Is Relativity Compatible with Quantum Theory?

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Literature in the Mathematical Sciences
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Outline

1. Problem and Results
   a. Physics problem: special relativity + quantum theory
   b. Proposal
   c. Mathematical obstructions
   d. Potential methods
   e. Existence theorems
   f. Properties of solutions

2. Brief Remarks about Methods
   a. Classical-quantum duality
   b. Perron-Frobenius analysis in infinite dimensions
   c. Phase cell localization and Independence
   d. Reflection positivity and Multiple reflection bounds

3. What’s Next?
Major pedestals of 20\textsuperscript{th} century physics

\[ E = mc^2 \]
Special Relativity 1905

\[ i\hbar \frac{d\psi}{dt} = H\psi \]
Quantum Theory 1926
Isolate:
Make: **Physics = Mathematics**

240-year history: 1687-1926 Newton to Schrödinger

- Newton: **Gravitational Physics Calculus**
- Maxwell: **Electromagnetism Symmetry**
- Gibbs/Boltzmann: **Statistical Physics Probability**
- Einstein: **Relativity Algebra, Geometry**
- Schrödinger: **Quantum Physics Analysis, PDE**
- Wigner 1960: “On the Unreasonable Effectiveness…”

- **SO WHAT ABOUT** Relativity + Quantum Theory?

2 December 2020 1a. Physics problem
Hilbert’s 6th Problem: Axiomatize Physics

• Formulated by Hilbert in 1900 at Paris ICM
• Einstein: “But the creative principle resides in mathematics…."
• Wightman: “A great physical theory is not mature until it has been put in precise mathematical form.”
“Relativistic wave equation” for the electron wave function $\psi$, given Coulomb potential $A(x)$.

$$\left( \sum_{\mu} \gamma^\mu \left( i\hbar \frac{\partial}{\partial x^\mu} - eA_\mu(x) \right) - mc \right) \psi(x) = 0$$

This explained hydrogen-atom energy levels—almost!

It omits interaction that would come if $\psi$ and $A$ satisfy coupled Dirac + Maxwell equations.
Quantum Field Theory
Born, Heisenberg, Jordan, Dirac (1925-27)

Forces between particles (or between particles and fields) arise from equations for the fields.

A particle is a state of a field (associated with irreducible representation of the special relativity group).
Representations of Poincaré group

Wigner (following earlier work of Majorana) classified the positive-energy, irreducible representations of the semi-direct product $\text{SL}(2,\mathbb{C}) \rtimes \mathbb{R}^4$ of the covering group $\text{SL}(2,\mathbb{C})$ of the Lorentz group $\text{SO}(3,1)$ with translations on $\mathbb{R}^4$.

Parameters of a particle emerge:

mass $m \geq 0$, half-integer spin $s$.

E. Majorana Nuovo Cimento (1932)
E. Wigner Annals of Mathematics (1939)
V. Bargmann and E. Wigner PNAS (1948)
An Ultimate Quantitative Test of QFT

Magnetic moment $\mu$ of an electron.

$$\mu = \kappa \frac{e\hbar}{mc}, \quad \kappa_{\text{Bohr}} = \frac{1}{2}, \quad \kappa_{\text{Dirac}} = 1.$$

1926 non-relativistic
1928 relativistic

$$\kappa = 1.001 \quad \text{Kusch (1947)}$$

70 years of measurements later:

$$\kappa = 1.00115965218073(\pm 28). \quad \text{(2008)}$$

part in $10^{12}$, Gerald Gabrielse

A.I.P. “Physics Achievement of the Year: 2006” Improved 2008
Field Revolution: Ultimate Quantitative Test
Magnetic moment $\mu$ of an electron.

$$\kappa = 1.00115965218073(\pm 28).$$

Amazing: These QED (Quantum Electrodynamics) experiments agree completely with the Feynman rules for calculation in perturbation theory, also carried out over 70 years!!

Many physics Nobel laureates related to this question.
(Central to physics)
The Big Mathematical Question

• As the “Feynman rules” for quantum electrodynamics agree with the most accurately measured phenomena in nature,

• And as the putative quantum field explanation of the measurements embodies special-relativity and quantum theory,

• Our belief in logical science mandates that we determine whether (or not) relativistic QFT makes mathematical sense.

• BIG Question: QED, QCD, YM, or Standard Model
What Do We Know?

\[ d = \text{dimension of space-time} \]

- **Goal:** Understand how to distinguish what we know, from what we believe we know.
  - ✓ \( d = 2 \)
  - ✓ \( d = 3 \)
  - ? \( d = 4 \)
  - ? \( d > 4 \)

I now describe some results.

Focus on Early Work; Tentacles to Today

Apologies in advance for important work I forget to mention
Formulate as Mathematics; 3 methods (+30 years)

   – Quantum theory (Hilbert space of vector states)
   – Field $\varphi(x)$ is an operator-valued distribution
   – Positive energy $H$ and unique ground state vector $\Omega$
   – Covariance under unitary representation of $\text{SL}(2,\mathbb{C}) \rtimes \mathbb{R}^4$
   – Locality $[\varphi(x), \varphi(y)]_{\pm} = 0$, for $x - y$ spacelike
   – Some technical regularity assumptions
   – Output: Spin/Statistics Theorem and the PCT Theorem

II. Axioms for Algebraic Quantum Theory:
   – Study bounded functions of the field and the von Neumann algebras that they generate.


Problem: NO INTERESTING MATHEMATICAL EXAMPLES

In 1950’s this goal appeared beyond mathematical reach.
4 Natural Methods: 2 Seem Promising

• ✗ Algebraic Solutions or Inverse Scattering
  – A few known examples (W. Thirring, J. Schwinger, …)
  – All have uninteresting scattering.
  – No indication physical theories are integrable

• ✗ Representations of $[q_k, p_j] = i\delta_{kj}$ (CCR)
  (A. Wightman, J. Lew) Too many inequivalent representations. (no Lagrangian connection)

• ✓ Hilbert Space & Specific Equations
  – Analysis of operators, and PDE  A. Wightman, I. Segal

• ✓ Path Integrals  K. Symanzik, J. Schwinger
  – Schrödinger → Diffusion Probability Theory, analysis of measures
Focus here on the Simplest Case, $d=2$

Free Field: (Klein Gordon equation): particles without interaction

$$\varphi_{tt}(x, t) - \varphi_{xx}(x, t) + \varphi(x, t) = 0$$

Canonical time-zero (CCR): $[q_k, p_j] = i\delta_{kj}$. Generalize to

$$[\varphi(x, 0), \varphi_t(y, 0)] = i\delta(x - y)$$

Well-Known Solution for linear equation by Fourier transformation. Particle interpretation arises from Fourier analysis of the field.

For interaction, need non-linearity. Introduce simplest non-linearity: Perturbation with parameter $0 \leq \lambda$ (to give positive energy).

$$\varphi_{tt}(x, t) - \varphi_{xx}(x, t) + \varphi(x, t) + \lambda\varphi^3(x, t) = 0$$

Clearly the non-linear term will be singular.
$\varphi^4$ Equation, $d = 2$

$\varphi_{tt}(x, t) - \varphi_{xx}(x, t) + \varphi(x, t) + \lambda \varphi^3(x, t) = 0$

Classical Hamiltonian: $H = \int_{t=0} \left( \frac{1}{2} \varphi_t^2 + \frac{1}{2} \varphi_x^2 + \frac{1}{2} \varphi^2 + \frac{\lambda}{4} \varphi^4 \right) dx$

$$= H_0 + \frac{\lambda}{4} \int_{t=0} \varphi^4 dx = H_0 + H_I$$

Quantum Hamiltonian: $H = H_0 + \frac{\lambda}{4} \int_{t=0} : \varphi^4 : dx = H_0 + H_I$

“Normal ordering” : $\varphi^4 : (x, 0) = \varphi^4(x, 0) - 6 c \varphi^2(x, 0) + 3 c^2$

$$= (\varphi^2(x, 0) - 3 c)^2 - 6 c^2$$

Hermite polynomial. $\langle \Omega_0, :\varphi^4:\Omega_0 \rangle = 0$, $\Omega_0$ ground state of $H_0$

Lower bound on quantum perturbation $H_I$: $-\frac{3\lambda}{2} |\text{Volume}| c^2$
Where does the constant $c$ come from?

1. Coulomb singularity in Yukawa Green’s function:

$$C(x - y) = (-\Delta + m^2)^{-1}(x, y) \sim \begin{cases} 
1, & \text{if } d = 1 \\
\ln |x - y|, & \text{if } d = 2, \text{ as } |x - y| \to 0 \\
\frac{1}{|x - y|^{d-2}}, & \text{if } d \geq 3
\end{cases}$$

Lowest orders of perturbation theory in $\lambda$ for the equation involves both

$$c = C(0) = \begin{cases} 
O(1), & \text{if } d = 1 \\
\infty, & \text{otherwise}
\end{cases}, \quad \text{and} \quad \|C\|_{L^4} = \begin{cases} 
O(1), & \text{for } d = 1, 2 \\
\infty, & \text{otherwise}
\end{cases}.$$  

2. Redefining $\varphi^4$ as :$\varphi^4$: is sufficient “renormalization” for the $d=2$ equation. Other renormalizations are needed with Dirac equation in $d=2$, or with $d > 2$. 

2 December 2020
Some Early Results: Hilbert Space Methods

• A.J. Theory on $T^s$ with mollification  Thesis (1965)
  • $H_i$ unbounded above and below; could $H_0$ stabilize?

• **E.Nelson  Endicott House Meeting Proceedings (1966)

Theorem: For $\varphi^4$ Hamiltonian on $S^1 \times \mathbb{R}$, there exists a constant $M = M(|S^1|) < \infty$, such that

$$0 \leq H + M = H_0 + H_I + M$$
More Early Results: Hilbert Space Methods

• **J.Glimm and A.J.**

**Theorem:** For Hamiltonian $H_L = H_0 + \lambda \int_{-L}^{L} :\varphi^4 : dx$, with $0 \leq L < \infty$,

$$H_L^* = H_L^{**}$$

Enables unitary evolution $e^{itH_L^*}$ and field in terms of initial values

$\varphi(x, t) = e^{itH_L^*} \varphi(x, 0) e^{-itH_L^{**}}$. This is the local Hamiltonian solution.

**Theorem:** For $f = \overline{f} \in C_0^\infty(\mathbb{R}^2)$ and $\varphi(f) = \int e^{itH_L} \varphi(x, 0) e^{-itH_L} f(x, t) dx dt$,

$$\varphi(f)^* = \varphi(f)^{**}$$

For $L$ sufficiently large, $\varphi(f)^{**}$ is independent of $L$. Also

$[\varphi(f)^{**}, \varphi(g)^{**}] = 0$, for $f$ and $g$ with space-like separated supports.

**Theorem:** The Hamiltonian $H_L$ has a unique ground eigenstate $\Omega_L$.

• **J.Cannon and A.J.**

**Theorem:** The Haag-Kastler axioms of algebraic QFT hold.
More Early Results: Hilbert Space Methods

- J. Glimm and A. J.

**Theorem:** The vacuum states $\omega_L$ defined by the ground state eigenvectors $\Omega_L$ of $H_L$

$$\omega_L(\cdot) = \langle \Omega_L, \cdot | \Omega_L \rangle,$$

have a weakly convergent subsequence to a vacuum state $\omega$ as $L \to \infty$.

**Theorem:** Representation of time zero fields locally, unitarily equivalent to Fock representation.

**Theorem:** There exists $0 < M < \infty$ such that

$$-ML \leq H_L = H_0 + \lambda \int_{-L}^{L} :\phi^4:dx$$
Lagrangian density for covariant fields. The feature exploited in EQFT is that the field equations and canonical commutation relations (CCR) imply differential equations of elliptic type for $S$-functions while the differential equations for $W$-functions or, more usual, Feynman amplitudes, are of hyperbolic type. For elliptic differential equations certain powerful analytic tools are available that have no (mathematically meaningful) counterpart for hyperbolic differential equations.

On the other hand, EQFT has the basic disadvantage of dealing with not directly physically interpretable quantities. However, the existence of $S$-functions with properties to be given later for a specific Lagrangian EQFT is a necessary condition for the existence of an MQFT to that Lagrangian. Thus, EQFT, provided its tools are sufficiently sharpened, may be an effective test for the admissibility of certain Lagrangians. In particular, the familiar renormalization constants, as far as they are not rigidly tied to the mass spectrum or other

<table>
<thead>
<tr>
<th>GIMM-JAFFE</th>
<th>EQFT</th>
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<tbody>
<tr>
<td>A: works directly with physically relevant quantities (Hamiltonian, field operators, state space).</td>
<td></td>
</tr>
<tr>
<td>D: analysis proceeds through non-covariant steps, such that relativistic invariants must be preserved separately. The vacuum and objects related to it, as well as divergences occurring in the non-covariant approach through canonical variables, present a nontrivial problem.</td>
<td></td>
</tr>
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<td>A: concepts employed (e.g. linear analysis) are relatively familiar. Many established techniques are available, and many results have been obtained.</td>
<td></td>
</tr>
<tr>
<td>A: Higher-spin theories, e.g. QED of spin-$j$ particles, offer little new complications compared with scalar theories. (A: advantage)</td>
<td></td>
</tr>
<tr>
<td>D: existence of $S$-functions only necessary for existence of (MQ) theory.</td>
<td></td>
</tr>
<tr>
<td>W-functions are only analytic continuations of $S$-functions.</td>
<td></td>
</tr>
<tr>
<td>A: covariants are preserved throughout. The vacuum presents no problem, and only the divergences familiar from covariant renormalization theory occur.</td>
<td></td>
</tr>
<tr>
<td>D: concepts (belonging to probability theory), are relatively unfamiliar. Few established techniques are available, and few results have (as yet) been obtained.</td>
<td></td>
</tr>
<tr>
<td>D: Most theories involving particles with spin, e.g. QED of spin-$j$ particles or four-fermion coupling, offer considerable complications compared with scalar theories or spin-$0$ QED.</td>
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Common features: Locality simplifies the analysis. Merely renormalizable theory (e.g. $g^4$-theory in 4 dimensions) presents much greater difficulties than a superrenormalizable theory.
Euclidean Revolution

• In spite of Symanzik’s negative assessment, E.Nelson persisted to work with his method. Four years later he showed that a random field, with a global Markov property, gives a QFT.  

• F.Guerra showed that Nelson’s method displayed an extraordinarily helpful symmetry.  

• Problems: Need global Markov property. Only scalar fields; no fermions (e.g. Yukawa interaction.)

• Other Methods: F.Guerra, L.Rosen, and B.Simon used correlation methods to study Euclidean infinite volume limits.
Amazing Elementary Discovery: RP

• I studied Nelson’s paper in 1972 with my postdoctoral fellows Konrad Osterwalder & Robert Schrader. I hoped for a more robust method. This led to their discovery of the “reflection-positivity (RP) property.”

• This began the IVth (Euclidean) theme for QFT axioms.

**Theorem:** Assume expectations are (i) Euclidean covariant, (ii) reflection positive, and (iii) obey some analytic regularity conditions. This is equivalent to the Wightman axioms, supplemented by additional analytic regularity conditions, but without uniqueness of the vacuum. If correlations decay, then the vacuum is unique.

\[
0 \leq \int_{\mathbb{R}^d} (\Phi)^2 \, d\mu \quad , \quad F(\Phi) \quad d\mu
\]

\[
d\mu = \frac{1}{Z_I} e^{-\mathcal{A}_I(\Phi)} d\mu_0(\Phi)
\]
Reflection Positivity

- RP solved many problems:
  - RP miracle: it gives the correct inner product for every quantum theory.
  - Markov properties unnecessary. Many different fields included.
  - RP automatically assures Hilbert space, positive $H$, and vacuum $\Omega$.
  - RP gives reflection inequality in quantum theory, useful for many things.
  - Ultimately RP became important in gauge theory, in statistical physics, in representation theory, and in other subjects including planar algebras.
Back to Seeking a Full Theory

Tom Spencer: PhD Courant 1972; Postdoctoral Fellow Harvard 1974-5
Finally, one has the first relativistic QFT with non-trivial scattering.

**Theorem: (Wightman Theory.)** [J. Glimm, A. J., and T. Spencer, *Annals of Mathematics* (1974)] There exists $\lambda_{\text{max}} > 0$, such that for all $\lambda \in [0, \lambda_{\text{max}}]$, the $\lambda \phi^4$ theory exists and satisfies the **Wightman axioms** (in strengthened form) and the **Haag-Ruelle axioms** for scattering theory. This includes:

1. The vacuum is unique: The Hamiltonian $H = H^\dagger > 0$ has a unique vector $\mathcal{I}$ (up to a multiple) for which $H \mathcal{I} = P \mathcal{I} = 0$.

2. The Hamiltonian has a lower and upper mass gap: The mass operator $M = p_H^2 P^2$ has an isolated eigenvalue $m > 0$ corresponding to single particle states.

3. The field is well-behaved: For real functions $f \in C_0^1(\mathbb{R}^2)$, the fields $\phi(f)$ have a domain on which their closures are self-adjoint, $\phi(f)^\dagger = \phi(f)^\dagger\dagger$.

4. The field is covariant: There is a positive-energy representation $U(g)$ of the Poincaré group on the Hilbert space of the theory, and the field $\phi(x)$ transforms covariantly, $U(g)\phi(x)U(g^{-1}) = \phi(gx)$.

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A Highpoint

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Examples of Axioms

Rudolf Haag and Arthur Wightman
Boulder 1982

What's Next?
Double-Well Potential: Non-Uniqueness?

Perron-Frobenius ensures “tunneling” and a unique ground state in usual quantum theory. (no spin) G-J showed that Perron-Frobenius still holds for finite volume scalar QFT.

The Ising model suggests this may not be true in infinite volume, and physicists believed this. If true, then symmetry \( \varphi \to -\varphi \) of \( H \) would be broken in the ground state, and \( \langle \varphi \rangle \neq 0 \).

Quantum Potential: Problem from \( c = \infty \).

Discussed during a 1972 meeting in Moscow with expert on phase transitions in lattice statistical physics, R. Dobrushin.

Classical Potential
\[
V(\varphi) = \lambda^{-2} \left( \lambda^2 \varphi^2 - 1 \right)^2
\]
\[
\lambda \to 0
\]
\[
V \to \text{“Ising” distribution}
\]
Theorem: \((\varphi \mapsto -\varphi\) Symmetry Breaking.) [J.Glimm, A.J., and T.Spencer, Comm. Math. Phys. (1975), and two papers on clustering in Ann. Phys. (1976)] There exists \(\lambda_{\text{max}} > 0\), such that for all \(\lambda \in [0, \lambda_{\text{max}}]\), the Euclidean \(\lambda^{-2} (\lambda^2 \varphi^2 - 1)^2\) measure exists as a weak limit of finite-volume measures with interaction on increasing space-time squares \(\Lambda\), and with the field on the boundary of \(\Lambda\) equal to one of \(-\lambda^{-1}\), 0, or \(\lambda^{-1}\). This gives three different limiting measures \(\omega_-, \omega_0, \omega_+\) respectively, for the same interaction.

\[
\begin{align*}
\lambda & \\
\text{Fix field on boundary to equal } 0 \text{ or } \pm \lambda^{-1}
\end{align*}
\]
Another Highpoint Phase Transitions Exist in QFT

Theorem: (φ → −φ Symmetry Breaking.) [J.Glimm, A.J., and T.Spencer, Comm. Math. Phys. (1975), and two papers on clustering in Ann. Phys. (1976)] There exists λmax > 0, such that for all λ ∈ [0, λmax], the Euclidean λ−2(λ2φ2 − 1)2 measure exists as a weak limit of finite-volume measures with interaction on increasing space-time squares Λ, and with the field on the boundary of Λ equal to one of −λ−1, 0, or λ−1. This gives three different limiting measures ω−, ω0, ω+ respectively, for the same interaction.

1. The states ω± give Wightman theories: The states ω± are extremal and satisfy the OS axioms. The corresponding two theories obtained from the OS-construction satisfy the Wightman axioms with a unique vacuum state.

\[ \text{Fix field on boundary to equal } 0 \text{ or } \pm \lambda^{-1} \]
Another Highpoint

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2. Broken symmetry \( \varphi \rightarrow -\varphi \): The field has vacuum expectation \( \omega_{+}(\varphi(x)) = -\omega_{-}(\varphi(x)) \) close to \( \pm \lambda^{-1} \).

\[ \Lambda \]

Fix field on boundary to equal 0 or \( \pm \lambda^{-1} \)
Another Highpoint
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2. Broken symmetry $\varphi \mapsto -\varphi$: The field has vacuum expectation $\omega_+(\varphi(x)) = -\omega_-(\varphi(x))$ close to $\pm \lambda^{-1}$.

3. $\omega_\pm$ theories have a non-zero mass gap: Correlations for the Euclidean theories from $\omega_\pm$ decay exponentially with a rate $m > 0$, corresponding to a mass gap $(0, m)$ in the spectrum of $H$. 

Theorem: (φ → −φ Symmetry Breaking.) [J.Glimm, A.J., and T.Spencer, Comm. Math. Phys. (1975), and two papers on clustering in Ann. Phys. (1976)] There exists \( \lambda_{\text{max}} > 0 \), such that for all \( \lambda \in [0, \lambda_{\text{max}}] \), the Euclidean \( \lambda^{-2} (\lambda^2 \varphi^2 - 1)^2 \) measure exists as a weak limit of finite-volume measures with interaction on increasing space-time squares \( \Lambda \), and with the field on the boundary of \( \Lambda \) equal to one of \(-\lambda^{-1}, 0, \lambda^{-1}\). This gives three different limiting measures \( \omega_-, \omega_0, \omega_+ \) respectively, for the same interaction.

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Another Highpoint

Phase Transitions Exist in QFT

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Phase Transitions in \(\varphi^6\) theory \cite{Gawedzki1978, Summers1981}
Weak-Coupling Phase Diagrams

For many lattice statistical mechanical models, S. Piragov and Ya. G. Sinai analyzed their phase diagrams. Could a similar analysis be relevant for QFTs?

Theorem: [“Pirogov-Sinai”-type phase diagram]
Let $0 \leq W$ be a polynomial of even degree with $r$ distinct zeros; let $V = W + P$, where $P$ is a polynomial perturbation of degree $(r - 1)$; and let $0 < \lambda$ be sufficiently small. Consider the $\lambda^{-2} V(\lambda \varphi)$ QFT in dimension $d = 2$. 
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This theory has $k \in \{r, r - 1, \ldots, 1\}$ distinct phases for the parameters of $P$ lying in $\binom{r}{k}$ hypersurfaces of dimension $(r - k)$ in the space of perturbations.

Let’s apply this to the prior $\lambda \varphi^4$ example. Then:

$$V = W + P, \quad W = (\varphi^2 - 1)^2, \quad P = -\mu \varphi$$
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$$\lambda^{-2}V(\lambda \varphi) = \lambda^{-2}(\lambda^2 \varphi^2 - 1)^2 - \lambda^{-1} \mu \varphi$$

$r = 2, k = 2$; 1 point (dim 0)
$r = 2, k = 1$; 2 lines (dim 1)

Topologically:
Phase diagram is the corner of a hypercube in $r-1$ dimensions
d=3

- Space-time dimension 3 is qualitatively harder: normal ordering is insufficient for renormalization.
- Wave function renormalization entails a “change of representation” in finite volume.
- This example gives the only known family of Euclidean-invariant, RP measures on $\mathcal{S}'(\mathbb{R}^3)$.


In $d = 3$ dimensional space-time, consider the renormalized $\lambda \varphi^4$ Hamiltonian $H_L$, with $0 < \lambda$ and with non-zero interaction localized in a fixed spatial square $[-L, L]^2$. Then $H_L$ is bounded from below.

Theorem: (Full Field Theory.) [J.Feldman and K.Osterwalder, *Ann. Phys.* (1976)] The renormalized $\lambda \varphi^4$ field theory, with $0 < \lambda$ sufficiently small, exists in $d = 3$ dimensional space-time. This theory satisfies the Wightman axioms with a positive mass gap. It comes from a Euclidean-invariant, RP-positive measure on $\mathcal{S}'(\mathbb{R}^3)$. 

2 December 2020 1.e. Existence Theorems
Celebration Exhibit in the Cabot Science Library at Harvard

Glimm-Jaffe “Quantum Physics”
Springer Monograph
Publication 1981
Some Other Work: Much with RP


- Complex Fields A.J., C.Jäkel, R.Martinez


Some Other Work: Models

- F. Guerra, L. Rosen, B. Simon
- J. Fröhlich
- Many results
Some Other Work: Phase Transitions

- Contin. symmetry unbroken d=2 Dobrushin, Shlossman
- Kosterlitz-Thouless transition J.Fröhlich, T.Spencer
- Critical dimension random field Ising J.Imbrie

Some Other Work: Hypercontractivity

- E.Nelson discovered this in 1965
- Hypercontractivity for fermions L.Gross
- Quantum Information settings C.King

Some Other Work: Phase Cell Loc.

- Many partial results J.Magnen, V.Rivasseau, R.Sénéor
Some Other Work: SUSY

• Many partial results on d=2 SUSY, but no Wightman theory. O.McBryan, A.Lesniewski, J.Weitsman, A.J., S.Janowsky, J.Imbrie

Non-Commutative Geometry


Algebraic QFT

Many results: S.Doplicher, R.Haag, J.Roberts, R.Longo, D.Buchholz, K.Fredenhagen,....
Some Other Work: Ren. Group


Other Approaches & Partial Results

• Scaling Limit: Exact Solutions  B.McCoy, T.T.Wu (1972), also Russian work.

• Stochastic PDE and Quantization (Regular Structures)  M.Hairer and collaborators (2015-2020)
Some Brief Remarks on Methods

Key Word: Estimates

1. Classical-Quantum Duality from OS

\[ \int_{S'} \psi(\Phi(0)) \psi(\Phi(t)) d\mu = \langle \psi(\varphi), e^{-tH} \psi(\varphi) \rangle_H \]

Classical Probability Theory with RP Property

Quantum Mechanical Hilbert Space

Evolution of Feynman-Kac-Symanzik-Nelson to Osterwalder-Schrader

Allow classical methods to estimate quantum operators.

2. Generalize Perron-Frobenius to Infinite Dimensions

3. Combine operator algebra methods with Hilbert space methods
More Brief Remarks on Methods

4. Localization of Estimates (refined from $d=2$ to $d=3$)
   Localization in phase space (need small position in 3d)
   Limit by uncertainty: borderline in $d=4$
   Phase-cell localization more robust—but less elegant
   than—renormalization group methods.
   Method Inductive, rather than iterative

5. RP & Multiple Reflection Bounds
   for example: $\pm \varphi(f, 0) \leq \|f\| (H + I)$, for $f = \bar{f} \in C_0^\infty$
   for example: Estimate QFT deviation from Peierls’ argument
   for $\varphi^4$ phase transition.
What’s Next?

Today: Most physicists believe that the $\phi^4$ and QED are mathematically inconsistent in $d=4$!

Physicists discovered a potential refinement of Sobolev inequalities for perturbation, in the case that the two sides of the inequalities have the same scaling properties—i.e. in the critical dimension. (2004 Nobel prize)

That is called “asymptotic freedom.” The belief is that asymptotically-free theories may exist (such as non-abelian gauge theory with a simple gauge group), but not $\phi^4$ or QED.
Yang-Mills Theory for $d=4$?

Summer 1966 tour of SLAC main accelerator, just before its opening, during a conference at Stanford University. H. Lehmann, C. N. Yang, W. Panofsky (director), A. Wightman, H. Joos, N. Byers
K. Wilson Action

\[ \mathcal{A} = \frac{1}{g^2} \sum_{\text{plaquettes}} \mathcal{A}(p) \]

Assign a unitary representation of \( g \in \mathcal{G} \) to oriented bonds \( b \), with \( U_{g^{-1}}(b) = U_g(b^{-1}) \).

\[ \mathcal{A}(p) = \text{Tr} \prod_{b \in \partial p} U_g(b) \]

Balaban seminar: Harvard, November 27, 2012

Reflection Positive: K.Osterwalder, E.Seiler (1976)
Study Effective Action \( d=2,3,4 \) T.Balaban (1982-1987) many papers several with co-authors; renormalization group (RG) methods gauge choice dependent on length scale; analyze effective actions BDH developed RG methods
Many partial $d=4$ results in finite volume

And Tomorrow

I believe:
1. The finite-volume SU(2) theory is within human reach.
2. The infinite volume limit will require new physics and new mathematics. See A.J and E.Witten (2001)
A. Weil on Riemann Hypothesis

1. When he was young he wanted to prove the RH.

2. As he got older, he wanted to see the proof.

3. Now he would only like to know that it has been proved.

I feel that way about YM$_4$!!
Some viewers of this meeting will be inspired to construct a mathematical YM-QFT in dimension 4, also with a mass gap! This will be interesting for both mathematics and physics! This probably requires some genius mathematicians, who also know physics!

Good Luck