Buildings, C*-algebras and new higher-dimensional analogues of the Thompson groups.

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Newcastle University

Harvard Picture Language Project Seminar
May 2020

Outline

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- Difficulties: higher dimensions require counter-intuitive geometry of buildings.
- Another application of buildings I recently developed is new Drinfeld-Manin solutions of Yang-Baxter equations.

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- in terms of operations on rooted binary trees;

Many generalizations, for example, Brin-Thompson groups, many approaches, but very hard to distinguish : work of C.Bleak, B.Nucinkis, and their collaborators.

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- Very general approach to further generalizations, such that all existing ones appear as a particular case (joint work with M.Lawson, just published in Adv. in Math.);
- C*-algebraic invariants (K-theory) to distinguish the groups.

C*-algebras

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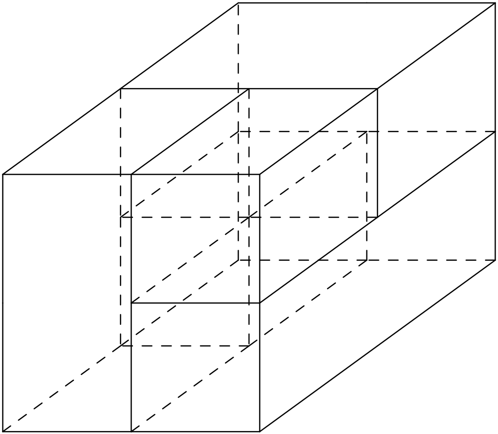
- For all $x \in B$: $\|x^* x\| = \|x\| \|x^*\|$.

Prefix codes

The most common definition of the Thompson group by the piecewise linear homeomorphisms of the unit interval does not extend well to higher dimensions.



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- R is essential if $R \cap I \neq \emptyset$ for every right ideal I of A^*
- $R = PA^*$ is essential if and only if P is a maximal prefix code.

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Definition

The Thompson group $V_{k,1}$ is the group consisting of maximal isomorphisms between finitely generated essential right ideals of A^* with multiplication: $\phi\psi = max(\phi \circ \psi)$, where ψ is composition of partial functions.

Connections of Thompson's groups and C*-algebras

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- V.Jones and his school: representations of Thompson's groups in C*-algebras are used in theoretical physics.
- My goal: higher D generalizations and their applications to distinguish groups.

Buildings and polyhedra

Buildings consist of **chambers** and **apartments**.

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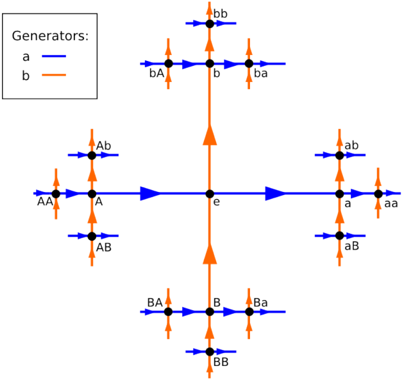
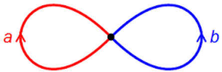
- for any two chambers there is an apartment containing both of them
- if two apartments F_1 and F_2 have non-trivial intersection, then there is an isomorphism from F_1 to F_2, fixing $F_1 \cap F_2$ pointwise.

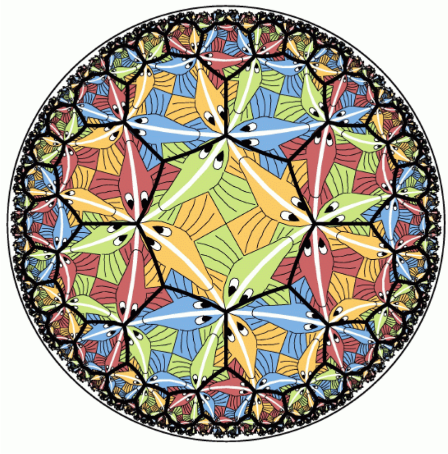
One-dimensional buildings: Cayley graphs of free groups





The four-valent tree is the *universal cover* of the wedge of two circles. Geometric interpretation of prefix codes: finite sub-trees partitioning the boundary of the infinite tree.

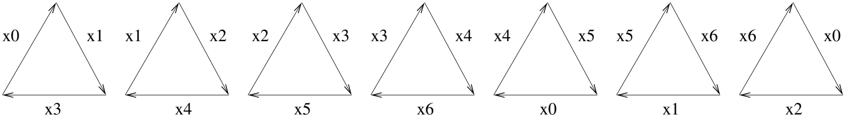
Example of an apartment: M.C.Escher - Circle Limit III



Polyhedra and links

Definition

A *polyhedron* is a two-dimensional complex which is obtained from several decorated p-gons by identification of corresponding sides.



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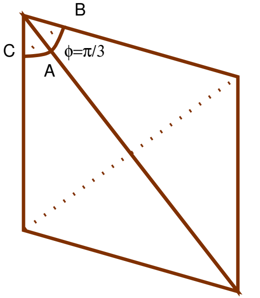
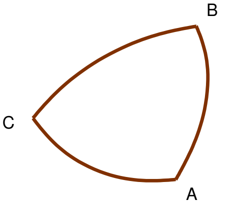
Take a sphere of a small radius at a point of the polyhedron. The intersection of the sphere with the polyhedron is a graph, which is called the *link* at this point.





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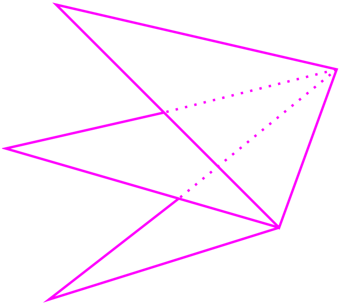
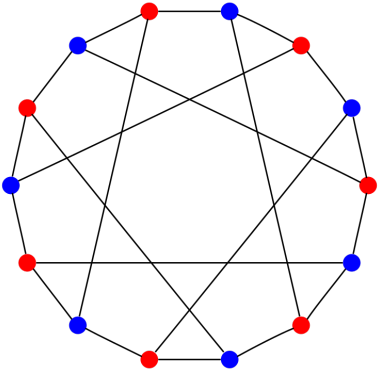
Polyhedra and links

We consider *thick* polyhedra, which means that each edge is contained in at least three polygons.



Example of a link



This graph has *diameter* (the maximal distance between two vertices) three and *girth* (the length of the shortest cycle) six.

Polyhedra and links

Theorem (Ballmann, Brin 1994)

Let X be a compact two-dimensional thick polyhedron. If all links are graphs of diameter m and girth $2m$, then the universal cover of the polyhedron is a two-dimensional building.

A polygonal presentation is a set of words satisfying certain combinatorial properties (AV, 2000).

Later work: many series of infinite families in arbitrary dimensions.

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Theorem (AV,2002)

A polyhedron with given links can be constructed explicitly using a polygonal presentation. Any connected bipartite graph can be realized as a link of every vertex a 2-dimensional polyhedron with $2k$-gonal faces.

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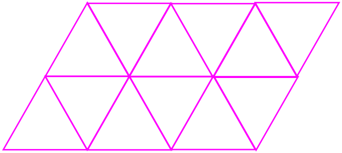
Let $\mathcal{B}$ be a n-dimensional Euclidean building equipped with a cocompact action of a group G, nD words are rectangular subsets of apartments in $\mathcal{B}$, decorated by the action of G.



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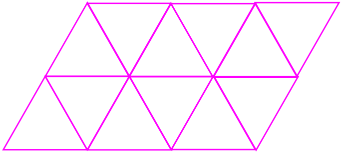

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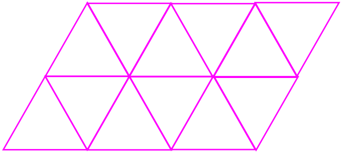

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nD (maximal) prefix code is a subcomplex of a building, which corresponds to a partition of the boundary into disjoint sets.

Cubes and Products of Trees

The four squares define a group G which belongs to a family constructed by J.Stix and AV

$$G = \langle a_1, a_2, b_1, b_2 \mid a_2 b_1 a_2 b_2^{-1}, a_1 b_2^{-1} a_2^{-1} b_2^{-1}, a_1 b_1 a_1 b_2, a_1 b_1^{-1} a_2 b_1^{-1} \rangle.$$

Let $S = \{a_1, a_2, b_1, b_2\}$. Then $Cay(G, S)$ is a one-skeleton of a thick Euclidean building (product of two trees) with the following properties:

- lots of (equilateral) squares (chambers)

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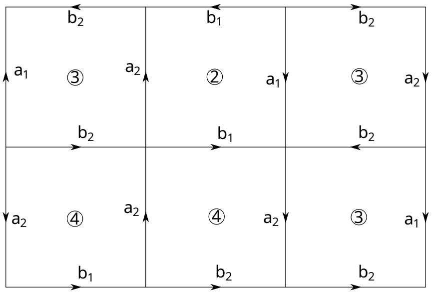
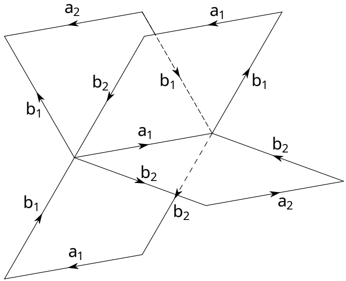
- link of every vertex is the complete bipartite graph on eight vertices
- G is an arithmetic lattice in $PGL(2, \mathbb{F}_3((t))) \times PGL(2, \mathbb{F}_3((t)))$

Cubes and Products of Trees

① $a_1 b_1 a_1 b_2$ ② $a_1 b_1^{-1} a_2 b_1^{-1}$ ③ $a_1 b_2^{-1} a_2^{-1} b_2^{-1}$ ④ $a_2 b_1 a_2 b_2^{-1}$





3D example

Quaternions can be used to get a cube complex of any dimension, for any set of odd primes (RSV 2018).

$a_1 = 1 + j + k$, $a_2 = 1 + j - k$, $a_3 = 1 - j - k$, $a_4 = 1 - j + k$,
$b_1 = 1 + 2i$, $b_2 = 1 + 2j$, $b_3 = 1 + 2k$, $b_4 = 1 - 2i$, $b_5 = 1 - 2j$, $b_6 = 1 - 2k$,
$c_1 = 1 + 2i + j + k$, $c_2 = 1 - 2i + j + k$, $c_3 = 1 + 2i - j + k$, $c_4 = 1 + 2i + j - k$,
$c_5 = 1 - 2i - j - k$, $c_6 = 1 + 2i - j - k$, $c_7 = 1 - 2i + j - k$, $c_8 = 1 - 2i - j + k$.

With this notation we have $a_i^{-1} = a_{i+2}$, $b_i^{-1} = b_{i+3}$, and $c_i^{-1} = c_{i+4}$, and using these abbreviations we find the explicit presentation.

3D example

$$\Gamma_{\{3,5,7\}} = \left\langle \begin{array}{c} a_1, a_2 \\ b_1, b_2, b_3 \\ c_1, c_2, c_3, c_4 \end{array} \middle| \begin{array}{c} a_1 b_1 a_4 b_2,\ a_1 b_2 a_4 b_4,\ a_1 b_3 a_2 b_1, \\ a_1 b_4 a_2 b_3,\ a_1 b_5 a_1 b_6,\ a_2 b_2 a_2 b_6 \\ a_1 c_1 a_2 c_8,\ a_1 c_2 a_4 c_4,\ a_1 c_3 a_2 c_2,\ a_1 c_4 a_3 c_3, \\ a_1 c_5 a_1 c_6,\ a_1 c_7 a_4 c_1,\ a_2 c_1 a_4 c_6,\ a_2 c_4 a_2 c_7 \\ b_1 c_1 b_5 c_4,\ b_1 c_2 b_1 c_5,\ b_1 c_3 b_6 c_1, \\ b_1 c_4 b_3 c_6,\ b_1 c_6 b_2 c_3,\ b_1 c_7 b_1 c_8, \\ b_2 c_1 b_3 c_2,\ b_2 c_2 b_5 c_5,\ b_2 c_4 b_5 c_3, \\ b_2 c_7 b_6 c_4,\ b_3 c_1 b_6 c_6,\ b_3 c_4 b_6 c_3 \end{array} \right\rangle.$$

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- The Cayley graph of Γ with respect to the generating set $\{a, b\}$, $Cay(\Gamma, \{a, b\})$, is a homogeneous tree of degree 4.
- The vertices of the tree are elements of Γ *i.e.* reduced words in $S = \{a, b, a^{-1}, b^{-1}\}$.

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- The boundary, Ω, of the tree can be thought of as the set of all semi-infinite reduced words $w = x_1 x_2 x_3 ....$, where $x_i \in S$

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- Ω has a natural compact (totally disconnected) topology :
- if $x \in \Gamma$ then let $\Omega(x)$ be all semi-infinite words with the prefix x
- $\Omega(x)$ is open and closed in Ω and the sets $g\Omega(x)$ and $g(\Omega \setminus \Omega(x))$, where $g \in \Gamma$ and $x \in S$, form a base for the topology of Ω.

Graph C*-algebras

Left multiplication by $x \in \Gamma$ induces an action α of Γ on $C(\Omega)$ by

$$\alpha(x)f(w) = f(x^{-1}w).$$

$C(\Omega) \rtimes \Gamma$ is generated by $C(\Omega)$ and the image of a unitary representation π of Γ
such that $\alpha(g)f = \pi(g)f\pi^*(g)$ for $f \in C(\Omega)$ and $g \in \Gamma$ and every such C^*-algebra is a quotient of $C(\Omega) \rtimes \Gamma$.

Graph C*-algebras

For $x \in \Gamma$, let p_x denote the projection defined by the characteristic function $\mathbf{1}_{\Omega(x)} \in C(\Omega)$.
For $g \in \Gamma$, we have

$$gp_xg^{-1} = \alpha(g)\mathbf{1}_{\Omega(x)} = \mathbf{1}_{g\Omega(x)}$$

and therefore for each $x \in S$,

$$p_x + xp_{x^{-1}}x^{-1} = \mathbf{1}.$$

$$p_a + p_{a^{-1}} + p_b + p_{b^{-1}} = \mathbf{1}$$

Partial isometries

For $x \in S$ we define a *partial isometry* $s_x \in C(\Omega) \rtimes \Gamma$ by

$$s_x = x(\mathbf{1} - p_{x^{-1}}).$$

Then,

$$s_x s_x^* = x(\mathbf{1} - p_x)x^{-1} = p_x$$

and

$$s_x^* s_x = \mathbf{1} - p_{x^{-1}} = \sum_{y \neq x^{-1}} s_y s_y^*.$$

These relations show that the partial isometries s_x, for $x \in S$, generate a C*-algebra $\mathcal{O}_A$.

Transition matrix

Where

$$A = \begin{pmatrix} \mathbf{1} & 0 & \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} \end{pmatrix}$$

relative to $\{a, a^{-1}, b, b^{-1}\} \times \{a, a^{-1}, b, b^{-1}\}$.

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- Boundary Ω is defined by an equivalence relation on sectors (just as in the case of trees it is given by an equivalence relation on words);
- Γ is a fundamental group of the polyhedron P defined earlier.

nD polyhedral algebras

The following definition is a natural genelalization of the Jones' Pythagorean algebra.

Definition

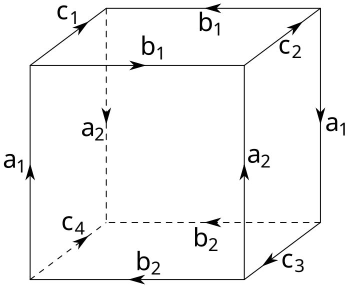
An *nD* polyhedral algebra is the universal $C^\star$-algebra generated by partial isometries $S_{u,v}$, where u and v are words in the given *nD* alphabet, with $t(u) = t(v)$, satisfying the relations

$$\begin{aligned} S^*_{u,v} &= S_{v,u} & S_{u,v}S_{v,w} &= S_{u,w} \\ S_{u,v} &= \sum S_{uw,vw} & S_{u,u}S_{v,v} &= 0, \ \forall u \neq v \end{aligned} \tag{1}$$

(The sum here is over n-dimensional words w with $o(w) = t(u) = t(v)$ and with shape $\sigma(w) = e_j$, for $j = 1, \dots, n$, where e_j is the j-th standard basis vector in $\mathbb{Z}^n$.)

We note, that *nD* words in an *nD*-alphabet need to satisfy certain compatibility conditions, these alphabets turn out to give new solutions to Yang-Baxter equations.

Example of a letter of an *nD*-alphabet



The set X is taken to be the set of labels on the edges of the cubes, the bijection R is induced by squares of the complex, namely is $x_i x_j x_k x_l$ is a label of a square, then $R(x_i, x_j) = (x_l^{-1}, x_k^{-1})$. In the (3,5,7) example the set X has 18 elements, so the R-matrix is of size 153x153.

$R^{12}R^{23}R^{12}(a_1, b_1, c_2) = R^{12}R^{23}(b_2^{-1}, a_2, c_2) = R^{12}(b_2^-1, c_3^{-1}, a_1^{-1}) = (c_4, b_2^{-1}, a_1^{-1})$. $R^{23}R^{12}R^{23}(a_1, b_1, c_2) = (c_4, b_2^{-1}, a_1^{-1})$.

Theorem (J.Konter,AV)

The order of the class [$\mathbf{1}$] of the identity element $\mathbf{1}$ of $C(\Omega) \rtimes \Gamma$ in $K_0(C(\Omega) \rtimes \Gamma)$ is $q-1$, where Γ is a group acting on a triangular Euclidean building with three orbits and $q = 2^{2l-1}, l \in \mathbb{Z}$.

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- Further applications of harmonic maps to study of buildings and higher-dimensional complexes;
- New applications of polygonal presentations to algebraic geometry: Beauville surfaces and fake quadrics.

Relevant references

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