Buildings, C*-algebras and new higher-dimensional analogues of the Thompson groups.

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Newcastle University

Harvard Picture Language Project Seminar
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Outline

Introduction

C*-algebras

Buildings

Higher dimensional words

Graph C*-algebras

$nD$ polyhedral C*-algebras

Further directions of research
The subject requires several fields:

- Operator Algebras;
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We discuss higher-dimensional generalizations of Thompson groups and relevant C*-algebras.

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- Very general approach to further generalizations, such that all existing ones appear as a particular case (joint work with M.Lawson, just published in Adv. in Math.).
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- Very general approach to further generalizations, such that all existing ones appear as a particular case (joint work with M.Lawson, just published in Adv. in Math.);
- C*-algebraic invariants (K-theory) to distinguish the groups.
We begin with the abstract characterization of C*-algebras given in the 1943 paper by Gelfand and Naimark.

**Definition**

A C*-algebra, $B$, is a Banach algebra over the field of complex numbers, together with a map $x \mapsto x^*$ for $x \in B$ with the following properties:

1. It is an involution, for every $x \in B$:
   
   $x^{**} = (x^*)^* = x$

2. For all $x, y \in B$:
   
   $(x + y)^* = x^* + y^*$

3. For every complex number $\lambda$ and every $x \in B$:
   
   $(\lambda x)^* = \lambda^* x^*$

4. For all $x \in B$:
   
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- For every complex number $\lambda$ and every $x \in B$: $(\lambda x)^* = \overline{\lambda}x^*$.
- For all $x \in B$: $\|x^*x\| = \|x\|\|x^*\|$. 

Prefix codes

The most common definition of the Thompson group by the piecewise linear homeomorphisms of the unit interval does not extend well to higher dimensions.
Prefix Codes

- Let $A = \{a_1, ..., a_k\}$ and $u, v \in A^*$; $u$ is a prefix of $v$ if $v = uw$ for some $w \in A^*$. 

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- If $R$ a right ideal of $A^*$, then $R = PA^*$ for a uniquely determined prefix code $P$; $P$ is the unique minimal set of generators for $R$. 
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- \( R \) is essential if \( R \cap I \neq \emptyset \) for every right ideal \( I \) of \( A^* \).
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- $R = PA^*$ is essential if and only if $P$ is a maximal prefix code.
Prefix Codes

- Let $R_1, R_2$ be right ideals of $A^*$. A bijection $\phi : R_1 \to R_2$ is an $A^*$-isomorphism if $\phi(uv) = \phi(u)v$ for all $u \in R_1, v \in A^*$. 
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- An extension of an $A^*$-isomorphism $\phi : R_1 \to R_2$ is an $A^*$-isomorphism $\psi : I_1 \to I_2$ of right ideals $I_1, I_2$ where $R_i \subseteq I_i (i = 1, 2)$ and $\psi(u) = \phi(u)$ for all $u \in R_1$. $\phi$ is maximal if it has no proper extension.
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**Definition**

The Thompson group $V_{k,1}$ is the group consisting of maximal isomorphisms between finitely generated essential right ideals of $A^*$ with multiplication: $\phi \psi = \text{max}(\phi \circ \psi)$, where $\psi$ is composition of partial functions.
Connections of Thompson’s groups and C*-algebras

- Birget (2004) Representations of Thompson’s groups in C*-algebras using words, aware of trees, but 1D situation does not require geometric methods.
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- My goal: higher D generalizations and their applications to distinguish groups.
Buildings and polyhedra

**Buildings** consist of **chambers** and **apartments**.

**Definition**
A $n$-dimensional euclidean (hyperbolic) building is a $n$-dimensional complex $X$ such that:

- $X$ is a union of tessellated $nD$-spaces (apartments)
- For any two chambers there is an apartment containing both of them
- If two apartments $F_1$ and $F_2$ have non-trivial intersection, then there is an isomorphism from $F_1$ to $F_2$, fixing $F_1 \cap F_2$ pointwise.
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One-dimensional buildings: Cayley graphs of free groups

The four-valent tree is the universal cover of the wedge of two circles. Geometric interpretation of prefix codes: finite sub-trees partitioning the boundary of the infinite tree.
Example of an apartment: M.C. Escher - Circle Limit III
Polyhedra and links

Definition
A polyhedron is a two-dimensional complex which is obtained from several decorated $p$-gons by identification of corresponding sides.
Polyhedra and links

**Definition**
Take a sphere of a small radius at a point of the polyhedron. The intersection of the sphere with the polyhedron is a graph, which is called the *link* at this point.

\[ AB = BC = CA = \frac{\pi}{3} \]
We consider \textit{thick} polyhedra, which means that each edge is contained in at least three polygons.
Example of a link

This graph has *diameter* (the maximal distance between two vertices) three and *girth* (the length of the shortest cycle) six.
Polyhedra and links

Theorem (Ballmann, Brin 1994)

Let $X$ be a compact two-dimensional thick polyhedron. If all links are graphs of diameter $m$ and girth $2m$, then the universal cover of the polyhedron is a two-dimensional building.

A polygonal presentation is a set of words satisfying certain combinatorial properties (AV, 2000).

Later work: many series of infinite families in arbitrary dimensions.
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**Theorem (AV,2002)**

A polyhedron with given links can be constructed explicitly using a polygonal presentation. Any connected bipartite graph can be realized as a link of every vertex a 2-dimensional polyhedron with $2k$-gonal faces.

Later work: many series of infinite families in arbitrary dimensions.
Definition
Let $\mathcal{B}$ be a $n$-dimensional Euclidean building equipped with a cocompact action of a group $G$, $nD$ words are rectangular subsets of apartments in $\mathcal{B}$, decorated by the action of $G$. 

![Diagram of a building](image-url)
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**Definition**
$nD$ (maximal) prefix code is a subcomplex of a building, which corresponds to a partition of the boundary into disjoint sets.
Cubes and Products of Trees

The four squares define a group $G$ which belongs to a family constructed by J.Stix and AV

$$G = \langle a_1, a_2, b_1, b_2 \mid a_2b_1a_2b_2^{-1}, a_1b_2^{-1}a_2^{-1}b_2^{-1}, a_1b_1a_1b_2, a_1b_1^{-1}a_2b_1^{-1} \rangle.$$ 

Let $S = \{a_1, a_2, b_1, b_2\}$. Then $\text{Cay}(G, S)$ is a one-skeleton of a thick Euclidean building (product of two trees) with the following properties:

- lots of (equilateral) squares (chambers)
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- link of every vertex is the complete bipartite graph on eight vertices
- $G$ is an arithmetic lattice in $PGL(2, \mathbb{F}_3(t)) \times PGL(2, \mathbb{F}_3(t))$
Cubes and Products of Trees

1 \ a_1 b_1 a_1 b_2
2 \ a_1 b_1^1 a_2 b_1^1
3 \ a_1 b_2^1 a_2^1 b_2^1
4 \ a_2 b_1 a_2 b_2^1

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3D example

Quaternions can be used to get a cube complex of any dimension, for any set of odd primes (RSV 2018).

\[ a_1 = 1 + j + k, \quad a_2 = 1 + j - k, \quad a_3 = 1 - j - k, \quad a_4 = 1 - j + k, \]
\[ b_1 = 1 + 2i, \quad b_2 = 1 + 2j, \quad b_3 = 1 + 2k, \quad b_4 = 1 - 2i, \quad b_5 = 1 - 2j, \quad b_6 = 1 - 2k, \]
\[ c_1 = 1 + 2i + j + k, \quad c_2 = 1 - 2i + j + k, \quad c_3 = 1 + 2i - j + k, \quad c_4 = 1 + 2i + j - k, \]
\[ c_5 = 1 - 2i - j - k, \quad c_6 = 1 + 2i - j - k, \quad c_7 = 1 - 2i + j - k, \quad c_8 = 1 - 2i - j + k. \]

With this notation we have \( a_i^{-1} = a_{i+2}, \) \( b_i^{-1} = b_{i+3}, \) and \( c_i^{-1} = c_{i+4}, \) and using these abbreviations we find the explicit presentation.
3D example

\[ \Gamma_{\{3,5,7\}} = \left< \begin{array}{c} a_1, a_2 \\ b_1, b_2, b_3 \\ c_1, c_2, c_3, c_4 \end{array} \right| \begin{array}{c} a_1b_1a_4b_2, a_1b_2a_4b_4, a_1b_3a_2b_1, \\ a_1b_4a_2b_3, a_1b_5a_1b_6, a_2b_2a_2b_6 \\ a_1c_1a_2c_8, a_1c_2a_4c_4, a_1c_3a_2c_2, a_1c_4a_3c_3, \\ a_1c_5a_1c_6, a_1c_7a_4c_1, a_2c_1a_4c_6, a_2c_4a_2c_7 \\ b_1c_1b_5c_4, b_1c_2b_1c_5, b_1c_3b_6c_1, \\ b_1c_4b_3c_6, b_1c_6b_2c_3, b_1c_7b_1c_8, \\ b_2c_1b_3c_2, b_2c_2b_5c_5, b_2c_4b_5c_3, \\ b_2c_7b_6c_4, b_3c_1b_6c_6, b_3c_4b_6c_3 \end{array} \right> \]
Introduction  C*-algebras  Buildings  Higher dimensional words  Graph C*-algebras  nD polyhedral C*-algebras  Further directions of research

Graph C*-algebras

Let $\Gamma = \mathbb{Z} \ast \mathbb{Z}$, the free group on two generators $a$ and $b$. 

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Graph $C^*$-algebras

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- The Cayley graph of $\Gamma$ with respect to the generating set $\{a, b\}$, $\text{Cay}(\Gamma, \{a, b\})$, is a homogeneous tree of degree 4.
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- The Cayley graph of $\Gamma$ with respect to the generating set $\{a, b\}$, $\text{Cay}(\Gamma, \{a, b\})$, is a homogeneous tree of degree 4.
- The vertices of the tree are elements of $\Gamma$ i.e. reduced words in $S = \{a, b, a^{-1}, b^{-1}\}$. 
The boundary, $\Omega$, of the tree can be thought of as the set of all semi-infinite reduced words $w = x_1 x_2 x_3 \ldots$, where $x_i \in S$. 
Graph C*-algebras

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- \( \Omega \) has a natural compact (totally disconnected) topology.
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Graph C*-algebras

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- $\Omega$ has a natural compact (totally disconnected) topology:
- If $x \in \Gamma$ then let $\Omega(x)$ be all semi-infinite words with the prefix $x$
- $\Omega(x)$ is open and closed in $\Omega$ and the sets $g\Omega(x)$ and $g(\Omega \setminus \Omega(x))$, where $g \in \Gamma$ and $x \in S$, form a base for the topology of $\Omega$. 
Graph $C^*$-algebras

Left multiplication by $x \in \Gamma$ induces an action $\alpha$ of $\Gamma$ on $C(\Omega)$ by

$$\alpha(x)f(w) = f(x^{-1}w).$$

$C(\Omega) \rtimes \Gamma$ is generated by $C(\Omega)$ and the image of a unitary representation $\pi$ of $\Gamma$ such that $\alpha(g)f = \pi(g)f\pi^*(g)$ for $f \in C(\Omega)$ and $g \in \Gamma$ and every such $C^*$-algebra is a quotient of $C(\Omega) \rtimes \Gamma$. 
Graph C*-algebras

For $x \in \Gamma$, let $p_x$ denote the projection defined by the characteristic function $1_{\Omega(x)} \in C(\Omega)$.

For $g \in \Gamma$, we have

$$gp_x g^{-1} = \alpha(g) 1_{\Omega(x)} = 1_{g\Omega(x)}$$

and therefore for each $x \in S$,

$$p_x + xp_{x^{-1}}x^{-1} = 1.$$
Partial isometries

For $x \in S$ we define a partial isometry $s_x \in C(\Omega) \rtimes \Gamma$ by

$$s_x = x(1 - p_{x^{-1}}).$$

Then,

$$s_x s_x^* = x(1 - p_x) x^{-1} = p_x$$

and

$$s_x^* s_x = 1 - p_{x^{-1}} = \sum_{y \neq x^{-1}} s_y s_y^*.$$

These relations show that the partial isometries $s_x$, for $x \in S$, generate a C*-algebra $\mathcal{O}_A$. 
Transition matrix

Where

\[
A = \begin{pmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1
\end{pmatrix}
\]

relative to \( \{a, a^{-1}, b, b^{-1}\} \times \{a, a^{-1}, b, b^{-1}\} \).
Polyhedral C*-algebras

- Buildings (instead of trees);
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- Boundary $\Omega$ is defined by an equivalence relation on sectors (just as in the case of trees it is given by an equivalence relation on words);
Polyhedral C*-algebras

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- Boundary $\Omega$ is defined by an equivalence relation on sectors (just as in the case of trees it is given by an equivalence relation on words);
- $\Gamma$ is a fundamental group of the polyhedron $P$ defined earlier.
**nD polyhedral algebras**

The following definition is a natural generalization of the Jones’ Pythagorean algebra.

**Definition**

An $nD$ polyhedral algebra is the universal $C^*$-algebra generated by partial isometries $S_{u,v}$, where $u$ and $v$ are words in the given $nD$ alphabet, with $t(u) = t(v)$, satisfying the relations

\[
S_{u,v}^* S_{v,u} = S_{u,v}, \quad S_{u,v} S_{v,w} = S_{u,w},
\]

\[
S_{u,v} = \sum S_{uw,vw}, \quad S_{u,u} S_{v,v} = 0, \quad \forall u \neq v
\]  

(1)

(The sum here is over $n$-dimensional words $w$ with $o(w) = t(u) = t(v)$ and with shape $\sigma(w) = e_j$, for $j = 1, \ldots, n$, where $e_j$ is the $j$-th standard basis vector in $\mathbb{Z}^n$.)

We note, that $nD$ words in an $nD$-alphabet need to satisfy certain compatibility conditions, these alphabets turn out to give new solutions to Yang-Baxter equations.
Example of a letter of an $nD$-alphabet

The set $X$ is taken to be the set of labels on the edges of the cubes, the bijection $R$ is induced by squares of the complex, namely is $x_ix_jx_kx_l$ is a label of a square, then $R(x_i,x_j) = (x_i^{-1},x_j^{-1})$. In the $(3,5,7)$ example the set $X$ has 18 elements, so the $R$-matrix is of size $153 \times 153$.

$$R^{12}R^{23}R^{12}(a_1,b_1,c_2) = R^{12}R^{23}(b_2^{-1},a_2,c_2) = R^{12}(b_2^{-1},c_3^{-1},a_1^{-1}) = (c_4,b_2^{-1},a_1^{-1}).$$

$$R^{23}R^{12}R^{23}(a_1,b_1,c_2) = (c_4,b_2^{-1},a_1^{-1}).$$
Theorem (J. Konter, AV)

The order of the class $[1]$ of the identity element $1$ of $C(\Omega) \rtimes \Gamma$ in $K_0(C(\Omega) \rtimes \Gamma)$ is $q - 1$, where $\Gamma$ is a group acting on a triangular Euclidean building with three orbits and $q = 2^{2l-1}, l \in \mathbb{Z}$. 
Further directions of research

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- New applications of polygonal presentations to algebraic geometry: Beauville surfaces and fake quadrics.
Relevant references