The quest of a finite purely-quantum group

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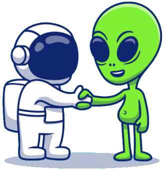
Slogan

We will explain what would be a *finite purely-quantum group*.
$\rightarrow$ its existence is related to famous open problems.

- Non-existence would be an excellent result,
- Existence/construction should be one of the most important discovery: a mathematical analogous to meeting an alien!



Based on joint works with Zhengwei Liu, Yunxiang Ren and Jinsong Wu, we will show that a construction could be workable.

Contents

Original motivation from subfactor theory

Without going into details, let think a subfactor as an inclusion $(N \subseteq M)$ which admits an index $|M : N|$, multiplicative with intermediate:

$$N \subseteq P \subseteq M \Rightarrow |M : N| = |M : P| \cdot |P : N|,$$

and satisfying:

Theorem (Jones, 1983)

The set of possible indices $|M : N|$ is exactly:

$$\left\{ 4\cos^2\left(\frac{\pi}{n}\right) \mid n \geq 3 \right\} \sqcup [4, \infty].$$

Sieve of Eratosthenes

If the index $|M : N|$ is in the set $S = \left\{4\cos^2\left(\frac{\pi}{n}\right) \mid n \geq 3\right\}$ or

$$(4,8) \setminus \left(2S \cup \{16\cos^4\left(\frac{\pi}{5}\right), 16\cos^2\left(\frac{\pi}{5}\right)\cos^2\left(\frac{\pi}{6}\right)\}\right)$$

then the subfactor $(N \subseteq M)$ admits no (proper) intermediate.

A subfactor without intermediate is called **maximal** (Bisch, 1994).
$\rightarrow$ "quantum analogous" of prime numbers!
What about natural numbers? See my papers on Ore's theorem.

Examples of maximal subfactors (irreducible and finite depth)

- index $< 4 \rightarrow$ ADE: A_m, D_{2m}, E_6, E_8,
- index in $(4, 5+\sqrt{3})$: Haagerup,... (12 ones, up to dual),
- maximal subgroup subfactors: $S_{n-1} \subset S_n$, ...

The finite group subfactors ($R^G \subset R$) are in a specific class called *finite index, irreducible, depth* 2, completely characterized by:

Theorem (Longo, Szymanski, 1994; David, 1996)

Every finite index, irreducible, depth 2 subfactor is of the form ($R^H \subseteq R$), where H is a finite quantum group (i.e. finite dimensional Hopf C*-algebra, also called Kac algebra).

The only known maximal ones of this class are the finite group subfactors ($R^G \subset R$), with G is cyclic of prime order.

Question 1 (P. 2012)

Is there a finite index, irreducible, depth 2 maximal subfactor which is not a group subfactor?

$\rightarrow$ such ones would be the **closest** analogues of the prime numbers.

Transition to fusion category theory

All the fusion categories mentionned in this talk are over $\mathbb{C}$.

Theorem (Izumi, Longo, Popa, 1998)

There is a 1-1 correspondence between

- intermediates of finite quantum group subfactor $R^H \subseteq P \subseteq R$,
- *left coideal* $*$-subalgebras $L \subseteq H$, i.e. $\Delta(L) = H \otimes L$,

given by $P = R^L$.

Theorem (Folklore, Burciu, 2012)

There is a 1-1 correspondence between

- *normal* left coideal $*$-sub. $L \subseteq H$, i.e. $h_1 L S(h_2) \subseteq L, \forall h \in H$,
- fusion subcategories $\mathcal{C}$ of $\mathrm{Rep}(H)$,

given by $\mathcal{C} \simeq \mathrm{Rep}(H//L)$.

The last theorem holds for any finite dimensional semisimple Hopf algebra over $\mathbb{C}$. Open problem: existence of a Kac structure.

Theorem (G. Kac, 1972)

Let H be a finite quantum group. Then $\mathrm{Rep}(H)$ is a unitary integral fusion category of *Frobenuis type*, i.e. $\dim(\text{irrep}) \mid \dim(H)$.

A fusion category without (non-trivial) fusion subcategory is called *simple*. Then, here is a weaker version of Question 1:

Question 2

Is there a (unitary) integral simple fusion category (of Frobenuis type) which is not group-like, up to Grothendieck equivalence?

A fusion category is *weakly group-theoretical* if its Drinfeld center is equivalent to the one coming from a sequence of group extensions.

Theorem (Etingof, Nikshych, Ostrik, 2011)

A weakly group-theoretical simple fusion category is Grothendieck equivalent to $\mathrm{Rep}(G)$, with G a finite simple group.

This leads to the *ultimate weaker version* of Question 1:

Question 3 (Etingof, Nikshych, Ostrik, 2011)

Is there an integral fusion category not weakly group-theoretical?

In particular, the existence of the following is open.

A finite purely-quantum group:

It would be a finite quantum group H such that $\mathrm{Rep}(H)$ admits no weakly group-theoretical fusion subcategory (other than Vec).

Here is the strategy of the quest:

- classify simple integral fusion rings of Frobenius type,
- filter them by all known (unitary) categorification criteria,
- try to (unitarily) categorify the remaining non group-like ones,
- find associated finite quantum groups (if exist).

Classification of simple integral fusion rings

From far, the SageMath code (15 pages) looks like:

Atlas: https://sites.google.com/view/sebastienpalcoux/fusion-rings

Computation result (Liu, P., Wu, 2019)

Under the following bounds

rank	≤ 5	6	7	8	9	10	all
$\mathrm{FPdim} <$	1000000	150000	15000	4080	504	240	132

there are exactly 34 simple integral fusion rings of Frobenius type (and $\mathrm{FPdim} \neq p^a q^b, pqr$ [ENO, 2011]), 4 of them are group-like:

#	rank	FPdim	type	group
1	5	60	$[1,3,3,4,5]$	$\mathrm{PSL}(2,5)$
1	6	168	$[1,3,3,6,7,8]$	$\mathrm{PSL}(2,7)$
2	7	210	$[1,5,5,5,6,7,7]$	
2	7	360	$[1,5,5,8,8,9,10]$	$\mathrm{PSL}(2,9)$
4	7	7980	$[1,19,20,21,42,42,57]$	
15	8	660	$[1,5,5,10,10,11,12,12]$	$\mathrm{PSL}(2,11)$
5	8	990	$[1,9,10,11,11,11,11,18]$	
2	8	1260	$[1,6,7,7,10,15,20,20]$	
2	8	1320	$[1,6,6,10,11,15,15,24]$	

Categorification criteria and TPE

In the framework of *Quantum Fourier Analysis*, we get:

Commutative Schur Product Criterion (Liu, P., Wu, 2019)

Let $\mathcal{F}$ be a commutative fusion ring, let (M_i) be its fusion matrices, and let $(\lambda_{i,j})$ be the table given by their simultaneous diagonalization, with $\lambda_{i,1} = \|M_i\|$. If $\exists (j_1, j_2, j_3)$ such that

$$\sum_i \frac{\lambda_{i,j_1}\lambda_{i,j_2}\lambda_{i,j_3}}{\lambda_{i,1}} < 0$$

then $\mathcal{F}$ admits no unitary categorification.

This criterion rules out 28 among the 30 non group-like simple integral fusion rings of the previous classification (more than 93%).

The remaining 2 are denoted $\mathcal{F}_{210}$ and $\mathcal{F}_{660}$ (according to FPdim).

In the framework of the *Triangular Prism Equation* (TPE), we get:

Zero Spectrum Criterion (Liu, P., Ren, 2020)

For a fusion ring $\mathcal{F}$, if there are indices i_j, $1 \leq j \leq 9$, such that $n_{i_4,i_1}^{i_6}$, $n_{i_5,i_4}^{i_2}$, $n_{i_5,i_6}^{i_3}$, $n_{i_7,i_9}^{i_1}$, $n_{i_2,i_7}^{i_8}$, $n_{i_8,i_9}^{i_3}$ are non-zero, and

$$\sum_k n_{i_4,i_7}^k n_{i_5^*,i_8}^k n_{i_6,i_9^*}^k = 0;$$

$$n_{i_2,i_1}^{i_3} = 1;$$

$$\sum_{k \in I} n_{i_5,i_4}^k n_{i_3,i_1^*}^k = 1 \text{ or } \sum_k n_{i_2,i_4^*}^k n_{i_3,i_6^*}^k = 1 \text{ or } \sum_k n_{i_5^*,i_2}^k n_{i_6,i_1^*}^k = 1,$$

$$\sum_{k \in I} n_{i_2,i_7}^k n_{i_3,i_9^*}^k = 1 \text{ or } \sum_k n_{i_8,i_7^*}^k n_{i_3,i_1^*}^k = 1 \text{ or } \sum_k n_{i_2^*,i_8}^k n_{i_1,i_9^*}^k = 1,$$

then $\mathcal{F}$ cannot be categorified (at all) over any field.

$\rightarrow$ This criterion rules out $\mathcal{F}_{660}$.
(see also our classification of mult. 1, rk ≤ 6, Grothendieck rings).

Theorem (Liu, P., Ren, 2020)

A unitary simple integral fusion category of Frobenius type with the above bounds is Grothendieck equivalent to:

- $\mathrm{Rep}(\mathrm{PSL}(2,q))$ with q prime power and $4 \leq q \leq 11$,
- a "possible" categorification of $\mathcal{F}_{210}$.

By TPE method, we also proved that such a categorification must have a simple object with a negative Frobenius-Schur indicator.

The simple integral fusion ring $\mathcal{F}_{210}$ is of rank 7, FPdim 210 and type $[1,5,5,5,6,7,7]$, with fusion matrices:

$$\begin{matrix} 1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1 \end{matrix}, \quad
\begin{matrix} 0&1&0&0&0&0&0\\1&1&0&1&0&1&1\\0&0&1&0&1&1&1\\0&1&0&0&1&1&1\\0&0&1&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1 \end{matrix}, \quad
\begin{matrix} 0&0&1&0&0&0&0\\0&0&1&0&1&1&1\\1&1&1&0&0&1&1\\0&0&0&1&1&1&1\\0&1&0&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1 \end{matrix}, \quad
\begin{matrix} 0&0&0&1&0&0&0\\0&1&0&0&1&1&1\\0&0&0&1&1&1&1\\1&0&1&1&0&1&1\\0&1&1&0&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1 \end{matrix}, \quad
\begin{matrix} 0&0&0&0&1&0&0\\0&0&1&1&1&1&1\\0&1&0&1&1&1&1\\0&1&1&0&1&1&1\\1&1&1&1&1&1&1\\0&1&1&1&1&2&1\\0&1&1&1&1&1&2 \end{matrix}, \quad
\begin{matrix} 0&0&0&0&0&1&0\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&2&1\\1&1&1&1&2&1&2\\0&1&1&1&1&2&2 \end{matrix}, \quad
\begin{matrix} 0&0&0&0&0&0&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&1\\0&1&1&1&1&1&2\\0&1&1&1&1&2&2\\1&1&1&1&2&2&1 \end{matrix}$$

The next section will explain a *very cool* reason why $\mathcal{F}_{210}$ is so robust to all the (known) categorification criteria...

As pointed out by Lübeck (in private), the generic character table of $\mathrm{Rep}(\mathrm{PSL}(2,q))$ is available on GAP4. We must distinghish 3 cases (q prime-power): q even, $q \equiv 1 \mod 4$, or $q \equiv 3 \mod 4$.

Case q even

classparam k / charparam c	$\{1\}$	$\{1\}$	$\{1,\dots,\frac{q-2}{2}\}$	$\{1,\dots,\frac{q}{2}\}$
$\{1\}$	1	1	1	1
$\{1,\dots,\frac{q}{2}\}$	$q-1$	-1	0	$-2\cos(\frac{2\pi kc}{q+1})$
$\{1\}$	q	0	1	-1
$\{1,\dots,\frac{q-2}{2}\}$	$q+1$	1	$2\cos(\frac{2\pi kc}{q-1})$	0
class size	1	q^2-1	$q(q+1)$	$q(q-1)$

Theorem (Schur's orthogonality relations for every finite group)

Let $(\chi_{i,j})$ be the character table. Then the fusion rules are:

$$n_{i,j}^k = \sum_s \frac{\chi_{i,s}\chi_{j,s}\overline{\chi_{k,s}}}{\sum_t |\chi_{t,s}|^2}$$

Theorem (Liu, P., Ren, 2020)

In the case q even, the fusion rules of $\mathrm{Rep}(\mathrm{PSL}(2,q))$ are:

$$x_{q-1,c_1}x_{q-1,c_2} = \delta_{c_1,c_2}x_{1,1} + \sum_{\substack{c_3 \text{ such that} \\ c_1+c_2+c_3 \neq q+1 \\ \text{and } 2\max(c_1,c_2,c_3)}} x_{q-1,c_3} + (1-\delta_{c_1,c_2})x_{q,1} + \sum_{c_3} x_{q+1,c_3},$$

$$x_{q-1,c_1}x_{q,1} = \sum_{c_2}(1-\delta_{c_1,c_2})x_{q-1,c_2} + x_{q,1} + \sum_{c_2} x_{q+1,c_2},$$

$$x_{q-1,c_1}x_{q+1,c_2} = \sum_{c_3} x_{q-1,c_3} + x_{q,1} + \sum_{c_3} x_{q+1,c_3},$$

$$x_{q,1}x_{q,1} = x_{1,1} + \sum_{c} x_{q-1,c} + x_{q,1} + \sum_{c} x_{q+1,c},$$

$$x_{q,1}x_{q+1,c_1} = \sum_{c_2} x_{q-1,c_2} + x_{q,1} + \sum_{c_2}(1+\delta_{c_1,c_2})x_{q+1,c_2},$$

$$x_{q+1,c_1}x_{q+1,c_2} = \delta_{c_1,c_2}x_{1,1} + \sum_{c_3} x_{q-1,c_3} + (1+\delta_{c_1,c_2})x_{q,1} + \sum_{\substack{c_3 \text{ such that} \\ c_1+c_2+c_3 \neq q-1 \\ \text{and } 2\max(c_1,c_2,c_3)}} x_{q+1,c_3} + \sum_{\substack{c_3 \text{ such that} \\ c_1+c_2+c_3 = q-1 \\ \text{or } 2\max(c_1,c_2,c_3)}} 2x_{q+1,c_3},$$

and the **interpolation** to q non prime-power is still a fusion ring.

$\rightarrow$ they automatically check all the known categorification criteria.

Cool observation: $\mathcal{F}_{210}$ corresponds to $q = 6$.

Here are the generic character tables for the two other cases:

Case $q \equiv 1 \mod 4$

charparam c \ classparam k	$\{1\}$	$\{1,2\}$	$\{1,\dots,\frac{q-5}{4}\}$	$\{\frac{q-1}{4}\}$	$\{1,\dots,\frac{q-1}{4}\}$
$\{1\}$	1	1	1	1	1
$\{1,2\}$	$\frac{q+1}{2}$	$\frac{1+(-1)^{k+c}\sqrt{q}}{2}$	$(-1)^k$	$(-1)^k$	0
$\{1,\dots,\frac{q-1}{4}\}$	$q-1$	-1	0	0	$-2\cos(\frac{4\pi kc}{q+1})$
$\{1\}$	q	0	1	1	-1
$\{1,\dots,\frac{q-5}{4}\}$	$q+1$	1	$2\cos(\frac{4\pi kc}{q-1})$	$2(-1)^c$	0
class size	1	$\frac{q^2-1}{2}$	$q(q+1)$	$\frac{q(q+1)}{2}$	$q(q-1)$

Case $q \equiv 3 \mod 4$

charparam c \ classparam k	$\{1\}$	$\{1,2\}$	$\{1,\dots,\frac{q-3}{4}\}$	$\{1,\dots,\frac{q-3}{4}\}$	$\{\frac{q+1}{4}\}$
$\{1\}$	1	1	1	1	1
$\{1,2\}$	$\frac{q-1}{2}$	$\frac{-1+i(-1)^{k+c}\sqrt{q}}{2}$	0	$(-1)^{k+1}$	$(-1)^{k+1}$
$\{1,\dots,\frac{q-3}{4}\}$	$q-1$	-1	0	$-2\cos(\frac{4\pi kc}{q+1})$	$-2(-1)^c$
$\{1\}$	q	0	1	-1	-1
$\{1,\dots,\frac{q-3}{4}\}$	$q+1$	1	$2\cos(\frac{4\pi kc}{q-1})$	0	0
class size	1	$\frac{q^2-1}{2}$	$q(q+1)$	$q(q-1)$	$\frac{q(q-1)}{2}$

Similarly, we can compute the generic fusion rules and interpolate them to q non prime-power.

Future step

Compute the generic F-symbols for q prime-power, and see how they could be interpolated to q non prime-power...

Philosophy

This interpolation method should work for every family $(G(q))$ of finite simple groups of Lie type (q is the order of the associated finite field, a prime-power), at least at the fusion ring level. The interpolated case could be seen "intuitively" as over a finite "virtual" field of non prime-power order, in the same flavor than

- the field with one element,
- the noncommutative geometry.

Joke (about these "virtual" fields): a "quantum" field theory...

Associated papers

- **Published**:
  - Ore's theorem for cyclic subfactor planar algebras, **PJM** (2018)
  - Euler totient of subfactor planar algebras, **PAMS** (2018),
  - Ore's theorem on subfactor planar algebras, **QT** (2020).
- **Under review** (with Zhengwei Liu and Jinsong Wu):
  - Fusion Bialgebras and Fourier Analysis , **arXiv**:1910.12059
- **Current works** (with Zhengwei Liu and Yunxiang Ren):
  - Triangular prism equations and categorification,
  - Classification of Grothendieck rings of complex fusion categories of multiplicity one up to rank six, **arXiv**:2010.10264.
  - Categorical approach to Izumi's equations of near groups,
  - Interpolated family of non group-like simple integral fusion rings of Lie type.

Thanks for your attention!

Formal table characterization of commutative fusion ring

Let $\mathcal{F}$ be a commutative fusion ring. Let (M_i) be its fusion matrices, and let $D_i = diag(\lambda_{i,j})$, be their simultaneous diagonalization. The *eigentable* of $\mathcal{F}$ is the table given by $(\lambda_{i,j})$.

Theorem

Let $(\lambda_{i,j})$ be a formal $r \times r$ table. Consider the space of functions from $\{1, \dots, r\}$ to $\mathbb{C}$ with some inner product $\langle f, g \rangle$. Consider the functions (λ_i) defined by $\lambda_i(j) = \lambda_{i,j}$, and assume that $\langle \lambda_i, \lambda_j \rangle = \delta_{i,j}$. Consider the pointwise multiplication $(fg)(i) = f(i)g(i)$, and the multiplication operator $M_f : g \mapsto fg$. Consider $M_i := M_{\lambda_i}$, and assume that for all i there is j (automatically unique, denoted i^*) such that $M_i^* = M_j$. Assume that M_1 is the identity. Assume that for all i, j, k, $n_{i,j}^k := \langle \lambda_i \lambda_j, \lambda_k \rangle$ is a nonnegative integer. Then $(n_{i,j}^k)$ are the structure constants of a commutative fusion ring and $(\lambda_{i,j})$ is its eigentable. Moreover, every eigentable of a commutative fusion ring satisfies all the assumptions above.

Braiding criterion

Theorem (Nikshych, 2013)

Let $\mathcal{C}$ be a non-degenerate integral braided fusion category. If there is a simple object of prime-power FPdim then there is a nontrivial symmetric subcategory.

Corollary

For q non prime-power. If the interpolated fusion ring has a simple object of prime-power FPdim then no braided categorification.

Here is the list of $q < 200$ such that the associated fusion ring has no simple object of prime-power FPdim (so that the existence of a braided categorification is not excluded by above corollary):

- q even: 34, 56, 76, 86, 92, 94, 116, 118, 134, 142, 144, 146, 154, 160, 176, 184, 186, 188,
- $q \equiv 1 \mod 4$: 69, 77, 153, 185, 189,
- $q \equiv 3 \mod 4$: 91, 111, 115, 155, 171, 175, 183, 187.

Simple integral fusion rings NOT of Frobenius type

Open: existence of a fusion $\mathbb{C}$-category not of Frobenius type (FT).

Theorem (ENO, 2011)

Any weakly group-theoretical fusion category is FT.

There are 21 simple integral fusion rings not FT of rank ≤ 7 and $\mathrm{FPdim} < 1500$ (and $\neq p^a q^b, pqr$), 4 ones pass the Schur criterion:

#	rank	FPdim	type	mult.
1	6	924	$[1, 7, 8, 12, 15, 21]$	10
1	6	1320	$[1, 9, 10, 11, 21, 24]$	11
1	7	560	$[1, 6, 7, 7, 10, 10, 15]$	6
1	7	798	$[1, 7, 8, 9, 9, 9, 21]$	8

Only the two in the middle pass the *Drinfeld center criterion* (i.e. each formal codegree divides FPdim).

Other problem: existence of a NC simple integral fusion ring.

Classification: multiplicity one up to rank 6

Theorem (Liu, P., Ren, 2020)

The complex Grothendieck rings of multiplicity one up to rank six:

- known fusion categories:
  - $\mathrm{Vec}(G)$ with finite group $G = C_n$ $(n \leq 6)$, C_2^2, S_3,
  - $\mathrm{Rep}(G)$ with finite group $G = S_3$, S_4, D_n $(4 \leq n \leq 7)$, D_9, Q_8, $C_3 \rtimes C_4$, $C_3 \rtimes S_3$,
  - near-group $C_n + 0$, $n \leq 5$ (Tambara-Yamagami $TY(C_n)$),
  - $\mathrm{SU}(2)_n$ $(n \leq 5)$, $\mathrm{PSU}(2)_n$ $(3 \leq n \leq 11)$, $\mathrm{SO}(3)_2$, $\mathrm{SO}(5)_2$,
  - even part of a 1-supertransitive subfactor of index $3 + 2\sqrt{2}$,
  - products of two ones above.
- new ones (# rings), none modular, some come from zesting*:

#	FPdim	rank	type	zesting(s) of
3	8	6	$[1,1,1,1,\sqrt{2},\sqrt{2}]$	$\mathrm{Vec}(C_2) \otimes \mathrm{SU}(2)_2$
1	12	5	$[1,1,\sqrt{3},\sqrt{3},2]$	$\mathrm{SO}(3)_2$
1	20	6	$[1,1,2,2,\sqrt{5},\sqrt{5}]$	$\mathrm{SO}(5)_2$
1	24	5	$[1,1,2,3,3]$	

*zesting construction: Delaney-Galindo-Plavnik-Rowell-Zhang, 2020.