Mikhlin type Fourier multipliers on free groups and free products of von Neumann algebras

Quanhua Xu

Université de Franche-Comté

and

Harbin Institute of Technology

Harvard University – 17 November 2020

Based on joint work with
Tao Mei and Eric Ricard

Group von Neumann algebras

Let Γ be a discrete group. The left regular representation λ of Γ is defined as: for $g \in \Gamma$,

$$\lambda(g) : \ell_2(\Gamma) \to \ell_2(\Gamma), \quad \delta_h \mapsto \delta_{gh}, \forall h \in \Gamma,$$

where δ_h is the Dirac mass at h. The group von Neumann algebra is (in quantum group notation)

$$\widehat{\Gamma} = \lambda(\Gamma)'' \subset B(\ell_2(\Gamma)).$$

$\widehat{\Gamma}$ is also the w*-closure of the algebras of polynomials

$$\mathbb{C}[\Gamma] = \left\{ \sum \alpha(g)\lambda(g) : \alpha(g) \in \mathbb{C} \right\}.$$

$\widehat{\Gamma}$ is equipped with a trace: $\tau(x) = \langle \delta_e, x\delta_e \rangle$ (e = identity of Γ). For $1 \leq p < \infty$ the noncommutative $L_p(\widehat{\Gamma})$ is the completion of $\widehat{\Gamma}$ relativet to the norm: $\|x\|_p = \left(\tau(|x|^p)\right)^{1/p}$. Put $L_\infty(\widehat{\Gamma}) = \widehat{\Gamma}$.

Fourier multipliers

Given $\varphi : \Gamma \to \mathbb{C}$ define the associated multiplier as

$$M_\varphi : \mathbb{C}[\Gamma] \to \mathbb{C}[\Gamma], \quad \sum_g \alpha(g)\lambda(g) \mapsto \sum_g \varphi(g)\alpha(g)\lambda(g).$$

Definition. M_φ is an L_p multiplier if it extends to a bounded map on $L_p(\widehat{\Gamma})$. Similarly, M_φ is a cb L_p multiplier if it extends to a completely bounded map on $L_p(\widehat{\Gamma})$.

In this talk, we will consider only cb multipliers, Recall that $M_\varphi : L_p(\widehat{\Gamma}) \to L_p(\widehat{\Gamma})$ is cb if

$$\mathrm{Id}_{S_p} \otimes M_\varphi : L_p(B(\ell_2) \otimes \widehat{\Gamma}) \to L_p(B(\ell_2) \otimes \widehat{\Gamma}) \text{ is bounded.}$$

The cb norm $\|M_\varphi\|_{\mathrm{cb}}$ is the norm of the above map. Let $\mathrm{M_{cb}}(L_p(\widehat{\Gamma}))$ denote the space of cb L_p multipliers.

Aim. Find criteria on φ so that $\varphi \in \mathrm{M_{cb}}(L_p(\widehat{\Gamma}))$.
Very hard task in general!

Basic properties

Before proceeding further, we give some basic properties of Fourier multipliers.
Let $1 \leq p \leq \infty$ and p' be the conjugate index of p. Then

- $\mathrm{M_{cb}}(L_2(\widehat{\Gamma})) = \ell_\infty(\Gamma)$;
- $\varphi \in \mathrm{M_{cb}}(L_p(\widehat{\Gamma})) \Leftrightarrow \varphi \in \mathrm{M_{cb}}(L_{p'}(\widehat{\Gamma}))$;
- $2 \leq p < q \leq \infty \Rightarrow \mathrm{M_{cb}}(L_q(\widehat{\Gamma})) \subset \mathrm{M_{cb}}(L_p(\widehat{\Gamma}))$ contractively.

Thus we need only to consider the case $2 < p \leq \infty$.

Known results: abelian groups

Abelian case. Assume Γ abelian. Then $\widehat{\Gamma}$ is the dual group of Γ and $L_p(\widehat{\Gamma})$ is the usual L_p-space on $\widehat{\Gamma}$. We have

$$\mathrm{M_{cb}}(L_\infty(\widehat{\Gamma})) = \{\widehat{\mu} : \mu \text{ bounded Borel measure on } \widehat{\Gamma}\}.$$

Moreover, bounded L_∞ multipliers are automatically cb.
In general, no nice criterion for L_p multipliers for $2 < p < \infty$.

Mikhlin-Hörmander theorem. Let $\Gamma = \mathbb{Z}^d$. Then $\widehat{\Gamma} = \mathbb{T}^d$, the d-torus. Assume $\varphi : \mathbb{Z}^d \to \mathbb{C}$ satisfies

$$(*) \qquad |n|^{|\alpha|}|\partial^\alpha \varphi(n)| \leq C, \quad \forall\, n \in \mathbb{Z}^d,\ \forall\, \alpha \in \mathbb{N}^d,\ |\alpha| \leq [d/2] + 1,$$

where $|n| = |n_1| + \cdots + |n_d|$ for $n = (n_1, \cdots, n_d) \in \mathbb{Z}^d$, and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$, partial derivations.
Then φ is an L_p multiplier for every $1 < p < \infty$.
Moreover, if $(*)$ holds for all $\alpha \in \{0, 1\}^d$, then φ is cb (Bourgain, Burkholder, McConnell, Zimmermann).

Examples of Mikhlin-Hörmander multipliers

Below are two fundamental examples.

- **Riesz transforms.**

$$\varphi_j(n) = \frac{n_j}{|n|}, \quad n = (n_1, \cdots, n_d) \in \mathbb{Z}^d,\ 1 \leq j \leq d.$$

  If $d = 1$, we have the Hilbert transform on the unit cercle.

- **Littlewood-Paley decomposition.** Let

$$\Delta_k = \{n \in \mathbb{Z}^d : 2^k \leq |n| < 2^{k+1}\}, \quad k \in \mathbb{N}$$

  and

$$\varphi(n) = \sum_{k=0}^{\infty} \varepsilon_k \chi_{\Delta_k} \text{ with } \varepsilon_k = \pm 1.$$

  Together with the Khintchine inequality, this multiplier implies the famous Littlewood-Paley decomposition.

Known results: Free groups

Let $\mathbb{F}_\infty$ be a free group on infinite generators (g_k). Every $h \in \mathbb{F}_\infty \setminus \{e\}$ is written in the reduced form

$$h = g_{i_1}^{k_1} \cdots g_{i_n}^{k_n}, \quad i_1 \neq i_2 \neq \cdots \neq i_n,\ k_j \in \mathbb{Z} \setminus \{0\}.$$

The natural length is $|h| = |k_1| + \cdots + |k_n|$.
Radial multipliers (Haagerup-Steenstrup-Szwarc, 2010). Let $\varphi : \mathbb{F}_\infty \to \mathbb{C}$ be given by $\varphi(h) = \psi(|h|)$ for some $\psi : \mathbb{N} \to \mathbb{C}$. Then

$$\varphi \in \mathrm{M_{cb}}(\widehat{\mathbb{F}}_\infty) \Leftrightarrow \big(\psi(i+j) - \psi(i+j+2)\big)_{i,j\geq 0} \text{ is of trace class.}$$

Ozawa extends this result to hyperbolic groups (or graphs).
Free Hilbert transform (Mei-Ricard, 2017). Let $\varepsilon = (\varepsilon_{\pm k})_{k\geq 1}$ with $\varepsilon_{\pm k} = \pm 1$. Define

$$H_\varepsilon(\lambda(h)) = \varepsilon_{\mathrm{sign}(k_1) i_1}\, \lambda(h) \ \text{ for } \ h = g_{i_1}^{k_1} \cdots g_{i_n}^{k_n} \ \text{ as above.}$$

Then H_ε is cb on $L_p(\widehat{\mathbb{F}}_\infty)$ for every $1 < p < \infty$.

Free multipliers on the first letters

Given $d \geq 1$ define an action of $\mathbb{T}^d$ on $\mathbb{C}[\mathbb{F}_\infty]$ as follows:

$$\pi_z^{(d)}\big(\lambda(h)\big) = z_1^{k_1} \cdots z_d^{k_d} \lambda(h), \; h = g_{i_1}^{k_1} \cdots g_{i_n}^{k_n} \; \text{ in reduced form}$$

for any $z = (z_1, \cdots, z_d) \in \mathbb{T}^d$. If $n < d$, k_d is interpreted as zero.

Theorem 1. $\pi^{(d)}$ extends to a cb action on $L_p(\widehat{\mathbb{F}}_\infty)$ for $1 < p < \infty$.

By transference, we immediately get the following

Corollary 2. Assume $\varphi : \mathbb{Z}^d \to \mathbb{C}$ satisfies

$$|k|^{|\alpha|} |\partial^\alpha \varphi(k)| \leq C, \quad \forall\, k \in \mathbb{Z}^d, \; \forall\, \alpha \in \{0, 1\}^d .$$

Define

$$M_\varphi\big(\lambda(h)\big) = \varphi(k_1, \cdots, k_d) \lambda(h) \; \text{ for } \; h = g_{i_1}^{k_1} \cdots g_{i_n}^{k_n} .$$

Then M_φ is cb on $L_p(\widehat{\mathbb{F}}_\infty)$ for any $1 < p < \infty$.

Free product of groups

Let $\Gamma_\infty = \Gamma^{*\infty}$ be the infinite free power of Γ (Note: $\mathbb{F}_\infty = \mathbb{Z}^{*\infty}$).
Define $\pi^{(d)} : \mathbb{C}[\Gamma_\infty] \to \mathbb{C}[\Gamma^d] \otimes \mathbb{C}[\Gamma_\infty]$ by

$$\pi^{(d)}\big(\lambda(h)\big) = \big(\lambda(\gamma_{i_1}) \otimes \cdots \otimes \lambda(\gamma_{i_d})\big) \otimes \lambda(h)$$

for $h = \gamma_{i_1} \cdots \gamma_{i_n} \in \Gamma_\infty$ in reduced form.
Theorem 3. $\pi^{(d)}$ extends to a cb map from $L_p(\widehat{\Gamma_\infty})$ to $L_p(\widehat{\Gamma^d} \otimes \widehat{\Gamma_\infty})$ for $1 < p < \infty$.

Corollary 4. Assume in addition that Γ is hyperlinear. Given $\varphi \in \mathrm{M_{cb}}(L_p(\widehat{\Gamma^d}))$, define

$$M_\varphi\big(\lambda(h)\big) = \varphi(\gamma_{i_1}, \cdots, \gamma_{i_d})\lambda(h)$$

for $h = \gamma_{i_1} \cdots \gamma_{i_n} \in \Gamma_\infty$ in reduced form. Then M_φ is cb on $L_p(\widehat{\Gamma_\infty})$ for any $1 < p < \infty$.
Remark. The hyperlinearity is to guarantee that $\widehat{\Gamma_\infty}$ is QWEP in order to use transference (QWEP=quotient of a WEP algebra).

Free product of von Neumann algebras

The previous results extend to the setting of free products of VNAs. In fact, it is easier to prove them in the latter setting.

Let $(\mathcal{A}_k, \phi_k)_k$ be a family of VNAs with normal faithful states. Let $\mathring{\mathcal{A}}_k = \{\mathring{x} = x - \phi_k(x) : x \in \mathcal{A}_k\}$. Then

$$\mathcal{W} = \mathbb{C} \bigoplus_{n \geqslant 1} \bigoplus_{\substack{(i_1, \cdots, i_n) \in \mathbb{N}^n \\ i_1 \neq i_2 \cdots \neq i_n}} \mathring{\mathcal{A}}_{i_1} \otimes \cdots \otimes \mathring{\mathcal{A}}_{i_n} = \bigoplus_{n \geqslant 0} \mathcal{W}_n$$

is a $*$-algebra. The ϕ_k's determine a state ϕ on $\mathcal{W}$. The free product $(\mathcal{A}, \phi)$ is the VNA obtained by the GNS construction from $(\mathcal{W}, \phi)$. $\mathcal{W}$ is w*-dense in $\mathcal{A}$ and $\phi\big|_{\mathcal{A}_k} = \phi_k$.

Note that $\mathcal{W}_n$ is the space of homogeneous polynomials of degree n. Let P_n be the projection onto $\mathcal{W}_n$ and

$$P_{\geq n} = \mathrm{Id} - (P_0 + \cdots + P_{n-1}).$$

Central theorem

Hypothesis. Fix $2 < p < \infty$. Assume that $T_k : \mathcal{A}_k \to \mathcal{A}_k$ is a unital state preserving map that extends to cb map both on $L_2(\mathcal{A}_k)$ and $L_p(\mathcal{A}_k)$ such that

$$C_q = \sup_k \| T_k \|_{cb(L_q(\mathcal{A}_k))} < \infty, \quad q = 2,\ p.$$

Given an integer d define a map $T^{(d)}$ on $\mathcal{W}$ by

$$T^{(d)}(x) = \begin{cases} a_1 \otimes \dots a_{d-1} \otimes T_{i_d}(a_d) \otimes a_{d+1} \otimes \dots a_n & \text{if} \quad d \leqslant n. \\ a_1 \otimes \dots \otimes a_n & \text{if} \quad d > n. \end{cases}$$

for $x = a_1 \otimes \dots \otimes a_n \in \mathring{\mathcal{A}}_{i_1} \otimes \dots \otimes \mathring{\mathcal{A}}_{i_n}$ with $i_1 \neq \cdots \neq i_n$.

Theorem 5. $T^{(d)}$ extends to a cb map on $L_p(\mathcal{A})$ with

$$\| T^{(d)} \|_{cb(L_p(\mathcal{A}))} \lesssim_p (C_2 + C_p).$$

Remark. We need only to assume that T_k maps $\mathcal{A}_k$ into $L_p(\mathcal{A}_k)$.

1^{st} ingredient of proof: Cotlar type formula

Special case: $d = 1$ and all T_k's are $*$-homomorphisms. Let

$$T = T^{(1)} \quad \text{and} \quad T^{\mathrm{op}}(x) = T(x^*)^* .$$

We need some paraproducts on $\mathcal{W} \times \mathcal{W}$:

$$x \dagger^{0,1} y = xy - \mathbb{E}_\varepsilon H_\varepsilon(H_\varepsilon(x) y), \quad x \dagger^{1,0} y = \left(y^* \dagger^{0,1} x^*\right)^*$$
$$x \dagger^{1,1} y = xy - x \dagger^{0,1} y - x \dagger^{1,0} y.$$

Here $H_\varepsilon = T^{(1)}$ for $T_k = \varepsilon_k \mathrm{Id}_{\mathcal{A}_k}$, and $\mathbb{E}_\varepsilon$ is the expectation on ε with ε_k's symmetric and independent random signs.

Remark. For elementary tensors, $x \dagger^{0,1} y$ collects in xy all reduced words that don't start in the same algebra as x.

Cotlar type formula. For any $x, y \in \mathcal{W}$

$$P_{\geq 2}[T(x) T^{\mathrm{op}}(y)] = P_{\geq 2}\Big[T\big(x T^{\mathrm{op}}(y)\big) + T^{\mathrm{op}}\big(T(x) y\big) - T\, T^{\mathrm{op}}(xy)\Big]$$
$$P_1[T(x) T^{\mathrm{op}}(y)] = P_1\Big[T\big(x \dagger^{1,0} T^{\mathrm{op}}(y)\big) + T^{\mathrm{op}}\big(T(x) \dagger^{0,1} y\big) + T\big(x \dagger^{1,1} y\big)\Big].$$

Outline of proof

Below are the main steps of the proof of the central theorem in the case where $d = 1$ and all T_k's are $*$-homomorphisms.
$T = T^{(1)}$ and $T^{\mathrm{op}}(x) = T(x^*)^*$.
Step 1. For any $x \in \mathcal{W}$, write

$$T(x)T(x)^* = T(x)T^{\mathrm{op}}(x^*) = \phi\big(T(x)T^{\mathrm{op}}(x^*)\big) + P_1\big(T(x)T^{\mathrm{op}}(x^*)\big) + P_{\geqslant 2}\big(T(x)T^{\mathrm{op}}(x^*)\big).$$

Step 2. Using the Cotlar formula and boundedness of the paraproducts from $L_{2p} \times L_{2p}$ to L_p (Mei-Ricard), show

$$T \text{ bounded on } L_p \Rightarrow T \text{ bounded on } L_{2p}\,.$$

Step 3. Starting with $p = 2$, iteration shows T is bounded on L_p for $p = 2^n$ with $n \in \mathbb{N}$.
Step 4. Complex interpolation allows us to conclude the proof.

2^{nd} ingredient of proof: length reduction formula

We need a length reduction formula for the proof of the central theorem for $d > 1$ and general T_k's. An element $x \in \mathcal{W}$ can be written as

$$x = x_0 + x_1 + \sum_{i,\alpha} a_i(\alpha) \otimes b_i(\alpha),$$

where $x_0 \in \mathbb{C}$, $x_1 \in \mathcal{W}_1$, $a_i(\alpha) \in \mathring{\mathcal{A}}_i$ and $b_i(\alpha) \in \mathring{\mathcal{W}}$ not starting with letters from $\mathring{\mathcal{A}}_i$.

Length reduction formula. For $2 < p < \infty$, we have

$$\begin{aligned}\|x\|_p \approx_p |x_0| + \|x_1\|_p + \Big\| \sum_{i,\alpha} a_i(\alpha) \otimes b_i(\alpha) \Big\|_{\mathcal{W}_1^c \otimes_p L_p(\mathcal{A})} \\ + \Big\| \sum_{i,\alpha} a_i(\alpha) \otimes b_i(\alpha) \Big\|_{L_p(\mathcal{A}) \otimes_p \mathring{\mathcal{W}}^r}.\end{aligned}$$

Here we use the usual notation of column and row norms.

Remark. This formula extends Junge-Parcet-Xu's previous for homogeneous polynomials.