The Representation Theory of the Clifford Group with Applications in Quantum Information

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Outline

▪ Clifford?

▪ Some representation theory of the Clifford group

▪ Applications

▪ [More representation theory]

Quantum homeopathy works: Efficient unitary designs with a system-size independent number of non-Clifford gates

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Clifford?

Set up dictionary:

**Continuous variables**

- many-body QM
- Gaussian states
- CCR, Weyl operators
- Canonical transformations

**Discrete variables**

- Stabilizer states
- Generalized Pauli operators / Weyl operators
- Clifford group
- Stabilizer codes

“Physics is that subset of human experience which can be reduced to coupled harmonic oscillators”

-- Michael Peskin

“Quantum information is that subset of physics which can be reduced to Clifford actions on stabilizers”

-- David Gross
Continuous variables: Gaussian states

- Hilbert space: $L^2(\mathbb{R}^n)$
- Gaussian wave functions

$$\psi(x) \approx \exp(-x^T C x + i x^T b), \quad \bar{x} \in \mathbb{R}^n, C \in \mathbb{C}^{n \times n}$$

- Ground states of Hamiltonians quadratic in $X_i, P_j$

$$H = \begin{bmatrix} P_1 & \vdots & \vdots & \vdots & P_n \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & X_{11} & \cdots & X_{12n} \\ Q_1 & \cdots & \cdots & \ddots & Q_n \\ \vdots & \vdots & \cdots & \ddots & \vdots \end{bmatrix} \begin{bmatrix} X_{11} & \cdots & X_{12n} \\ \vdots & \ddots & \vdots \\ \vdots & \vdots & X_{2n1} & \cdots & X_{2n2n} \\ Q_1 & \cdots & \cdots & \ddots & Q_n \\ \vdots & \vdots & \cdots & \ddots & \vdots \end{bmatrix} \begin{bmatrix} P_1 \\ \vdots \\ P_n \\ Q_1 \\ \vdots \\ Q_n \end{bmatrix}$$

- Characterized by $(2n)^2$ values of covariance matrix

$$\langle \psi | X_i P_j | \psi \rangle$$
Continuous variables: Symmmetries 1/2

- Position & momentum shifts: Weyl operators

\[ w(p, q) = e^{i(p \hat{q} - q \hat{p})}, \quad (p, q) \in \mathbb{R}^2 \]

- Group law involves *symplectic form*

\[ w(p, q)w(p', q') = w(p + p', q + q')e^{i \pi (p'q' - qp')} \]

- *Canonic transformations*: Unitaries preserving group law

\[ U \ w(p, q) \ U^\dagger = w(S(p, q)) \quad S \in \text{Sp}(\mathbb{R}^{2n}) \]

- ...represent *symplectic group* (up to phases).

- Metaplectic representation, oscillator representation, Bosonic Bogoliubov transformation, canonic transformations...
Continuous variables: Symmmetries 2/2

(Simple) Fact:
Gaussian wave functions form orbit under affine symplectic group $\text{Sp}(2n) \ltimes \mathbb{R}^{2n}$.

\[
\psi(\tilde{x}) \approx \exp(-\tilde{x} C \tilde{x} + i\tilde{x}b), \quad \tilde{x} \in \mathbb{R}^n
\]
Discrete variables: stabilizer states 1 / 2

- **Hilbert space:**
  \[
  \mathbb{C}^d \otimes \cdots \otimes \mathbb{C}^d \simeq L^2(\mathbb{Z}_d^n)
  \]

- **Stabilizer states** include **discrete Gaussians**...
  \[
  \psi(x) \simeq \exp\left(\frac{2\pi i}{d} (-x C x + xb)\right)
  \]

  \[x, b \in \mathbb{Z}_d^n, C \in \mathbb{Z}_d^{n \times n}\]

- ...and a bit more.

[Disclaimer: Will sweep some technicalities in even characteristic ($d = 2$) under the rug]
Symmetries: Discrete

- Position & momentum shifts by \((p, q) \in \mathbb{Z}_d^{2n}\):
  
  \[
  X: |x\rangle \mapsto |x + 1\rangle, \quad Z: |x\rangle \mapsto e^{i\frac{2\pi}{d} x} |x\rangle, \quad W(p, q) = Z^p X^q.
  \]

- Group law involves symplectic form

\[
 w(p, q)w(p', q') = w(p + p', q + q')e^{i \frac{2\pi}{d} (pq' - qp')}
\]

- Clifford transformations: Unitaries preserving group law

\[
 U w(p, q) U^\dagger = w(S(p, q)) \quad S \in \text{Sp}(\mathbb{Z}_d^{2n}).
\]

- Generated by phase gate, Hadamard, controlled-Z

\[
 S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \text{C-Z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]
Discrete variables: stabilizer states 2 / 2

\[ w(p, q)w(p', q') = w(p + p', q + q')e^{i \frac{2\pi}{d} (p'q - qp')} \]

- Weyl operators commute ⇔ symplectic form vanishes \( p q' - q p' = 0 \).

**Definition**
- Subspace \( M \subseteq \mathbb{R}^{2n} \) is *isotropic* if symplectic form vanishes on \( M \)
- Abelian group \( w(M) \) is a stabilizer group
- Joint +1 subspace is stabilizer code
- If \( \dim M = n \), subspace is one-dimensional ⇒ stabilizer state
- Result: Stabilizer states are one affine symplectic orbit; all Gaussians arise this way

\[
\begin{align*}
\chi \otimes \chi (|00\rangle + |111\rangle) & = |00\rangle + |111\rangle \\
\zeta \otimes \zeta (|00\rangle + |111\rangle) & = |00\rangle + |111\rangle
\end{align*}
\]
Application: Quantum Fault Tolerance

- Noise challenges quantum computers

**Fault tolerance:**
- Encode each *logical qubit* into many physical qubits
- ...s.t. few-qubit errors can be detected & corrected

**Fact:**
- all known encodings are stabilizer codes
- all robust gates are Clifford gates
  (all = virtually all 😊)
Discrete variables: stabilizer states $3/2$

$$\psi(x) \approx \exp \left( \frac{2\pi i}{d} (-x \ C \ x + xb) \right)$$

$$\psi(x) \approx \exp (-x \ C \ x + ix \ b)$$

How are covariance matrices related to stabilizer groups?

Stabilizer group = invariant direction of “infinitely squeezed states”.
Clifford?

**Continuous variables**
- many-body QM
  - Gaussian states
  - CCR, Weyl operators
  - Canonical transformations

**Discrete variables**
- Quantum information
  - Stabilizer states
  - Generalized Pauli operators / Weyl operators
  - Clifford group
  - Stabilizer codes

- Rich enough to capture interesting phenomena
- Well-behaved enough to allow explicit analysis
Some representation theory of the Clifford group
Central question

Want to understand:

Representation theory of $t$-fold tensor powers $U^\otimes t$ of Clifford unitaries $U$.

But why should I care?
Quantum Info: Representation theory of $t$-th tensor powers used in, e.g.:

- Randomized benchmarking
- Decoupling technique
- Non-malleable quantum one-time pads
- Variance bounds for randomized benchmarking
- Stabilizer POVM optimal state-independent measurement for pure states
- Simulating quantum circuits via low-rank stabilizer decompositions
- Efficient spherical/unitary designs
- Connected to *discrete Howe-Kashiwara-Vergne duality*. 
Stabilizer testing (Guess whether Gauss)

Guess whether Gauss:
- Given $t$ copies of unknown state $\psi^{\otimes t}$ decide:
  - $\psi$ is a stabilizer state, or
  - $\psi$ is far away from the set of stabilizer states?

Problem [Montanaro, de Wolf]:
- Possible for dimension-independent $t$?
Guess whether Gauss

Ansatz
- Fix some \( t \).
- Let \( P_{\text{stab}} \) project onto span of \( t \)-th tensor power of stabilizer states
- Use POVM \( \{P_{\text{stab}}, I - P_{\text{stab}}\} \)

Rep-theory: \( P_{\text{stab}} \) is in commutant of \( \text{Cliff}^t \).

Negative results: \( P_{\text{stab}} = P_{\text{Sym}} \) for
- \( t = 1 \) [trivial]
- \( t = 2 \) [DiVincenzo, Leung, Terhal 2002]
- \( t = 3 \) [Webb; Zhu; Kueng, DG 2015]
- \( t = 4 \) [Zhu, Grassl, Kueng, DG 2016]
- \( t = 5 \) [Walter, Nezami, DG 2018]

Later: Optimal solution for \( t = 6 \).
Lemon to Lemonade: Stabilizer rank

- [Gottesman-Knill] Cliffords on stabs are efficiently classically simulable
- [Bravyi-Kitaev] Cliffords on non-stabilizers are universal.

- Magic state model:

Q: What’s best exponent $c$ such that one can simulate in $O(2^c t)$ time?
Stabilizer rank

- Expand magic states into superposition of stabilizers...

\[ |T\rangle^{\otimes t} = \sum_{i=1}^{r} c_i |s_i\rangle \]

- ...efficient algorithm if \( r \ll 2^t \).

- Maybe best current classical simulation algorithm.

**Q: Can one bound stabilizer rank?**

[Bravyi, Brown, Calpin, Campbell, Gosset, Howard `18]
Lemon to Lemonade: Stabilizer rank

Theorem [Nezami, Walter, DG 18; Zhu, Grassl, Kueng, DG 16]

- For $t \leq 5$, the powers $|S\rangle^\otimes t$ of stabilizer states span symmetric space $\text{Sym}^t (\mathbb{C}^2)$. 

- For powers of single qubit states:
  
  \[
  \text{stabrank}(|\psi\rangle^\otimes 5) \leq \dim \text{Sym}^5 (\mathbb{C}^2) = 6 \ll 2^5 = 32. 
  \]

- $\Rightarrow$ Best-known general bound on stabilizer rank.
Central question

Want to understand:

Representation theory of $t$-fold tensor powers $U^\otimes t$ of Clifford unitaries $U$.

What would qualify as “understanding”?
Schur-Weyl duality 1

- On $t$-th tensor power $\mathcal{H} \otimes^t$ of a Hilbert space $\mathcal{H}$, commuting actions:

$$U(\mathcal{H}) \ni U \mapsto U \otimes \cdots \otimes U,$$

$$S_t \ni \pi: |\psi_1\rangle \otimes \cdots \otimes |\psi_t\rangle \mapsto |\psi_{\pi_1}\rangle \otimes \cdots \otimes |\psi_{\pi_t}\rangle.$$

- Operator $A$ commutes with $U^\otimes t$ iff

$$A = \sum_{\pi \in S_t} c_\pi \pi$$

and vice versa.

- Under action of $S_t \times U(\mathcal{H})$:

$$\mathcal{H} \otimes^t \simeq \bigoplus_\lambda S_\lambda \otimes U_\lambda$$

- $S_\lambda$ irrep of $S_t$, $U_\lambda$ irrep of $U(\mathcal{H})$.

[Nezami, Walter, DG 18] [Montealegre-Mora, DG 19]
Schur-Weyl duality 2: Transversality

Both algebras

- Generated *uniformly* by *tensor powers*!
- Which are also *unitary*!

⇒ Often, results are *independent of n*!
On $L^2(\mathbb{R}^n)$, what’s commutant of canonical trasfos?

- Smaller group $\Rightarrow$ larger commutant...
- ...turns out $O(t) \supset S_t$ does the job.

**Example ($m \times t = 2$):**

\[
\begin{pmatrix}
\cos \beta & -\sin \beta \\
\sin \beta & \cos \beta
\end{pmatrix}
\begin{pmatrix}
\cos \alpha & -\sin \alpha \\
\sin \alpha & \cos \alpha
\end{pmatrix}
\begin{pmatrix}
\cos \alpha & \sin \alpha \\
-\sin \alpha & \cos \alpha
\end{pmatrix}
\]

**Diagram:**

- $t$ is a line segment with arrows indicating transformations $u$, $\alpha$, $\beta$, and $\gamma$.
- $p_1$, $p_2$, $q_1$, $q_2$ as axes with arrows indicating direction changes.
Central question

Want to understand:

Representation theory of \( t \)-fold tensor powers \( U^\otimes t \) of Clifford unitaries \( U \).

OK, let’s go!
For real, now.

Theorem [Nezami, Walter, DG 18]
The commutant algebra of $t$-th tensor powers of the Clifford group over $d^n$ is generated by $t$-th tensor powers of:

- Discrete orthogonal transformations
- Self-orthogonal CSS code projections
Discrete orthogonal transformations

A $t \times t$ matrix $O$, entries in $\mathbb{Z}_d$, is orthogonal if

$$O^T O = \text{Id} \mod d$$
Discrete orthogonal transformations

Example: Anti-permutations

- Binary complement of permutation matrices
Calderbank-Shor-Steane codes

Let \( N \subset \mathbb{Z}_d^t \) be self-orthogonal:

\[
\sum_i u_i v_i = 0 \mod d, \quad u, v \in N.
\]

\[
\begin{align*}
X^u & := X_{i_1}^{u_1} \otimes \ldots \otimes X_{i_t}^{u_t} \\
\bar{X}^v & := X_{i_1}^{v_1} \otimes \ldots \otimes X_{i_t}^{v_t}
\end{align*}
\]

\[
\Rightarrow \quad X^u \bar{X}^v = \bar{X}^v X^u
\]

A self-orthogonal CSS code is the common eigenspace of these commuting Weyl operators.
Calderbank-Shor-Steane codes

\[ E_X : \]

\[ X \otimes X (|00\rangle + |11\rangle) = |00\rangle + |11\rangle \]

\[ Z \otimes Z (|00\rangle + |11\rangle) = |00\rangle + |11\rangle \]

diagram: toric code (not quite self-dual)
The commutant

**Theorem [Nezami, Walter, DG 18]**
Commutant generated by tensor powers of:

- Finite orthogonal transformations
- Self-orthogonal CSS code projections

- Transversal! 😊
- Product generators not *quite* a group. ⇒ (mild) failure of finite *Howe duality*
Back to applications
Consider stabilizer state $|s\rangle$ on $n$ qudits...

...and its $t$-th tensor power.

Tensor powers of stabilizer states are invariant under the stochastic orthogonal group.

Proof: True for $|s\rangle = 0, \ldots, 0,s\rangle$:

$$|\sigma \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix}\rangle = |\begin{bmatrix} 0 & \cdots & 0 \end{bmatrix}\rangle$$

$$\Rightarrow \sigma \otimes^n |s\rangle \otimes^t = \sigma \otimes^n u \otimes^t |0\rangle \otimes^t$$

$$= u \otimes^t \sigma \otimes^n |0\rangle \otimes^t = |s\rangle \otimes^t$$
Guess whether Gauss

**Thm. [Nezami, Walter, DG 18]**

Let $\psi$ be state on $n$ qubits.

Measure projection on $(+1)$-eigenspace of anti-identity on $\psi \otimes^6$.

If $\psi$ is stabilizer, will accept with $p = 1$.

If

$$\max_S |\langle S | \psi \rangle|^2 \leq 1 - \epsilon,$$

accepts with $p \leq 1 - 4\epsilon$.

(Solves open pro. in q. property testing).
Application: Homeopathy

Quantum homeopathy works: Efficient unitary designs with a system-size independent number of non-Clifford gates

J. Haferkamp,¹ F. Montealegre-Mora,² M. Heinrich,² J. Eisert,¹ D. Gross,² and I. Roth¹
Random walk on groups

Old question:
- Take group $G$ and
- set of generators $\{g_1, \ldots, g_k\}$.

How big does $m$ have to be such that the distribution of random words 
$$g_{i_1} \ldots g_{i_m}$$
approximates Haar measure on $G$?

Famous example [Diaconis]:

To shuffle 52 cards, $m = 7$ necessary and sufficient.
Quantum shuffling

New instance of old question:
- Take $n$ qubits and
- set of gates $\{g_1, \ldots, g_k\}$.

How big does $m$ have to be such that the distribution of circuit $g_{i_1} \ldots g_{i_m}$ approximates Haar measure on $U(2^n)$?
Quantum shuffling: Applications

Exciting applications, like
- Black hole information paradox!

Boring applications, like
- Quantum encryption
Quantum shuffling: Crypto application

Need:
- Short key length $m$
- Easy, fault-tolerant implementation, ideally Clifford-based.
Quantum shuffling: Distance measure

**Definition**
Distribution $\nu$ on $U(2^n)$ is **unitary $t$-design** if

$$\rho \mapsto \int d\nu \, U^\otimes t \rho U^{-\otimes t} = \int d\nu_{\text{Haar}} \, U^\otimes t \rho U^{-\otimes t}$$

It is $\epsilon$-**approximate** $t$-design if the maps are $\epsilon$-close in cb norm.

**Prior results**
- $m = O(n^2 t^{10} \log 1/\epsilon)$ is sufficient
- If one restricts to Clifford gates, can only reach $t=3$
Quantum shuffling: Result

Thm [many authors, 2020]
$O(t^4 \log 1/\epsilon)$ non-Clifford gates are sufficient.

Yepp, density of non-Cliffords $\rightarrow 0$ as $n$ grows.

You see, a homeopathic dose is enough.
Quantum shuffling: Proof ingredients

Proof by blood, toil, tears, and sweat...
(Using uniform description of commutant)
Quantum shuffling: How it *should* look like

Schur-Weyl case [Collins, Sniady]:

$$\sum_{\pi} \pi \tr(\pi^{-1} \rho) \to \int dV_{\text{Haar}} U \otimes t \rho U^{-\otimes t}$$

Want good understanding of:

$$\sum_{E \in O(t),CSS} E \tr(E^{-1} \rho) \to \int dV_{\text{Clifford}} U \otimes t \rho U^{-\otimes t}$$
Summary of Quantum Information results

We have found the commutant of tensor powers of the Clifford group.

- Generated by stochastic orthogonal transformations and CSS code projections

- Many applications: Stabilizer rank, stabilizer testing, unitary designs, ...

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
More representation theory

Howe-Kashiwara-Vergne duality
Howe Duality – Continuous Variables

- Consider *metaplectic* representation
  \[ \mathcal{H} = L^2(\mathbb{R}^n), \; \mu: \text{Sp}(\mathbb{R}^{2n}) \to U(\mathcal{H}) \]

- Tensor power \( \mu \otimes t \ldots \)
  ...commutes with \( O(t) \supset S_t \).

- Under \( O(t) \times \text{Sp}(\mathbb{R}^{2n}) \):
  \[ \mathcal{H}^\otimes t \cong \bigoplus_{\tau} \tau \otimes \Theta(\tau) \]

- \( \tau \) irrep of \( O(t) \), \( \Theta(\tau) \) irrep of \( \text{Sp}(\mathbb{R}^{2n}) \).
Consider *metaplectic* representation
\[ \mathcal{H} = (\mathbb{C}^d)^{\otimes n}, \mu: \text{Sp}(\mathbb{Z}_{d}^{2n}) \to U(\mathcal{H}) \]

- Tensor power \( \mu \otimes t \ldots \)
- \( \ldots \)commutes with \( O(t) \supset S_t \).

- Under \( O(t) \times \text{Sp}(\mathbb{Z}_{d}^{2n}) \):
  \[ \mathcal{H}^{\otimes t} \simeq \bigoplus_{\tau} \tau \otimes \theta(\tau) \]

- \( \tau \) irrep of \( O(t) \), \( \theta(\tau) \) reducible rep of \( \text{Sp}(\mathbb{Z}_{d}^{2n}) \).
Howe Duality – finite (and odd) dimensions

\[ \mathcal{H} \otimes t \cong \bigoplus_{\tau} \tau \otimes \Theta(\tau) \]

- \( \tau \) irrep of \( O(t) \), \( \Theta(\tau) \) reducible.
- Failure of Howe duality over finite fields known since 70s...
- ...building on Nezami-Walter-DG, Gurevich-Howe 2016...
- we can reduce out this space 😊

Theorem [Montealegre, DG 2019]

\[ \mathcal{H} \otimes t \cong \bigoplus_{r} \bigoplus_{\tau} \eta(\tau) \otimes \text{Ind}(\eta(\tau)) \]
Rank of \( \text{Sp}(V) \)-representations

- \( \text{Sp}(V) \) contains a large Abelian subgroup

\[
\begin{bmatrix}
A_1 & A_2 \\
0 & A_1
\end{bmatrix}
\begin{bmatrix}
A_1 & A_2 \\
0 & A_1
\end{bmatrix}
= 
\begin{bmatrix}
A_1 & A_1 + A_2 \\
0 & A_1
\end{bmatrix}
\]

- \( \Rightarrow \) Restriction of any rep \( \pi \) to Abelian group decomposes Hilbert space into 1D irreps:

\[
\pi \left( \begin{bmatrix}
A_1 & A \\
0 & A_1
\end{bmatrix} \right) | \overline{\Phi}_B \rangle = \exp(i \Lambda_{AB}) | \overline{\Phi}_B \rangle
\]

**Def.:** \[ \text{rank } \pi = \max_B \text{rank } B \]

[Gurevich-Howe 2017]
The rank of $\text{Sp}(V)$-representations

**Def.:** \[ \text{rank } \pi = \max_B \text{rank } B \]

**Fact.:** The rank of $\mu^\otimes t$ is $t$.

\[ t = 1: \]

\[ \mu \left( \begin{pmatrix} \frac{1}{d} & A \\ 0 & \frac{1}{d} \end{pmatrix} \right) |x\rangle = \omega \begin{pmatrix} (x, A x) \\ \frac{1}{d} \end{pmatrix} |x\rangle = \omega \text{tr}_B A \begin{pmatrix} x & x^T \end{pmatrix} |x\rangle. \]

[Gurevich-Howe 2017]
The $\eta$-correspondence

Thm [Gurevich-Howe `17]
- $\Theta(\tau)$ contains exactly one rank-\(t\) irrep $\eta(\tau)$.
- The map $\tau \mapsto \eta(\tau)$ is injective.
Where do the rank-deficient reps come from?

Idea: Can one “imbed lower tensor powers into $t$-th tensor power”? 

...that’s what transversal gates on quantum codes do!
Thm [Montealegre-Mora, DG]
Let $N \subset \mathbb{Z}_d^t$ be isotropic, let $C_N$ be the associated CSS code.
- Then $C_N^\otimes t$ is isomorphic to $\mu^\otimes s$, $s = t - 2 \dim N$.
- All rank-deficient subreps arise this way!

\[ X^\otimes t = \bigoplus_{\tau \in \text{irr. } O_t} \mathbb{Z} \otimes \eta(\tau) \bigoplus \bigoplus_{\tau \in \text{irr. } O_{t-2}} \mathbb{Z} \otimes \eta(\tau) \]

$\# N's \quad \otimes \bigoplus \quad \vdots$

irred. and inequ.
Summary: Howe-Kashiwara-Vergne duality

We have looked at the representation theory of tensor powers of the Clifford group.

- Clifford group results motivates to consider action on CSS codes...

- ...transversal action of the Clifford group on these codes explains failure of Howe duality over finite fields. 😊
Thank you!

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With:

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Application: Robust Hudson

Thm. [Nezami, Walter, DG 18]
Pure $\psi$ on $n$ qudits, $d$ odd.

Wigner sum negativity for pure state:

$$sn(\psi) = \sum_{v, W_\psi(v) \leq 0} |W_\psi(v)|.$$  

Then

$$\max_S |\langle \psi | S \rangle|^2 \leq 1 - d^2 \text{sn}(\psi),$$

independent of $n.$
Application: Exponential de Finetti

**Thm.**

Let $\psi \in (\mathbb{C}^{2^n})^{\otimes t}$ be invariant under stochastic orthogonal group.

Let $\rho$ be the reduction to the first $s$ copies.

There is a distribution over stabs s.t.:

\[
\left\| \mathbf{g} - \sum_{S} \mathbf{g} \otimes^s \rho(S) \right\|_{tr} \leq \exp \left( n^2 - (t-s) \right)
\]

Finite analogue of [Leverrier 2017]