K-theory of Operator Algebras, Orbifolds, and Conformal Field Theory

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dedicated to the memory of Vaughan Jones

CARDIFF UNIVERSITY
PRIFYSGOL CAERDYDD

Overview

Verlinde ring

- as representation theory of loop group
- construction of CFT from modular tensor category

- K-theory and higher equivariant twisted K-theory
- subfactors and conformal nets

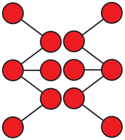
- orbifolds

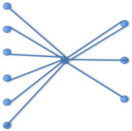
subfactors

$N^G \subset N$; G group $\quad \rightsquigarrow \quad N \subset M$

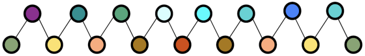
$\rho : M \to N, \quad \bar{\rho} : N \to M \qquad \rho\bar{\rho} \succeq id_N$

ρ generates N-N, N-M, M-N, M-M sectors via $\rho\bar{\rho}$, $\rho\bar{\rho}\rho$, $\bar{\rho}\rho\bar{\rho}$ etc

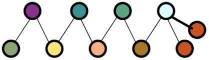




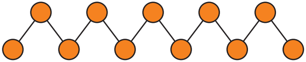
A series for SU(2)

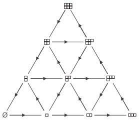


orbifold of A_{4n+1} $N^{\mathbb{Z}_2} \subset M^{\mathbb{Z}_2}$ is D_{2n}



WZW loop group



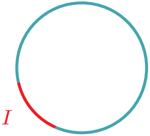


$$\mathrm{Ver}_k(SU(2)) \cong \mathbb{Z}[\rho]/\langle S_{k+1}\rangle$$

- $\pi_\lambda(L_I SU(n))'' \subset \pi_\lambda(L_{I'} SU(n))'$
  $\lambda = 0$: $N = N$ $\lambda N \subset N$ Wassermann

  λ endomorphism, N type III_1 factor: $\lambda\mu = \sum_\nu N^\nu_{\lambda\mu}\nu$

representations of conformal net of factors $N(I)$ $I \subset S^1$



braided system of endomorphisms $Rep(N)$

CFT - the search for the exotic

Verlinde ring as representation theory of loop group at level k

modules or representations of

- vertex operator algebra
- conformal nets of von Neumann algebras

$Rep(G)$ or $Rep(LG)_k$ is a category

tensor $\lambda \otimes \mu$ trivial object id

morphism $\lambda \otimes \mu \simeq \mu \otimes \lambda$

tensor category $\mathcal{N} = Rep(\Phi)$ where $\Phi =$???

CFT - the search for the exotic

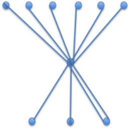
Beyond 4: Haagerup subfactor at index $(5 + \sqrt{13})/2$

quantum double of finite group, Haagerup subfactor etc

$\mathcal{X}$ on N : $\exists A \subset N \otimes N^{opp}$, $\iota\bar{\iota} = \Sigma_{\nu \in \mathcal{X}} \nu \otimes \nu^{opp}$

A-A is MTC — Ocneanu, Longo-Rehren, Izumi, Popa

CFT - the search for the exotic



$\mu^2 = \sum_g g$
$\mu g = \mu$
Tambara-Yamagami

Orbifold
$g \leftrightarrow -g$

Verlinde Ring and K-theory

WZW Verlinde ring as ${}^{\text{twist}}K^G(G)$ Freed-Hopkins-Teleman

G a finite group, $\mathcal{D}(G)$ is q double:

Verlinde ring $K_G(G) \simeq Rep(\mathcal{D}(G))\,,$ G on G by conjugation

G-graded vector spaces $Vect_G \rightsquigarrow K(G) = \mathbb{Z}^G$

$\mathrm{K}_G(\bullet) = Rep(G)$

q double of $N^G \subset N$ is $(N \otimes N)^{G\times G} \subset (N \otimes N)^G\,,$ $G \subset G \times G$

$$G \nearrow G \times G \nwarrow G$$

$G \curvearrowright G$ by conjugation

Temperley Lieb as Hilbert modules

For Temperley-Lieb C*-tensor category $\mathcal{C} = \mathcal{TLJ}_k$ constructed a C*-algebra $\mathcal{B}_k$ such that as braided C*-tensor categories

$$\mathcal{TLJ}_k \cong \mathrm{Mod}_{\mathcal{B}}{}^f$$

for right Hilbert modules which admit a finite orthonormal basis

Aaserud–E

Virasoro with central charge $c_k = 1 - 6/(k+2)(k+3)$:

$$\mathrm{su}(2)_{k+1} \subset \mathrm{su}(2)_k \oplus \mathrm{su}(2)_1$$

$$\mathcal{B}_k \otimes \mathcal{B}_1 \subset \mathcal{B}_{k+1} \quad ??$$

higher equivariant twists

$SU(n)$-equivariant $\mathrm{M}_F^\infty \otimes \mathcal{K}$-bundles E-Pennig

$\mathrm{M}_F^\infty = \otimes^\infty \mathrm{End}(F(\mathbb{C}^n))$

F :Vect$\rightarrow$Vect ; $F(V \oplus W) = F(V) \otimes F(W)$, e.g. Λ^{top}, Λ^*

$$\mathcal{G} = \{(g, z_1, z_2) \in G \times Z \times Z \mid z_1\,, z_2 \notin EVg\}\,, \quad Z = \mathbb{T} \setminus \{1\}$$

$$\mathcal{G} \longrightarrow G$$

$SU(n)$ adjoint action groupoids, equivariant Fell bundles

$$E(g, z_1, z_2) = \bigoplus_{\substack{z_1<\lambda<z_2 \\ \lambda \in \mathsf{EV}(g)}} \mathsf{Eig}(g, \lambda)$$

$$\mathcal{E}_{(g,z_1,z_2)} = F(E(g, z_1, z_2)) \otimes \mathsf{M}_F^{\infty}$$

$$\mathsf{M}_F^{\infty} \cong End(F(E(g, z_1, z_2))) \otimes \mathsf{M}_F^{\infty}$$

Fell bundle $\mathcal{E} \to \mathcal{G}$

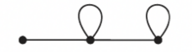
$C^*(\mathcal{E}) \otimes \mathcal{K} \cong$ section alg. of locally trivial bundle, fibre $\mathsf{M}_F^{\infty} \otimes \mathcal{K}$

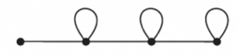
higher equivariant $SU(2)$ twists

$$K_0^G(C^*(\mathcal{E})) = 0 \ , \quad R_F(SU(2)) \cong K_0^{SU(2)}(\mathsf{M}_{\mathcal{F}}^{\infty}) \cong \mathbb{Z}[\rho][F(\rho)^{-1}]$$
$$K_1^G(C^*(\mathcal{E})) = R_F(SU(2))/(g_2(F))$$

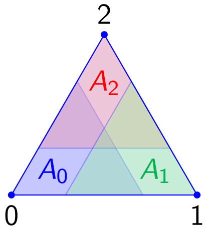
for $F(V) = \Lambda^*(V)^{\otimes k}$, $F(\rho) = (\rho+2)^k$, $g_2(F) = \sum_{\ell=1}^{k} \binom{k}{\ell} \rho_{\ell-1}$

$$k = 3: \quad 3 + 3\rho_1 + \rho_2 = \rho^2 + 3\rho + 2 = (\rho+2)(\rho+1)$$
$$k = 4: \quad 4 + 6\rho_1 + 4\rho_2 + \rho_3 = \rho(\rho+2)^2 \ ,$$
$$k = 5: \quad 5 + 10\rho_1 + 10\rho_2 + 5\rho_3 + \rho_4 = (\rho+2)^2(\rho^2+\rho-1)$$
$$k = 6: \quad 6 + 15\rho_1 + 20\rho_2 + 15\rho_3 + 6\rho_4 + \rho_5 = (\rho+2)^3(\rho^2-1) \ ,$$
$$k = 7: \quad 7 + 21\rho_1 + 35\rho_2 + 35\rho_3 + 21\rho_4 + 7\rho_5 + \rho_6 = (\rho+2)^3(-1-2\rho+\rho^2+\rho^3)$$





higher equivariant SU(3) twists



Mayer-Vietoris $\longrightarrow$ spectral sequence

exponential twist — computable K-theory for:

- twist $F = (\Lambda^*)^m$, $m = 1, 2, 3, 4, 5, 6, 7, 8$ with Mathematica
- non-trivial twist F over $\mathbb{Q}$

higher equivariant SU(3) twists

$$K_0^G(C^*(\mathcal{E}))\otimes\mathbb{Q}\cong(R_F(SU(3))\otimes\mathbb{Q})/J_F \quad , \quad K_1^G(C^*(\mathcal{E}))\otimes\mathbb{Q}\cong 0$$

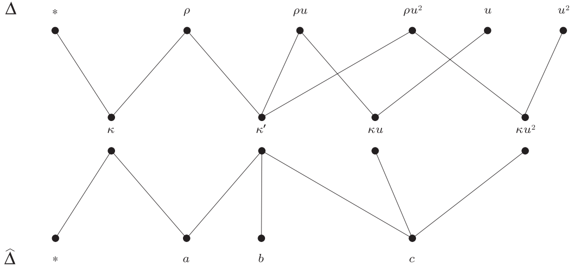
J_F is the submodule generated by σ_1^F and σ_2^F

$$\sigma_1^F=\frac{1}{\Delta}\det\begin{pmatrix} F(t_1) & F(t_2) & F(t_3) \\ t_1 & t_2 & t_3 \\ 1 & 1 & 1\end{pmatrix}, \quad \sigma_2^F=\frac{1}{\Delta}\det\begin{pmatrix} F(t_1)t_1 & F(t_2)t_2 & F(t_3)t_3 \\ t_1 & t_2 & t_3 \\ 1 & 1 & 1\end{pmatrix}.$$

Full twist $F=(\Lambda^*)^{\otimes m}$, $F(t_i)=(1+t_i)^m$, J_F generated by

$$\sigma_1^F=\sum_{l=2}^{m}\binom{m}{l}\mathrm{Sym}^{l-2}(\rho) \quad , \quad \sigma_2^F=\sum_{l=1}^{m}\binom{m}{l}\mathrm{Sym}^{l-1}(\rho)$$

Principal graphs of the Haagerup $(5+\sqrt{13})/2$ subfactor



$$\alpha^3 = 1\,, \quad \rho\alpha = \alpha^2\rho\,, \quad \rho^2 = 1 + \rho + \rho\alpha + \rho\alpha^{\mathbf{2}}$$

$$d_\lambda = [M, \lambda M]^{1/2} \qquad d_\rho^2 = 1 + 3d_\rho; \quad d_\rho = (3+\sqrt{13})/2$$

Modular data for Haagerup $\mathcal{D}$Hg

$$S = \frac{1}{3}\begin{pmatrix}
x & 1-x & 1 & 1 & 1 & 1 & y & y & y & y & y & y \\
1-x & x & 1 & 1 & 1 & 1 & -y & -y & -y & -y & -y & -y \\
1 & 1 & 2 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & -1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
y & -y & 0 & 0 & 0 & 0 & c(1) & c(2) & c(3) & c(4) & c(5) & c(6) \\
y & -y & 0 & 0 & 0 & 0 & c(2) & c(4) & c(6) & c(5) & c(3) & c(1) \\
y & -y & 0 & 0 & 0 & 0 & c(3) & c(6) & c(4) & c(1) & c(2) & c(5) \\
y & -y & 0 & 0 & 0 & 0 & c(4) & c(5) & c(1) & c(3) & c(6) & c(2) \\
y & -y & 0 & 0 & 0 & 0 & c(5) & c(3) & c(2) & c(6) & c(1) & c(4) \\
y & -y & 0 & 0 & 0 & 0 & c(6) & c(1) & c(5) & c(2) & c(4) & c(3)
\end{pmatrix}$$

$$T = \mathrm{diag}(1,1,1,1,\xi_3,\overline{\xi_3},\xi_{13}^{6},\xi_{13}^{-2},\xi_{13}^{2},\xi_{13}^{5},\xi_{13}^{-6},\xi_{13}^{-5})$$

$$x = (13 - 3\sqrt{13})/26 \qquad y = 3/\sqrt{13} \qquad c(j) = -2y\,\cos(2\pi j/13) \qquad \xi = e^{2\pi i/13}$$

$$S_{jj'} = (-2y/3)\cos(2\pi j j'/13)$$

E-Gannon

Potts model and orbifold $\quad H(\sigma) = -\sum_{i,j\ n \cdot n} J\delta(\sigma_i, \sigma_j)$

$$W_i W_{i+1} = e^{2\pi i/Q} W_{i+1} W_i\,, \quad W_i W_j = W_j W_i\,, |i-j| > 1 \text{ in } M_{Q^\infty}$$

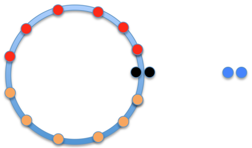
$$e_i = \text{Spectral}(W_i, 1) \qquad e_i e_{i\pm 1} e_i = e_i / Q$$

$$V = \exp L \sum e_{2i+1}\,, \quad W = \exp L^* \sum e_{2i} \quad (e^L - 1)(e^{L^*} - 1) = Q$$

$$\mu \text{ on } \mathcal{O}_Q \qquad \mu^2 = \textstyle\sum_g \alpha_g \qquad g \in \mathbb{Z}_Q$$

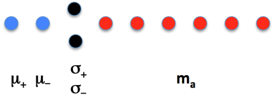
Orbifold $g \leftrightarrow -g$; $\qquad$ take $\times \mathbb{Z}_2$

$$\longrightarrow \hat{\mathbb{Z}}_2 = \sigma_\pm \qquad \mu_\pm \qquad m_g = \alpha_g \oplus \alpha_{-g}$$

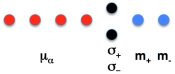


$$m_a m_b = m_{a+b} + m_{a-b} \qquad m_0 = \sigma_+ + \sigma_-$$
$$\mu_\tau \mu_{\tau'} = \sigma_{\tau+\tau'} + \sum m_a \qquad a \sim -a \neq 0$$
$$\mu_\tau m_a = \mu_+ + \mu_- \qquad \sigma_\pm m_a = m_a$$

fusion rules of double of Haagerup



$\mathbb{Z}_{13}$

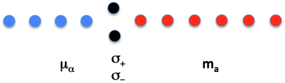


$\mathbb{Z}_3 \times \mathbb{Z}_3$

$$\sigma_\pm \simeq \mathbb{Z}_2 \quad m_a = e^{ia} + e^{-ia} \ : \ m_a m_b = m_{a+b} + m_{a-b}$$

$$\mu_\tau \mu_{\tau'} = \sigma_{\tau+\tau'} + \sum m_a \qquad a \sim -a \neq 0$$

$$\mu_\tau m_a = \mu_+ + \mu_- \qquad \sigma_\pm m_a = m_a$$



$$\sigma_+ = \text{identity}$$

$$\sigma_-^2 = \sigma_+ + \sigma_- + \sum \mu_\alpha + \sum m_a = R$$

$$\mu_\alpha m_b = R - \sigma_+ = R_-$$

$$\mu_\alpha \mu_\beta = R_- + \mu_{\alpha+\beta} + \mu_{\alpha-\beta} \qquad \mu_0 = \sigma_+ + \sigma_-$$

$$m_a m_b = R_- - m_{a+b} - m_{a-b} \qquad m_0 = -\sigma_+ + \sigma_-$$

$$\sigma_- \mu_\alpha = R_- + \mu_\alpha \qquad \sigma_- m_a = R_- - m_a$$

Groups and Orbifolds

- for $\omega \in H^3(G)$, $\exists$ a conformal net with $Rep(\mathcal{A}) = Rep\,\mathcal{D}^\omega(G)$
- If $\mathrm{Rep}(\mathcal{A}) \simeq \mathcal{D}^\omega(G)$, then $\mathcal{A} \simeq \mathcal{V}^G$ for a holomorphic net $\mathcal{V}$
- $\mathcal{V}$ holomorphic conformal net, $G \subset S_k$
  Rep $(\mathcal{V}^{k\otimes})^G = \mathrm{Rep}\mathcal{D}^\omega(G)$ - with $\omega^3 = 1$

E-Gannon

input : classification of group actions Jones thesis

$\cdots \to H^2(K/N) \to H^2(K) \to \Lambda \to H^3(K/N) \to H^3(K) \to \cdots$

- for G finite abelian odd, $\exists$ conformal net $Rep(\mathcal{A}) = TY(G)^{\mathbb{Z}_2}$
- for G finite abelian, $\exists$ conformal net with $Rep(\mathcal{B}) = \mathcal{D}TY(G)$

Bischoff, E-Gannon

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- A. Aaserud, DEE. Realising the braided Temperley-Lieb-Jones C*-tensor categories as Hilbert C*-modules. arXiv:1908.02674[math-ph] *Commun Math Phys* to appear
- A. Aaserud. DEE. K-theory and fusion rings from rank 2 compact Lie groups. arXiv:1803.03227[math-ph]