Harvard Mathematical Picture Language Seminar

Is any compact Lie group uniformly doubling?

L. Saloff-Coste

Cornell University

Doubling refers to the property that the volume of a ball of radius $2r$ is bounded above by a constant times the volume of the concentric ball of radius r. The multiplicative constant is 2^n in the Euclidean space of dimension n. This property does not hold in hyperbolic space.

1 An introduction to Geometric Analysis

Let (M, g) be a compact Riemannian manifold. Let Δ be the (positive) Laplacian on M. Note that M is also equipped with a (geodesic) distance function and a measure μ under which

$$\int_M u \Delta v d\mu = \int g(\nabla u, \nabla v) d\mu$$

(again, the Laplacian is *positive*). Consider the following objects:

- The diameter, d, and volume function $V(x, r)$, $x \in M, r > 0$.
- The spectrum $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \dots$
- The heat kernel $h(t, x, y)$, a positive function of $t > 0$ and $x, y \in M$ which solves the heat equation

  $$\partial_t u + \Delta u = 0$$

  in (t, y) with initial condition $u(0, y) = \delta_x$.

Loosely speaking, the purpose of Geometric Analysis is to understand these objects (among many others) and their relationships to the geometry of M.

Questions:

1. Is true that $\quad V(x, 2r) \leq CV(x, r)$? (this is doubling)
2. Is it true that $\frac{a}{d^2} \leq \lambda_1 \leq \frac{A}{d^2}$?
3. Is it true that

   $$0 < c \int_M \frac{d\mu(x)}{V(x, \sqrt{1/s})} \leq \#\{i : \lambda_i < s\} \leq C \int_M \frac{d\mu(x)}{V(x, \sqrt{1/s})}?$$

4. Is it true that, for $x, y \in M$ and $t > 0$,

   $$\frac{c_1}{V(x, \sqrt{t})} e^{-c_2 d(x,y)^2/t} \leq h(t, x, y) \leq \frac{c_3}{V(x, \sqrt{t})} e^{-c_4 d(x,y)^2/t}?$$

If we ask these questions for a single (M, g), the answer is YES; and only the last two have real content.

Here are three known theorems (the first two are folklore whereas the last one is due to Peter Li and S-T. Yau. None of them is immediately obvious).

Theorem 1.1 (Folklore)**.** *Fix n. For bounded convex domains in $\mathbb{R}^n$, the answers to the four questions above is YES, uniformly.*

Theorem 1.2 (Folklore)**.** *Fix n. For flat tori of dimension n, the answers to the four questions above is YES, uniformly.*

Theorem 1.3 (P. Li and S-T. Yau)**.** *Fix n. For compact Riemannian manifolds of dimension n with non-negative Ricci curvature, the answers to the four questions above is YES, uniformly.*

The key point in the case of bounded convex domains is:

Theorem 1.4 (Folklore)**.** *Fix n. For any convex domain Ω in $\mathbb{R}^n$ (n is fixed),*

$$\frac{\mathrm{Vol}(B_x^\Omega(2r))}{\mathrm{Vol}(B_x^\Omega(r))} \leq 2^n.$$

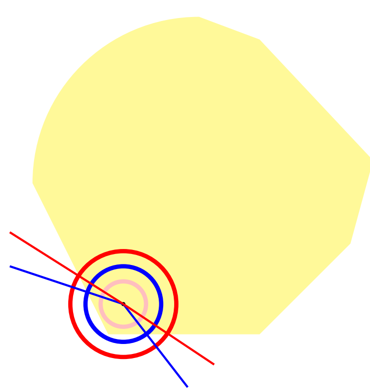


Figure 1: Volume comparison in a convex set

2 Compact Lie groups

There is a structure theorem that describes compact connected Lie groups. If K is such a group, there is a finite cover $\widetilde{K}$ of K (i.e., $K = \widetilde{K}/F$ where F is a finite group which must in fact lie in the center of $\widetilde{K}$) such that

$$\widetilde{K} = \Sigma_1 \times \cdots \times \Sigma_k \times \mathbf{T}$$

where $\mathbf{T}$ is a finite dimensional torus and

$$\Sigma_1, \dots, \Sigma_k$$

are compact, connected, simply connected, simple Lie groups. In particular, each of the groups Σ_i is of one of the types

A_ℓ	B_ℓ	C_ℓ	D_ℓ	G_2	F_4	E_6	E_7	E_8
$SU(\ell+1)$	$SO(2\ell+1)$	$Sp(\ell)$	$SO(2\ell)$	G_2	F_4	E_6	E_7	E_8

For any compact Lie group K, let $\mathfrak{L}(K)$ be *the set of all left-invariant Riemannian metrics* on K.

The natural measure associated with any such metric is the Haar measure (or, more precisely, a constant multiple of it).

3 The conjecture

Conjecture 1. Fix a compact connected Lie group K. The answers to the four questions above is YES, uniformly over $\mathfrak{L}(K)$.

For instance, if true, this conjecture would imply that there exist constants $0 < c(K), C(K) < \infty$ such that, for any metric g in $\mathfrak{L}(K)$,

$$c(K)\frac{\mu_g(K)}{V_g(1/\sqrt{s})} \leq W_g(s) = \#\{i : \lambda_{g,i} < s\} \leq C(K)\frac{\mu_g(K)}{V_g(1/\sqrt{s})}.$$

Proposition 3.1. Conjecture 1 *holds true on a given compact connected Lie group K whenever*

$$\sup_{g\in\mathfrak{L}(K)} \sup_{r>0} \frac{V_g(2r)}{V_g(r)} = D(K) < \infty.$$

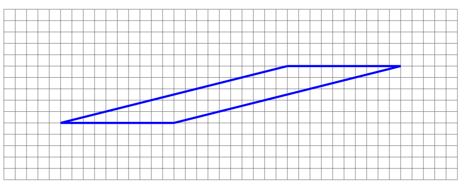
In words, the problem raised by Conjecture 1 reduces to the following.

Main conjecture

Fix a compact connected Lie group K. The exists a constant $D(K)$ such that, for any metric $g \in \mathfrak{L}(K)$ and all $r > 0$,

$$\frac{V_g(2r)}{V_g(r)} \leq D(K).$$

Tori?



Curvature?

Sub-Riemannian geometries?

Degenerated geometries?

Reduction: Proof of Proposition 1

Fix a metric $g \in \mathfrak{L}(K)$. For any smooth function f on K, any x, y in K, we can write

$$|f(xy) - f(x)| \leq \int_0^T |\nabla f(x\gamma_y(t))| dt$$

where $T = d_g(e, y)$ and $\gamma : [0, T] \longrightarrow K$ is a geodesic parametrized by arc length joining e to y in K. Square this inequality, use the Cauchy-Schwarz inequality, and integrate over all x, then all y, to get

$$2 \int_K |f - f_K|^2 d\mu \leq d^2 \int_K |\nabla f|^2 d\mu.$$

Here, f_K is the mean of f over K and this inequality is equivalent to the spectral gap estimate

$$\lambda_1 \geq 2/d^2$$

which thus holds for any left-invariant metric on any compact connected group K (d is the diameter of K under the considered metric).

Instead of integrating

$$|f(xy) - f(x)| \leq \int_0^T |\nabla f(x\gamma_y(t))| dt$$

over all x and all y, integrate over all $x, y \in B$ where B is a metric ball or radius r. Doing this carefully yields, for any $f \in \mathcal{C}^\infty(B(2r))$,

$$\int_B |f - f_B|^2 d\mu \leq \frac{V(2r)}{V(r)} r^2 \int_{2B} |\nabla f|^2 d\mu.$$

This inequality is a version of the Poincaré inequality on metric balls discussed earlier. Note that expression in front of the integral on the right-hand side depends on

$$V(2r)/V(r)$$

where $V(r) = V_g(r)$ denotes the volume of the metric ball of radius r for the given metric g. Because of left-invariance, we only have to consider the balls centered at the identity element $e \in K$.

4 The case of $SU(2)$

Theorem [Eldredge, Gordina, SC]: The case of $SU(2)$

Left-invariant metrics on $SU(2)$ are uniformly doubling. Consequently, Conjecture 1 holds true on $SU(2)$.

So, in particular, there are constants $0 < c \leq C < \infty$ such that Weyl spectral function is bounded by

$$c\,\frac{\mu_g(SU(2))}{V_g(1/\sqrt{s})} \leq \#\{i : \lambda_i < s\} \leq C\,\frac{\mu_g(SU(2))}{V_g(1/\sqrt{s})}.$$

The theorem is proved by estimating the volume function of any metric $g \in \mathfrak{L}(K)$. This is made easier by the following result.

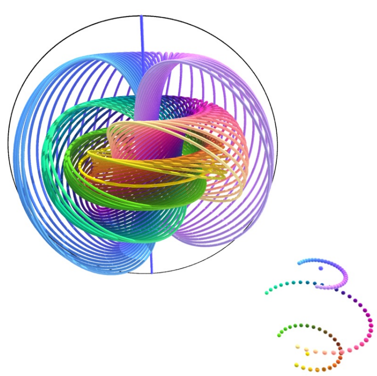
Definition 4.1. A Milnor basis for $SU(2)$ is a basis e_1, e_2, e_3 of $\mathfrak{s}u(2)$ such that

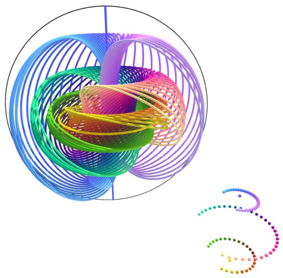
$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2.$$

For instance, the Pauli matrices

$$\frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \frac{1}{2}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad \frac{1}{2}\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

form a Milnor basis.





Lemma: Reduced parametrization of $\mathfrak{L}(G)$ by $0 < a_1 \le a_2 \le a_3$.

For any $g \in \mathfrak{L}(SU(2))$ there is a Milnor basis that is an orthogonal basis for g with $a_1 = \sqrt{g(e_1,e_1)} \le a_2 = \sqrt{g(e_2,e_2)} \le a_3 = \sqrt{g(e_3,e_3)}$.

Theorem (N. Eldredge, M. Gordina, S.C.): Volume estimate, uniform over all $0 < a_1 \le a_2 \le a_3$. Uniform doubling follows.

For any $g \in \mathfrak{L}(SU(2))$ with parameters $a_1 \le a_2 \le a_3$ as above, we have (uniformly over $\mathfrak{L}(SU(2))$)

$$\text{Diameter estimate:} \quad d_g(SU(2)) \simeq a_2$$

and

$$\text{Volume estimate:} \quad \forall\, r > 0, \;\; V_g(r) \simeq \overline{V}_g(r)$$

where

$$\overline{V}_g(r) = \begin{cases} r^3 & \text{if } 0 \le r \le a_1a_2/a_3 \\ (a_3/a_1a_2)r^4 & \text{if } a_1a_2/a_3 \le r \le a_1 \\ (a_1a_3/a_2)r^2 & \text{if } a_1 \le r \le a_2 \\ a_1a_2a_3 & \text{if } a_2 \le r < \infty. \end{cases}$$

For the Weyl spectral function $W_g(s) = \{i : \lambda_i < s\}$, this translates into

$$W_g(s) \simeq \begin{cases} 1 & \text{if } 0 < s \le 1/a_2^2 \\ a_2^2 s & \text{if } 1/a_2^2 \le s \le 1/a_1^2 \\ a_1^2a_2^2s^2 & \text{if } 1/a_1^2 \le s \le a_3^2/(a_1^2a_2^2) \\ a_1a_2a_3s^{3/2} & \text{if } a_3^2/(a_1^2a_2^2) \le s < \infty. \end{cases}$$

$$V_g(r) \simeq \begin{cases} r^3 & \text{if } 0 \le r \le a_1a_2/a_3 \\ (a_3/a_1a_2)r^4 & \text{if } a_1a_2/a_3 \le r \le a_1 \\ (a_1a_3/a_2)r^2 & \text{if } a_1 \le r \le a_2 \\ a_1a_2a_3 & \text{if } a_2 \le r < \infty. \end{cases}$$

Tools?

Campbell-Baker-Hausdorff-Dynkin formula and its long product version.
Jacobian estimates.

For a path $\gamma : [0,1] \to SU(2)$ with $\gamma(0) = e$, $\gamma(1) = g = \exp(z)$ with

$$\dot{\gamma}(t) = \sum_{i=1}^{3} \lambda_i(t) \widetilde{e_i}(\gamma(t)),$$

we have

$$z = \sum_{n=1}^{\infty} \sum_{I \in \{1,2,3\}^n} \left(\sum_{\sigma \in S_n} \left(\frac{(-1)^{e(\sigma)}}{n^2 \binom{n-1}{e(\sigma)}} \right) \int_{\Delta^n} \prod_{m=1}^{n} \lambda_{i_m}(s_{\sigma(m)}) ds \right) e_I$$

where $I = (i_1, \dots, i_n)$, and e_I is the n-fold iterated bracket $e_I = [[\dots[e_{i_1}, e_{i_2}], \dots], e_{i_n}]$.

Heuristics? (Ball/box argument)

Euclidean/sub-Riemannian/ collapsed geometries.

5 Coming soon

The case of $U(2) = SU(2) \times \mathbb{T}$ (also, $SU(2) \times \mathbb{T}^n$)

There is a constant D such that for any $g \in \mathfrak{L}(U(2))$, are uniformly doubling. Consequently, Conjecture 1 holds true on $SU(2)$.

Same strategy but several new difficulties arise. More parameters and more zones $(r, r^2, r^3, r^4, r^5, r^6, r^7)$ zones appear due to the role of higher brackets, that is, more complex sub-Riemannian geometries). Lack of "universal" orthogonality leads to dealing with parallelogram shaped boxes instead of rectangular boxes.

$SU(2) \times SU(2)$? How to treat the general case?

References:

Torus picture: Plus magazine (Andrea Gambassi and Corinna Ulcigrai.)

Hopf fibration picture: Niles Johnson The Ohio State University, Newark. `https://nilesjohnson.net/hopf.html`

Hopf fibration video: Collectif Henri Paul de Saint Gervais (Analysis situs) `https://www.youtube.com/watch?v=CxTWEM6RnjA`

Judge, Chris; Lyons, Russell Upper bounds for the spectral function on homogeneous spaces via volume growth. Rev. Mat. Iberoam. 35 (2019), no. 6, 1835–1858.

Lauret, Emilio A. On the smallest Laplace eigenvalue for naturally reductive metrics on compact simple Lie groups. Proc. Amer. Math. Soc. 148 (2020), no. 8, 3375–3380.

Gordina, M., Eldredge, N., SC: Left-invariant geometries on SU(2) are uniformly doubling. Geom. Funct. Anal. 28 (2018), no. 5, 1321–1367.

Thank you!