(2)$ denote random phases. Consider the Gibbs weight

$$e^{Re \beta \sum_{jj'} Tr(U_j^* \Omega_{jj'} U_{j'})} \prod_j d\mu(U_j)$$

Invariant under $U_j \rightarrow U_j V$, $V \in SU(2)$

Disorder distribution: $\Omega_{jj'}: \prod_{j,j'} e^{Re \bar{\beta} Tr(\Omega_{jj'})} d\mu(\Omega_{jj'})$.

Nishimori Line: $\beta = \bar{\beta}$

**Theorem** (Garban-Sp) If $d \geq 3$, long range order holds for $\beta = \bar{\beta}$ large.
Advantages and Drawbacks

Advantages:

▶ Translation invariance not needed: $\beta_{j,j'} = \bar{\beta}_{j,j'}$ can vary.
▶ Reflection positivity not used. Next to nearest neighbor OK
▶ For XY we can add a random field $\sum_j h_j \cos(\theta_j - \psi_j)$ where the distribution of $\phi_j$ is $\propto e^{\sum_j h_j \cos \psi_j}$

Drawbacks:

▶ Works only for $d \geq 3$, (For $d=2$ cannot get LRO of Ising or KT for XY)
▶ For SU(2) LRO proved only with Nishimori disorder.
▶ Does not apply to Heisenberg. Spins must be group valued.
Nishimori gauge invariance - XY case

Let \( f_{j,j'}(u) \) be periodic smooth functions eg. \( f(u) = e^{iu} \).

**Theorem** For all \( \beta = \bar{\beta} \)

\[
\left[ \prod_{j,j'} f_{j,j'}(\theta_j - \theta_{j'} + \omega_{j,j'}) \right](\omega, \beta) \big|_{\beta = \bar{\beta}} = \prod_{j,j'} [f_{j,j'}(\omega_{j,j'})]_{\beta}
\]

**Proof:**

\[
\left[ \prod_{j,j'} f_{j,j'}(\theta_j - \theta_{j'} + \omega_{j,j'}) \right] = \int \prod_{j,j'} d\omega_{j,j'} e^{\beta \cos \omega_{j,j'}} \int \prod_{j,j'} f_{j,j'} e^{\beta \cos(\theta_j - \theta_{j'} + \omega_{j,j'})} \prod d\theta_j \frac{Z(\omega, \beta)}{Z(\beta)}
\]

Change of variables: \( \theta_j \to \theta_j - \phi_j, \quad \omega_{j,j'} \to \omega_{j,j'} + \phi_j - \phi_{j'} \)
Key Observation

Gibbs weight is invariant but the disorder is shifted: \( \omega_{j,j'} + \phi_j - \phi_{j'} \).

Since the integral is independent of \( \phi_j \) we integrate over \( \phi_j \).

This integral exactly cancels \( Z(\omega, \beta) \).

Now the remaining integrals factorize and can be computed exactly.

**Corollary:**

\[
\frac{d}{d\beta} |\Lambda|^{-1} \ln Z_{\Lambda}(\omega, \beta) = [\cos(\omega)]_{\beta}
\]

is analytic on the Nishimori line.

Nevertheless there is a phase transition on NL in 3D.
Sketch of proof of LRO for XY following Abbe et al.

Consider family of paths $p : (0, 0, 0)$ to $\vec{n} = (n, n, n)$ formed from sums of vectors: $e_1 = (0, 0, 1)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$. Each path has $3n$ edges.

**Theorem** (Benjamini, Pemantle, Peres '98) In 3D there exists a measure $M(p)$ on such paths with the property that the probability two independent paths $p_1, p_2$ intersect at $K$ edges

$$\text{Prob}_M\{|p_1 \cap p_2| = K\} \leq C e^{-c_0 K}, \ c_0 > 0.$$ 

In 3D, uniform measure on such paths does not work.

These paths are more unpredictable than random walk.

Result cannot hold in 2D.
Proof of LRO: For paths $p: 0 \to \vec{n}$:

$$Y_p \equiv \prod_{jj' \in p} e^{i(\theta_j - \theta_{j'} + \omega_{j,j'})} = e^{i(\theta_0 - \theta_{\vec{n}})} \prod_{jj' \in p} e^{i\omega_{j,j'}} \equiv e^{i(\theta_0 - \theta_{\vec{n}})} R_p(\omega).$$

By the Nishimori gauge property, for every $p$,

$$[\langle Y_p \rangle] = \prod_{j,j' \in p} [e^{i\omega_{j,j'}}] \equiv \lambda(\beta)^{3n}$$

$$[\langle e^{i(\theta_0 - \theta_x)} \rangle(\omega, \beta) \hat{R}_p(\omega)] = 1, \quad \text{where} \quad \hat{R}_p(\omega) = \lambda(\beta)^{-3n} R_p(\omega)$$

$$[\langle e^{i(\theta_0 - \theta_x)} \rangle(\omega, \beta)] = 1 - [\langle e^{i(\theta_0 - \theta_x)} \rangle(\omega, \beta) \mathbb{E}_M (1 - \hat{R}_p(\omega))]$$

Thus it suffices to show:

$$[|\mathbb{E}_M(1 - \hat{R}_p(\omega))|^2]^{1/2} = [|\mathbb{E}_M \hat{R}_p(\omega)|^2 - 1]^{1/2} \leq O\left(\frac{\ln(\beta)}{\beta^{1/2}}\right)$$
Recall $\lambda^{-1}(\beta)[e^{i\omega_{jj'}}] = 1$ and $\lambda \approx e^{-1/2\beta}$.

$$\mathbb{E}_{M\times M}[\hat{R}_{p_1}\hat{R}^*_{p_2}] = \mathbb{E}_{M\times M}[\{\lambda^{-3n} \prod_{jj' \in p_1} e^{+i\omega_{jj'}}\} \{\lambda^{-3n} \prod_{jj' \in p_2} e^{-i\omega_{jj'}}\}]$$

For every common edge we get a factor of $\lambda(\beta)^{-2} = e^{1/\beta}$.

Let $P_K$ be the probability $p_1, p_2$ have $K$ common edges wrt $M$.

$$[|\mathbb{E}_{M}\hat{R}_p(\omega)|^2] = \sum_{K} P_K \lambda^{-2K} = 1 + \sum_{K} P_K(e^{K/\beta} - 1)$$
\[= 1 + \sum_{K \leq \ln^2 \beta} P_K(e^{K/\beta} - 1) + \sum_{K > \ln^2 \beta} P_K(e^{k/\beta} - 1)\]

\[= 1 + \frac{\ln^2 \beta}{\beta} + C \sum_{K > \ln^2 \beta} e^{-c_0 K + K/\beta} \]

\[= 1 + \frac{\ln^2 \beta}{\beta} + O(\beta^{-1}) \quad \text{for } \beta \gg 1 \]
Similar proof works for SU(N) case.

Define paths $p$ as before and

$$Y_p \equiv \prod_{j,j' \in p} U_j^* \Omega_{jj'} U_{j'} = U_0^* \{ \prod_{jj' \in p} \Omega_{jj'} \} U_{\vec{n}}$$

where the product is ordered along the path.
Concluding Remarks

According to Nishimori, one referee of his fundamental paper wrote: “the model is too artificial and thin to warrant publication.”

There is interesting work on dynamics along the Nishimori line (Ozeki ’95) He shows that there is no ageing (growth of time correlations). This is another indication that NL does not enter the SG phase.

For the Ising case, one can add small correlated (gauge invariant) disorder and still keep analyticity along NL. (Nishimori’22)

Nishimori Line and Bayesian statistics + error correcting codes.

2D Ising bond disorder - SUSY (Gruzberg, Ludwig, Read ’01), Network model (Merz-Chalker ’02)