The wondrous world of hyperfinite subfactors

Mathematical Picture Language Seminar

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Subfactors: an inclusion of II_1 factors

- **M ⊆ B(H)** von Neumann algebra (\(*\)-subalgebra, \(1 \in M\), closed in top. of ptwise conv. on vectors)
- **M** is a factor if \(\mathcal{Z}(M) = C\mathbb{1}\).
- A factor \(M\) is of type **II_1** if \(\dim M = \infty\) and if \(M\) has a tracial state \(\text{tr}: M \to \mathbb{C}\).
- M is **hyperfinite** if there are fin. dim. \(*\)-subalgs \(A_n \subseteq \text{Aut}(M)\) with \(\bigcup_{n=1}^{\infty} A_n^w = M\). **Construction:**
  \[
  A_n = M_{2n}(\mathbb{C}) \ni x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \in M_{2n+1}(\mathbb{C}) = A_{n+1}
  \]
\[ A = \bigcup_{n \geq 1} M_{2n}(\mathbb{C}) \text{, } C^*\text{-algebra with trace } \tau \text{ (normalized matrix trace)} \]
\[ (x(y)) = \tau(y^* x) \quad \| x \|_2^2 = \tau(x^* x) \quad x, y \in A \]
\[ H = \overline{A}^{\Pi_2}, \quad A \cong H \] left multiplication
\[ R = \overline{A}^W \leq B(H) \quad \text{hyperfinite } II_1 \text{ factor} \]

- \( \tau \) extends to \( R \), \( Z(R) = C \), \( \tau(\text{all projections}) = [0, 1] \) continuous dimension

- Murray-VN (1943): There is a unique hyperfinite II_1 factor.
- If a II_1 factor, then \( R \rightarrow M \) (\( R \) is the smallest \( \tau \)).
**Connes (1976):** $N \leq R$ subfactor ($\dim N = \omega$), then $N \leq R$.

G amenable ICC group, $\neq \mathbb{C}$, then $L(G) \leq R$.

**Jones (1983):** Position of $N$ in $R$, i.e., $N \leq R$ up to isom.

**Invariants?** $H = \overline{R}^{|\cdot|_1} =: L^2(R)$, $N$-$R$ bimodule:

$n \in N$, $m \in R$, $n \cdot \hat{m} = \hat{nm}$, $\hat{R} \leq L^2(R)$ dense

$S = L^2(R)_R$ standard rep. of $N \leq R$

$S$ is irreducible iff $N \cap R = G$ (we say $N \leq R$ is irreducible)
Coupling constant measures size of $L^2(R)$ as left $N$-mod:

$$L^2(R)^\oplus_{N\text{-module}} \cong \bigoplus_{i=1}^{k} L^2(N) \oplus L^2(N) \rho, \quad \rho \in N \text{ a proj}.$$ 

Jones index: $[R:N] = k + \text{tr}(\rho) \in [1, \infty]$

($[R:N] \in \{4, \infty\} \cup \{n \geq 3 \} \cup [4, \infty]$, Jones' rigidity theorem, 1983)

We assume $[R:N] < \infty$.

$$\mathcal{S} := L^2(R) \cong \text{standard rep.}$$

$$\mathcal{S} \cong L^2(R) \cong \text{contragredient rep.}$$
Proceed à la H. Weyl:

\[(\otimes R)^{\otimes k} \triangleleft \text{decompose} \rightarrow \text{irred. } N-N \text{ bimodules}\]

\[(\otimes \nu S)^{\otimes k} \triangleleft \text{decompose} \rightarrow \text{irred. } R-R \text{ bimodules } (*)\]

\text{irred. } N-R, R-N \text{ bim.}

\[\text{Standard invariant:}\]

\[g_{NR} := \begin{cases} 
\text{Hom}_{N-N} ((\otimes S)^{\otimes}) \circ \text{id} \\
\text{Hom}_{N-R} (\otimes S)^{\otimes} \circ \text{id} \\
\text{Hom}_{R-N} (\otimes S)^{\otimes} \circ \text{id} \\
\text{Hom}_{R-R} (\otimes S)^{\otimes} \circ \text{id} 
\end{cases} \]

\text{planar algebra, } L\text{-lattice, rigid } \text{C*-tensor cat. } / \text{fusion cat. etc.}
$G_{NCR}$ consists of a double-sequence of multi-matrix algebras + orthogonality (commuting squares).

**Fusion graphs**

$$\text{id} = L^2(N)$$

$$A \otimes A \cong B \otimes C$$

Irred. $N-N$ bim.

Irred. $N-R$ bim.

$(\Gamma, \Gamma')$ bipartite, connected, possibly infinite graphs + standard eigenvectors (e.g. PF-e.v.)
Example: $\Gamma = \Gamma' = A_\infty$ and $\|A_\infty\|^2 = 4$.

Standard invariant consists of Temperley–Lieb–Jones algebras with parameter $S = [R:V]^{1/2}$ (minimal or "trivial").

Popa (1990'): 

Amenability of $N_cR$.

$\Gamma$ is amenable if $\| \Gamma \|^2 = [R:V] = ||\Gamma'\|^2$.

Theorem (Popa, 1994): $G_{N_cR}$ is a complete invariant for $N_cR$ with $\Gamma$ amenable.

Thus, $N_cR$ with $[R:V] > 4$ and $TLJ (= A_\infty)$ standard invariant is non-amenable.
Finite depth subfactors (\(\cong\) finite graph) are amenable. Classification led to many "exotic" fusion categories (Haagerup \(\frac{5+\sqrt{13}}{2}\), EH, \ldots). Up to index 5.25, countably many finite depth subfactors, and the only stand.\(^\dagger\) with infinite graphs are \(A_\infty, D_\infty (A_{-\infty,\infty})\) and:

**Thm (B. Jones):** There are non-amenable, irreducible subfactors with index \(3 + \sqrt{5} \approx 5.23606\ldots\) and fusion graphs

\[\gamma_{A_3} \ast \gamma_{A_4} \text{ (Popa exist as hyper-finite subf.)}\]
Some open problems

1) Which stand. inv./planar alg. arise from hyperfinite subf.?
   • What is $I(R) = \{ [R:N] \mid N \leq R \text{ irreducible subf.} \}$? (Jones)

2) Given $N \leq R$ with $A_\infty$-stand. inv. and $[R:N] > 4$, is there $Q \leq R$ with same stand. inv., but $(N \leq R) \neq (Q \leq R)$?
   (compare Ocneanu, Jones result on group actions)

3) How many hyperfinite subf. are there with $G_{A_3} \times G_{A_4}$ stand. inv. (under $3 + \sqrt{5}$)? Many?

We need to look for invariants beyond $G_{N \leq R}$.
We need to construct ($\infty$ depth) hyperfinite subfactors.
Constructions (of irreducible hyperfinite subfactors)

• Use group actions, group reps, quantum groups etc.

Bisch–Haagerup subfactors: \( R^H \leq R^{\neq K} \)

(\( H, K \) finite groups with outer actions on \( R \))

Thm (B–Haagerup, 1996): Let \( G = \langle H, K \rangle \leq \text{Out} R = \text{Aut} R / \text{Int} R \).

1) \( R^H \leq R^{\neq K} \) is irreducible \( \iff \) \( H \cap K = \{ e \} \) in \( \text{Out} R \)
2) \( R^H \leq R^{\neq K} \) has infinite depth \( \iff \) \( |G| = \infty \).
3) \( R^H \leq R^{\neq K} \) is amenable \( \iff \) \( G \) is amenable.
Commuting squares

\[ B_0 = B_1, \quad K = \text{normal faithful trace} (\text{weight vector}) \]
\[ A_0 \leq A_1, \quad \text{Ai, Bi: multi-matrix algebras} \]

It is a commuting square if

\[ A_0^+ \cap A_1 \perp A_0^+ \cap B_0 \quad \text{in } B_1 \quad \text{w.r.t. to } (x(g) = tr(y^* x)) \]

\[ (\Leftrightarrow E_{B_0} E_{A_1} = E_{A_0} \Leftrightarrow E_{A_1} E_{B_0} = E_{A_0} \Leftrightarrow E_{A_1} (B_0) = A_0 \quad \text{etc.}) \]

Natural structure: \[ A \subset B \subset C \subset D \quad \text{fulfills UV alg., tr n.f. trace alg.} \]
then
\[ A^c C \subset A^c \cap D \]
\[ B^c \cap C \subset B^c \cap D \]

is a commuting square.
It is a commuting square iff $|U_{ij}| = \frac{1}{\sqrt{n}}$ for $1 \leq i, j \leq n$.

**Complex Hadamard matrix**

$$A_0 \subset A_1 \subset A_2 \subset A_3 \subset \ldots \subset \overline{U A_n^T}^v =: A$$

$$B_0 \subset B_1 \subset B_2 \subset B_3 \subset \ldots \subset \overline{UB}^v =: B$$

$G \mathcal{H} = KL$

$H L^T = G^* K$

+ easy condit. + \text{Markov trace} (PF...) \rightarrow \text{AcB irreducible, hyperfinite subf. with } [B:A] = \|K\|^2 = \|H\|^2.$
Note: K. H finite graphs $\rightarrow$ index is an algebraic integer.

Smallest known ($\mathcal{O}_{c, \text{H-S}}$): $[\mathcal{B}: \mathcal{A}] = 11 E_{10} 11^2 = 4.026...$

Bisch (1994): There is an irreducible hyperfinite subfactor with index 4.5.

Built from a c.s.:

\[
\begin{align*}
\mathbb{Z}_G & \xrightarrow{v} \mathbb{Z}^G \\
\mathbb{Z}^G & \xrightarrow{\mathcal{A}_0} \mathbb{Z}^G \\
\mathbb{Z}^G & \xrightarrow{\mathcal{B}_0} \mathbb{Z}^G \\
\mathbb{Z}^G & \xrightarrow{\mathcal{B}_1} \mathbb{Z}^G \\
\end{align*}
\]

By classification, it has $\mathbb{A}_\infty$-standard invariant.

Rem: Interestingly (infinite) c.s. have been constructed.

"Bare hand constructions"
How noncommutative can a subfactor be?

$N_c M (\subseteq R)$, commutativity of $M$ relative to $N$

Def: $N_c M$ has property $(\mathbb{M})$ if there are nontrivial central sequences for $M$ in $N$, i.e. $\exists (x_n) \in \ell^\infty(N, M)$ with $\|x_n y - y x_n\|_2 \to 0, n \to \infty, \forall y \in M$ and $\inf_n \|x_n - \text{tr}(x_n)I\|_2 > 0$.

Central sequence algebras: $\omega$ a free ultrafilter on $\mathbb{N}$ (\omega \in \beta\mathbb{N} \setminus \mathbb{N}),$ let

$$I_\omega = \{ (x_n) \in \ell^\infty(N, M) \mid \lim_{n \to \omega} \|x_n\|_2 = 0 \}, \quad M^\omega = \ell^\infty(N, M) / I_\omega.$$

$M^\omega$ is a VN algebra with trace $\text{tr}_\omega((x_n)) = \lim_{n \to \omega} \text{tr}(x_n)$. 
Prop: \( NcM \text{ in } \text{McDuff} \iff M' \cap N^w \neq \emptyset \). In this case, \( M' \cap N^w \) contains a diffuse, abelian subalg.

Def: \( NcM \text{ in } \text{McDuff} \) (or stable) if \( (NcM) \cong (NcM) \otimes R \).

Thm (B., 1990): \( NcM \text{ in } \text{McDuff} \iff M' \cap N^w \) is non-abelian.

Degrees of non-commutativity for \( NcM \):

- \( M' \cap N^w = \emptyset \)
- \( M' \cap N^w \) abelian, diffuse
- \( M' \cap N^w \) non-abelian, hence II_1 VN alg.

\( NcM \) more commutative

faktor \quad \text{non-faktor}
Examples:

- If $\text{fin. depth of } \{\text{strongly amenable}\}$, then $M \cap N$ is a II$_1$ factor.

- Non R constructed from commuting squares are McDuff (the sequences $(e_i)_{i \geq 1}$ and $(e_{0i})_{i \geq 1}$ are nontrivial, central seq. for $M$, contained in $N$).

$\Rightarrow$ "very commutative" (irred.) hyperfinite subfactors

Non-commutative examples?
Note: If $NcM$ satisfies $M \cap N^w = G$, then $N \otimes R \otimes M \otimes R$ has the same stand. inv., but big relative central sequ. algebra (hence $\neq (NcM)$) by my thm.

Thm(B.): Let $G = \langle H, K \rangle$, $H, K$ fin. groups, $G \vartriangleright R$ outer action. Then

1) $R^H \subset R \otimes K$ has prop. (P) iff $R \subset R \otimes \otimes G$ has prop. (P).

2) $R^H \subset R \otimes K$ has McDuff iff $R \subset R \otimes \otimes G$ has McDuff.

This allows us to construct many "non-commutative" irredu. hyperf. subfactors.
• By 1), it suffices to look for $G \subset \mathbb{R}$ s.t.
$R_x G$ does not have property (M).
• Lots of examples: Strongly ergodic actions of
non-inner amenable groups on $\mathbb{R}$
• Concrete examples (index 6): $G = \langle \mathbb{Z}_2, \mathbb{Z}_3 \rangle$,
$G$ property (T) (e.g. $G = SL(n, \mathbb{Z}), n \geq 29$),
$\sigma =$ Bernoulli shift on $\mathbb{R}$. Then $R_x G$ does not
have property (M) (Choda), hence $R^{\mathbb{Z}_2} \subset R \times \mathbb{Z}_3$ are
very non-commutative (do not have prop. (M)).
They cannot be constructed from a
commuting square!