Applied von Neumann algebra.

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The main ingredients of the material though, are physics-related. Hilbert space and von Neumann algebras.
The Hilbert spaces of this talk might be a little unfamiliar to the physics community these days since they consist of *holomorphic* L^2 functions. These differ in a significant (but rather pleasant) way from the usual L^2 spaces in that *point evaluation* is continuous-hence given by the inner product with a vector, called a *reproducing kernel*.

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M is called *finite* if there is a trace $tr : M \to \mathbb{C}$ satisfying

$$tr(ab) = tr(ba) \text{ and } tr(a^*a) > 0 \text{ for } a \neq 0$$

For instance the $n \times n$ matrices.

Here is a large family of von Neumann algebras which are the ones relevant to this talk:
If Γ is a countable discrete group then it acts on the Hilbert space $\ell^2(\Gamma)$ by left translation of the orthonormal basis of characteristic functions of group elements. $(\lambda_\gamma(e_\nu) = e_{\gamma\nu})$. This is the left regular representation which generates a von Neumann algebra $vN(\Gamma)$ whose elements are convergent sums of the form

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Here is a theorem whose proof uses von Neumann algebras in an essential way and whose proof is so easy that we give it in full. (Kaplansky)

Theorem

Let Γ be a discrete group and $\mathbb{F}$ a field of characteristic zero. Let $\mathbb{F}\Gamma$ be the group algebra. Then $ab = 1 \iff ba = 1$ in $\mathbb{F}\Gamma$.

Proof.

Since the relations $ab = 1$ and $ba = 1$ only involve finitely many scalars we may embed $\mathbb{F}$ in $\mathbb{C}$ and work in $\mathbb{C}\Gamma$ which embeds into $vN(\Gamma)$.
So the result follows from $ab = 1 \iff ba = 1$ in a finite von Neumann algebra M with trace tr. Let M act on some $\mathfrak{H}$.
Suppose $ab = 1$. Then for any $\xi \in \mathfrak{H}$, $ba(b\xi) = b\xi$ so since ba is bounded it suffices to show that the range of b is dense. But if $b = u|b|$ is the polar decomposition of b then u is a partial isometrey from the orthogonal complement of the kernel of b to the closure of the image of b. But $u^*u = 1$ since $ker(b) = 0$ (since $ab = 1$). So $tr(uu^*) = 1$ forcing $uu^* = 1$ since $tr(1 - uu^*) = 0$ so the image of b is dense. □

The conclusion of the theorem remains an open problem if one drops the condition that the characteristic of the field be zero.

The structure of von Neumann algebras is deep and there are several instances of that structure having strongly influenced other fields. Ergodic theory-Murray von Neumann, Dye, Weiss, Connes, Feldman Moore, Popa, Group theory-Connes, Haagerup, Popa, Ozawa,Lück,... Probability-Voiculescu, Shyakhtenko, Guionnet,...Geometry Atiyah, Connes.... Low dimensional topology-Wenzl, Ocneanu,.... Algebra-von Neumann,Kaplansky,... Quantum theory-Haag, Kastler, Longo,Wassermann,....

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I have become interested in results in mathematics, not internal to the theory of von Neumann algebras but whose proof actually uses them, and other proofs are either non-existent or rather more complicated. Here there are far fewer examples though I will not attempt to make a complete list. Notable are Connes' results on foliations. In this talk I want to explain another such result.

So I will state a theorem, for which I need very little background.

Definition

Weighted Bergman space A^2_α (for $\alpha > -1$)is the Hilbert space of holomorphic functions on the upper half plane $\mathbb{H}$ which are square summable with respect to the measure $y^\alpha dxdy$.

Definition

A Fuchsian group is a discrete (cofinite) subgroup of $PSL_2(\mathbb{R})$.

Theorem

Let Γ be a torsion free Fuchsian group and $\Gamma(z)$ be an orbit in $\mathbb{H}$. Then there is a non-zero function in A^2_α vanishing on $\Gamma(z)$ iff

$$\alpha > \frac{4\pi}{covolume(\Gamma)} - 1$$

Let us give a little context. Hardy space H^2 is contained in all the Bergman spaces and is defined as those holomorphic functions ξ on the open unit disc such that $lim_{r \to 1} \int_{\{z:|z|=r\}} |f(z)|^2 dz < \infty$. If $Z = \{z_n : n = 1, 2, \cdots\}$ is a subset of the disc, it is known (Szego) via the Blaschke product that there is a Hardy space function vanishing on Z iff

$$\sum_n (1 - |z_n|) < \infty$$

.

(Thus it is a corollary of our theorem that the sum over an orbit of Γ of $1 - |z|$ diverges- a known result of course.) Attempts to extend the Hardy space result to Bergman space have been only partially successful. One does find in a book by Hedenmalm, Korenblum and Zhu a necessary and sufficient condition in terms of a complicated entropy-related density for Z. The result also exhibits a critical value for which the answer is unknown. Our theorem actually calculates this density for orbits of Fuchsian groups! And settles the question for the critical value itself.

There is no mention of ordered groups, II_1 factors or cusp forms in the statement of the theorem, but the proof will rely essentially on them.
The torsion free condition can be relaxed but the statement is a bit more complicated and so is the proof (though the idea is basically the same) so I'll leave that till later.
Cusp forms are perhaps not so familiar to this audience so let's start with them. They will also allow us to give a constructive proof of the existence part of the theorem.

Definition

A modular function for $PSL(2, Z)$ of (even) weight p is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfying

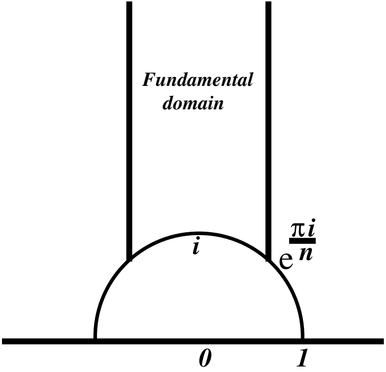
$$*** \qquad f\left(\frac{az+b}{cz+d}\right) = (cz+d)^p f(z)$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z)$

In particular $f(z+1) = f(z)$ so we may consider f as a function of $q = e^{2\pi iz}$. If $f = \sum_{n=0}^{\infty} a_n q^n$, f is called a modular form and a modular form is called a cusp form if $a_0 = 0$.
The simplest cusp form is the modular discriminant $\Delta(q) = q\prod_{n=1}^{\infty}(1-q^n)^{24}$ of weight 12.
There are lots of other Fuchsian groups besides $PSL_2(\mathbb{Z})$. For instance any Riemann surface is a quotient of $\mathbb{H}$ by a Fuchsian group acting freely so Γ in this case is the fundamental group of the surface.

It is obvious what a "modular function" should mean for any Γ.

$PSL_2(\mathbb{Z})$ does not quite act freely. A fundamental domain is



where $n = 3$. I'll talk about the case $n > 3$ later on.

The "cusp" in cusp form refers to the behaviour as z $\to i\infty$ in the above picture.
How that might be generalised algebraically is unclear but we are saved by the analysis....

The modular invariance property can and should be thought of as a fixed point property. If $SL(2,\mathbb{R})$ acts on functions $\xi:\mathbb{H}\to\mathbb{C}$ by

$$(g^{-1}\xi)(z)=(cz+d)^{-p}\xi\left(\frac{az+b}{cz+d}\right)$$

(where $g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$) Then f satisfies *** iff it is fixed by $PSL_2(\mathbb{Z})$ for this action.

We even go one step further. If $\mathcal{F}_s$ denotes the vector space of all functions acted on by $SL(2,\mathbb{R})$ as above with $p=s$ then multiplication by a modular function f determines a linear map

$$M_f:V_s\to V_{p+s}$$

which intertwines the action of $SL(2,\mathbb{Z})$.
Cusp forms are of great interest in number theory and our program certainly aims to exploit this, but for the moment our goals and achievements are more modest.

That is the algebra. What about the analysis? It is easy to show that, for a cusp form,

$$|f(z)| \leq (constant)(Imz)^{-p/2}$$

This solves the problem of how to define "cusp form" in general. Just take this inequality as a definition. For us, much more significiant is that the MULTIPLICATION OPERATORS by cusp forms, between function spaces, will be bounded exactly when they intertwine the actions of $SL(2, \mathbb{Z})$!!!
(And, for all Γ, cusp forms will define bounded operators. A special example is that of a compact Riemann surface where "modular functions" are sections of the symmetric tensor powers of the canonical line bundle, and they are always "cusp forms" by compactness.)
Recall that Bergman space A^2_{s-2} is the closed subspace of holomorphic functions in $L^2(\mathbb{H}, y^s \frac{dxdy}{y^2})$. This space is defined for any real $s > 1$.

Let M_f, for f a cusp form of weight p, be the multiplication operator by f. It is a bounded linear operator between Bergman spaces, for any $s > 1$.

$$M_f : A^2_{s-2} \to A^2_{s+p-2}$$

As such it has an adjoint M_f^*:

$$M_f^* : A^2_{s+p-2} \to A^2_{s-2}$$

and $M_f^* M_f$ commutes with the action of $SL(2,\mathbb{Z})$ on A^2_{s-2}. Note that M_f^* is not a multiplication operator as one has to project onto A^2_{s-2}. The combination $M_f^* M_f$ is what is called a "Toeplitz operator". Radulescu used more general Γ invariant Toeplitz operators in his penetrating analysis of the commutant of $PSL_2(\mathbb{Z})$ on Bergman space.

Now let's give a way to get part (ii) of our main theorem, for $PSL_2(\mathbb{Z})$:
(Rolen and Wagner had a big input here!)

If $\alpha > \frac{4\pi}{covolume(\Gamma)} - 1$ there is a non-zero function in A^2_{s-2} vanishing on $\Gamma(z)$.

Obviously $M_f\xi$ has zeros whenever f does. So take a cusp form of weight p with a zero at z and any $\xi \in A^2_{s-2}$ for any $s > 1$ and you'll get an element of A^2_{s+p-2} vanishing on the orbit of z. Life would be simple if Δ had zeros but it doesn't.... But the Eisenstein series G_2, a modular form of weight 4 has a zero at $e^{\frac{\pi i}{3}}$ so ΔG_2 can be used to produce functions vanishing on zero sets in $A_{17+\epsilon}$ for any $epsilon > 0$. Let me return to the Rolen-Wagner way of reducing s later on if I have time.
That'll do for cusp forms in the meantime.

Now let's turn to ordered groups.

A group Γ is (left) orderable if there is a total order $<$ on Γ (any two elements are comparable) which is invariant under left translation: $\alpha < \beta \iff \gamma\alpha < \gamma\beta$. This is the same as having a semigroup $P \subset \Gamma$ for which $\Gamma = P \coprod \{id\} \coprod P^{-1}$ in which case $\alpha < \beta$ means $\alpha^{-1}\beta \in P$. Orderable groups are obviously torsion free.

$\mathbb{Z}$ obviously orderable. $\mathbb{Z}^2$ by lexicographic order.
Free groups are left orderable, fundamental groups of surfaces are left orderable. Left orderable groups are torsion free so their actions as Fuchsian groups on $\mathbb{H}$ are free.

A Fuchsian group is either a free product of cyclic groups or has a surface subgroup of finite index so torsion free subgroups of Fuchsian groups are orderable.

Now let's start towards the proof of the nonexistence part of the main theorem. This will use von Neumann dimension.

A II_1 factor is an algebra M which acts typically on Hilbert spaces $\mathcal{H}$ in such a way that there is a unique *dimension function*

$$dim_M \mathcal{H} \in [0, \infty]$$

This real valued dimension characterizes $\mathcal{H}$ in the same sense that an ordinary Hilbert space is characterized by its dimension. All numbers in $[0, \infty]$ occur.

The most immediate example of a II_1 factor factor arises when we take the left regular representation of a group Γ on $\ell^2(\Gamma)$. Γ acts unitarily by permuting the obvious orthonormal basis of ℓ^2 indexed by group elements. If Γ were finite, the centre of the algebra generated by the left regular representation would be spanned by functions constant on conjugacy classes. But infinite groups can have no nontrivial finite conjugacy classes and this centre disappears. (icc) Free groups are icc. $PSL(2, \mathbb{Z})$ is icc.

The closure of the group algebra on $\ell^2(\Gamma)$ is called $vN(\Gamma)$, the group von Neumann algebra. Its centre is trivial if Γ is icc, which earns it the name of a "factor".
The "type II" part comes from the existence of the *trace tr* on $vN(\Gamma)$. It is very simple. If λ_γ is the left regular representation then the map

$$tr(\lambda_\gamma) = \begin{cases} 1 & \text{if } \gamma = id \\ 0 & \text{otherwise.} \end{cases}$$

extends by linearity and continuity to all of $vN(\Gamma)$.

$vN(\Gamma)$should be considered as an algebra in its own right, capable of acting on other Hilbert spaces, e.g. direct sums of $\ell^2(\Gamma)$.
One can obtain $\ell^2(\Gamma)$ from $vN(\Gamma)$and its trace. Defining the inner product $\langle x, y\rangle = tr(y^*x)$ it is trivial (it really is!) that the completion of $vN(\Gamma)$is the Hilbert space $\ell^2(\Gamma)$. It is alternatively called $L^2(M)$ where M is $vN(\Gamma)$.
If you get nothing else out of this part of the talk, get this:

$L^2(M) = \ell^2(\Gamma)$ is the left module of rank 1 for vN(Γ). All other Hilbert space modules can be obtained as subspaces of direct sums of $L^2(M)$. As such there is a unique dimension function for such Hilbert spaces $dim_M(\mathcal{H})$ with the properties

$$dim_M(L^2(M)) = 1 \text{ and } dim(\oplus_i \mathcal{H}_i) = \sum_i dim_M(\mathcal{H}_i)$$

All non-negative numbers and ∞ arise as $dim_M(\mathcal{H})$. Any two Hilbert spaces over M with the same dimension are unitarily equivalent M-modules.

It was Atiyah who first noticed the relevance of von Neumann algebra in geometry with his index theorem for covering spaces. Here is the simple idea.
If V is a manifold which is a galois covering of another one, say $W = V/\Gamma$ with deck transformation group Γ, choose some nice measure on V invariant under Γ. Then from the point of view of Γ, $L^2(V)$ looks like $\oplus_{\gamma\in\Gamma} L^2(W)$ by choosing a fundamental domain. And the action of Γ is as (left regular)$\otimes id$ on this Hilbert space when it is rewritten as $\ell^2(\Gamma) \otimes L^2(W)$ which is nothing but a direct sum of copies of $\ell^2(\Gamma)$.

So any Γ-invariant closed subspace of $L^2(V)$ has a dimension over $vN(\Gamma)$. One may thus immediately define the index of an elliptic operator on $L^2(V)$, commuting with Γ, as the difference between the dimensions of the kernel and cokernel, measured by $vN(\Gamma)$. Atiyah explicitly mentioned the case of Fuchsian groups acting on $\mathbb{H}$ but was more interested in vast generalisations of the theory, such as constructing the discrete series for lots of locally compact groups, and so did not go into more depth in this special case beyond remarking that his index calculation implied the existence of L^2 holomorphic functions.

Our context is this: Γ is a Fuchsian group (covolume ones are all icc-Akemann) acting unitarily on the Bergman space of L^2 holomorphic functions A^2_{s-2} with respect to the measure $y^s \frac{dxdy}{y^2}$ on the upper half plane. The measure is not invariant but that is corrected for by the $\frac{1}{(cz+d)^s}$ factor in the formula for $g\xi$. So the representation of Γ on $L^2(\mathbb{H}, y^p \frac{dxdy}{y^2})$ is just an infinite number of copies of the regular representation and any geometric subspaces will have a von Neumann dimension.
In particular we can speak of $dim\ _{vN(\Gamma)}(A^2_{s-2})$. We have:

$$dim\ _{vN(\Gamma)}(A^2_{s-2}) = \frac{s-1}{4\pi}(\text{hyperbolic area of } \mathbb{H}/\Gamma)$$

(Full disclosure-one needs to be careful when s is not an even integer.)

Special case

$$dim_{\mathrm{vN}(PSL_2(\mathbb{Z})\ )}(A^2_{s-2}) = \frac{s-1}{12}$$

Note that if we set $s = 13$ we get

$$dim_{\mathrm{vN}(PSL_2(\mathbb{Z})\ )}(\mathcal{H}_{13}) = 1$$

(If Γ is the fundamental group of a compact Riemann surface of genus g the formula is

$$dim\ _{vN(\Gamma)}(A^2_{s-2}) = (s-1)(g-1))$$

What is the significance of $dim_M(\mathcal{H}) = 1$ for a II_1 factor M?
From the properties of the dimension function we know that this means that $\mathcal{H}$ is isomorphic as a left M-module to $L^2(M) = \ell^2(\Gamma)$. The map $\gamma \mapsto \gamma^{-1}$ extends to an involution J on $L^2(M)$ with the property that $JvN(\Gamma)J$ is the II_1 factor given by the *right* regular representation, which is the commutant of M: $JMJ = M'$.
Rephrasing this, if ξ is used to denote the identity of M as an element of $L^2(M)$, J is the map

$$J(x\xi) = x^*\xi$$

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But any other vector η could be used in the same way provided it is a TRACE VECTOR, that is to say

$$tr(x) = \langle x\eta, \eta\rangle$$

a property that is obvious for ξ. Just define another J, call it J_η by $J_\eta(x\eta) = x^*\eta$. Then it will still be true that J_η identifies M and M' via an antiisomorphism. Returning to our situation of $\mathcal{H}$ with $dim_M\mathcal{H} = 1$, there is no canonical identification of $\mathcal{H}$ with $L^2(M)$ nor hence of M with M', but the slogan is

"Identifications of M with M' (or $\mathcal{H}$ with $L^2(M)$) are the same thing as TRACE VECTORS."

And a) trace vector for M iff trace vector for M' if $dim_M\mathcal{H} = 1$,

b) for M on $\mathcal{H}$, there is a trace vector iff $dim_M(\mathcal{H}) \geq 1$.

We see that, for $PSL_2(\mathbb{Z})$, there is a trace vector for its action on the space $\mathcal{H}_{13}$ of L^2 holomorphic functions on $\mathbb{H}$.
Challenge-find one!!!
Our main theorem will be proved using a method of producing trace vectors! But that method will fail precisely when we want them the most-i.e. when the von Neumann dimension is equal to 1.

It's time to move on to the proof of the theorem, which is where the new idea comes in.

Theorem

Let Γ *be a torsion free Fuchsian group and* $\Gamma(z)$ *be an orbit in* $\mathbb{H}$*. Then*
(i)If $s \leq 1 + \dfrac{4\pi}{covolume(\Gamma)}$ *there is no non-zero function in* A^2_{s-2} *vanishing on* $\Gamma(z)$*.*
(ii)If $s > 1 + \dfrac{4\pi}{covolume(\Gamma)}$ *there is a non-zero function in* A^2_{s-2}*vanishing on* $\Gamma(z)$*.*

We already saw that cusp forms should provide a proof of part (ii). Now we are armed with von Neumann dimension let's prove it almost trivially as an existence theorem!
Note that condition (ii) is the same as $dim_{vN(\Gamma)} A^2_{s-2} > 1$.

(I would like to acknowledge frequent and fruitful conversations with Curt Mc Mullen on just about everything from here on.)

Bergman spaces are "reproducing kernel Hilbert spaces" which means that for every $z \in \mathbb{H}$ there is an $\epsilon_z \in A^2_{s-2}$ such that

$$\langle \xi, \epsilon_z \rangle = \xi(z)$$

Consider the closure of $vN(\Gamma)\epsilon_z$ for any z. This has von Neumann dimension at most 1 so therefore there is a non-zero vector ξ orthogonal to it. Since ϵ_z is the reproducing kernel ξ vanishes on $\Gamma(z)$.

THAT'S IT!!

Now to (i)If $s \leq 1 + \dfrac{4\pi}{covolume(\Gamma)}$ there is no non-zero function in A^2_{s-2}vanishing on $\Gamma(z)$.
We will have to use the hypothesis on Γ.
We will use another terminology for a trace vector for $vN(\Gamma)$.

Definition

If G is a group acting on a Hilbert space we call a vector ξ a *wandering* vector if

$$\langle g\xi, \xi \rangle = 0 \quad \forall g \in G, g \neq id$$

Obviously a wandering unit vector for Γ in A^2_{s-2} is a trace vector for $vN(\Gamma)$.

What we are going to do is construct a wandering vector from an element of A^2_{s-2} which vanishes on an orbit of Γ. For convenience suppose it is the orbit of 0 (in the disc picture).
Choose a left ordering $<$ of Γ and let $\eta \in A^2_{s-2}$ have a zero of order 1 (for simplicity only) at each point of the orbit of 0 under Γ.Define the closed subspaces V and W of A^2_{s-2} to be

$$V = \{\xi | \xi(\gamma(0)) = 0 \text{ for } \gamma \leq id\}$$

and

$$W = \{\xi | \xi(\gamma(0)) = 0 \text{ for } \gamma < id\}$$

η is in V so that W and V are non-zero. Write

$$\eta = \sum_{n=k}^{\infty} c_n e_n$$

where $e_n(z) = \sqrt{\frac{s-1}{4\pi}}\sqrt{\frac{s(s+1)....(s+n-1)}{n!}}z^n$ is an orthonormal basis. We know that c_n is square summable.

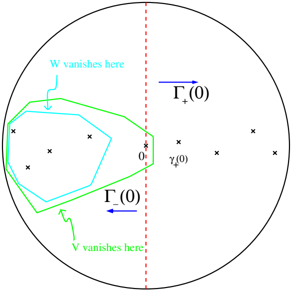
The limit of the sequence

$$a_n = \sqrt{\frac{s(s+1)....(s+n+k-1)}{s(s+1)....(s+n-1)} \frac{n!}{(n+k)!}}$$

is 1 so a_n is bounded. The holomorphic function $z^{-1}\eta(z)$ (weighted unilateral shift) has Taylor series

$$\sum_{n=0}^{\infty} c_{n+1} z^{-1} e_{n+1}(z) = \sum_{n=0}^{\infty} a_n c_{n+1} e_n(z)$$

Thus $z^{-1}\eta(z) \in A^2_{s-2}$, it vanishes on $\Gamma(0) \setminus \{0\}$ but is nonzero at 0. Thus V is strictly contained in W. (Recall $V = \{\xi | \xi(\gamma(0)) = 0 \text{ for } \gamma \leq id\}$ and $W = \{\xi | \xi(\gamma(0)) = 0 \text{ for } \gamma < id\}$.)

Schematic (very...)  We will now show that a

vector in the orthogonal complement $V^\perp \cap W$ of V in W is a wandering vector for Γ. For suppose $\xi \in V^\perp \cap W$. Then for $\gamma < id$ and any other $\lambda \leq id$,

$$\gamma\lambda \leq \gamma id = \gamma < id$$

$$\text{so} \qquad \pi_s(\gamma^{-1})\xi(\lambda(0)) = 0$$

which means that $\pi_s(\gamma^{-1})\xi \in V$ and thus

$$\langle \pi_s(\gamma^{-1})\xi, \xi \rangle = 0$$

which also means

$$\langle \pi_s(\gamma)\xi, \xi \rangle = 0$$

But any nonzero wandering vector for Γ is a trace vector for the von Neumann algebra it generates. Our von Neumann dimension calculation gave $dim_{\ vN(\Gamma)}(A^2_{s-2}) = \frac{s-1}{4\pi}$(hyperbolic area of $\mathbb{H}/\Gamma$) which is less than 1 under the hypothesis of the theorem. Thus such a trace vector cannot exist for $s < 1 + \dfrac{4\pi}{covolume(\Gamma)}$, hence no such η can exist. To treat the case $s = 1 + \dfrac{4\pi}{covolume(\Gamma)}$ we simply observe that in our construction of a wandering vector, the function vanishing on the orbit is orthogonal to the wandering vector ξ itself, hence all of its translates under Γ. Thus the von Neumann dimension of A^2_{s-2} is strictly greater than 1 which is the von Neumann dimension of the closed subspace spanned by $\Gamma(\xi)$.

This completes the proof of the theorem.

Here is an example showing that the torsion free hypothesis is necessary. Let G_2 be the Eisenstein series modular form for $\Gamma = PSL_2(\mathbb{Z})$ of (smallest) weight 4. Then $G_2(e^{\frac{\pi i}{3}}) = 0$ so G_2 vanishes on the Γ orbit of $e^{\frac{\pi i}{3}}$. Using the method of Rolen-Wagner, multiply G_2 by some branch of $\eta(z)^r$ for r real, small and positive. The resulting holomorphic function f will satisfy

$$|f(z)| \leq (constant) Im(z)^{-(2+r/4)}$$

and so defines by multiplication a bounded map from A^2_{s-2} to $\mathcal{H}_{s+4+r/2}$. So if s is slightly bigger than 1 we obtain elements of $\mathcal{H}_{4+\epsilon}$ vanishing exactly on the Γ orbit of $e^{\frac{\pi i}{3}}$ for all $\epsilon > 0$.

Here is the full theorem which takes torsion/fixed points into account:

Theorem

If Γ is any Fuchsian group and $O_1, O_2, \cdots, O_n$ are disjoint orbits in $\mathbb{D}$ of Γ. Then there is a non-zero function in A^2_{s-2} with a zero of order at least v_i on all points of O_i iff

$$s > 1 + \frac{4\pi}{covolume(\Gamma)} \sum_i \frac{v_i}{|stab_i|}$$

The proof goes much as in the simplified version. The order trick works across orbits, and multiplicity of zeros is handled by using "reproducing kernels" for point evaluation of derivatives of functions in A^2_{s-2}.

In a book by Hedenmalm Korenblum and Zhu on Bergman space, a density $D^+(S)$ called the "upper asymptotic κ-density" is defined for subsets S of the unit disc. It is shown on page 131 of that book that the condition $D^+(S) \leq \frac{1+\alpha}{p}$ is necessary and the condition $D^+(S) < \frac{1+\alpha}{p}$ is sufficient for A to be an A^p_α-zero set.

Corollary

If Γ and z are as in 4 then

$$D^+(\Gamma(z)) = \frac{2\pi}{covolume\ (\Gamma)}$$

An amusing corollary is that, in the case of $PSL_2(\mathbb{Z})$, one can compute the whole algebra of modular forms (of course this is elementary and not difficult to prove but this method is quite different in one aspect). The main thing is to prove that there is no cusp form of weight less than 12. For this one needs the Rolen-Wagner trick which I have not yet described so here it is.

For a $w \in \mathbb{H}$ for which $j(w)$ is not a fixed point for anything, start with $j(z) - j(w)$. This is $PSL_2(\mathbb{Z})$ invariant and vanishes on the orbit of w and has a q expansion with a simple pole at $q = 0$. This divergence can be "cured" by multiplying first by a putative cusp form f of weight less than 12, which kills the pole, then by some small positive real power of $\Delta(z)$. The absolute value of the resulting function satisfies the bound to make it a bounded map from A^2_{s-2} to $A^2_{s-2+(weight(f))+\epsilon}$. The image of this map consists of functions vanishing on the orbit of w.By choosing s slightly bigger than 1 one obtains a function in $\mathbb{H}_t$ vanishing on the orbit of w for $t < 13$ which is not allowed by the theorem. Note that this also gives us explicit functions vanishing on orbits for $s > 13$.

Note how the construction of trace vectors fails when we would most want it not to..... How can one get a trace vector when the von Neumann dimension is 1??

I have thought about this on and off for many years without success. But why would such a vector be so significant?

We have seen it would allow us to identify $vN(\Gamma)$with its commutant.

Which brings us to the beautiful result of Radulescu. We know that every pair (f,g) of cusp forms of the same weight gives us an element $M_f^* M_g$ of the commutant of $vN(\Gamma)$on A^2_{s-2}.

Theorem

(Radulescu) For any $s > 1$ the vector space of operators $M_f^* M_g$ is dense in the commutant $vN(\Gamma)$'.

Thus cusp forms give us a model for $vN(\Gamma)$'. If we had an explicit trace vector for $s = 13$ we would have an explicit identification of $vN(\Gamma)$and $vN(\Gamma)$'.

The commutant of $PSL_2(\mathbb{Z})$ also has a trace and one has

$$tr(M_f^* M_g) = \int_{\text{fundamental domain} F} f\bar{g} y^p \frac{dxdy}{y^2} \quad \text{(Petersson inner product)}$$

Thus a trace vector for $vN(\Gamma)$' is a holomorphic L^2 function ξ with

$$\langle M_f^* M_g \xi, \xi \rangle = \int_{\mathbb{H}} f\bar{g}(z)|\xi(z)|^2 y^s \frac{dxdy}{y^2} = \int_F f\bar{g}(z) \sum_{\gamma \in \Gamma} \frac{|\xi(\gamma(z))|^2}{|cz+d|^{2s}} y^s \frac{dxdy}{y^2}.$$

The series converges to a continuous function (Poincaré) which must be a constant times y^{-s} because of the abundance of cusp forms.

Trace vectors for M and M' are complementary except when the von Neumann dimension is 1 in which case they are the same notion. So we get the curious result:

Theorem

There is a function $\xi \in A^2_{s-2}$ *with* $\sum_{\gamma\in\Gamma} \frac{|\xi(\gamma(z))|^2}{|cz+d|^{2s}} = Im(z)^{-s}$ *iff* $s \leq 1 + \dfrac{4\pi}{covolume(\Gamma)}$. *Moreover if* $s = 1 + \dfrac{4\pi}{covolume(\Gamma)}$, *the condition is equivalent to* ξ *being a wandering vector for* Γ*!*

The dream.
We see that just the calculation of the von Neumann dimensions gives us an abstract identification of cusp forms and the group von Neumann algebra of $PSL_2(\mathbb{Z})$. But by results of Voiculescu (and Radulescu-Dykema) we know that there is a *random matrix* model for vN($PSL_2(\mathbb{Z})$). This establishes a direct connection between cusp forms and random matrices. This is a theorem. Many detailed statistical observations also show a connection but much remains unproven. (Keating ++). Unfortunately our connection is rather useless unless we can get an explicit map, i.e. a TRACE VECTOR.
Although having a trace vector would just be a start, the difficulty in finding one suggests that knowledge of one might reveal deep facts. We are tempted to call an explicit trace vector a "holy grail" vector.

History:
A particular instance of von Neumann dimension occurs if you have a II_1 *subfactor* $N \subseteq M$ of a II_1 factor. We can consider

$$[M : N] = dim_N(M)$$

In the early '80's it was shown that

$$[M : N] \in \{4cos^2\pi/n : n = 3, 4, 5, \cdots\} \cup [4, \infty]$$

exhibiting both discrete and continuous behaviour.

Hecke groups.
Here is a simple question about 2x2 matrices:

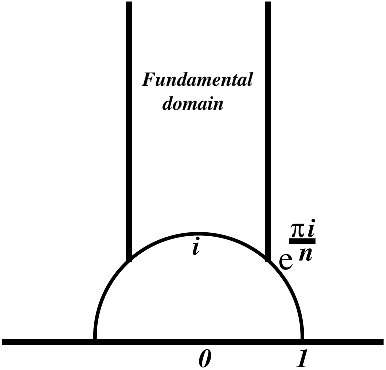
For what (positive real) values of λ do the two matrices $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ generate a discrete subgroup of $SL(2, \mathbb{R})$?

Answer (Fricke Klein) iff $\lambda \in \{2cos\pi/n : n = 3, 4, 5, \cdots\} \cup [2, \infty)$.

The identity of this set and the set of index values for subfactors has been an obsession of mine for almost 40 years. Is this a coincidence or is there a direct connection between the two results?

For instance, von Neumann algebras are a part of representation theory so there should be a representation theoretic construction of the subfactors. The Hecke groups scream out as an obvious place to look!

Note that the case $n = 3$ i.e. $\lambda = 1$ is the usual group $SL(2, \mathbb{Z})$.
In passing here is a picture of a fundamental domain for the action on the upper half plane $\mathbb{H}$:



In general a discrete subgroup of $SL(2, \mathbb{R})$ is called a Fuchsian group though sometimes the term also implies finite covolume. The Hecke groups have finite covolume iff $\lambda \leq 2$. A "discrete series".

The attempt to construct subfactors of the correct indices using Hecke groups has so far been a failure.

A representation theoretic construction of the subfactors was achieved by Wassermann in the 1990's. But the groups involved are loop groups rather than discrete or locally compact groups.
In the last year or so I have returned to this question, motivated by the desire to construct more subfactors and their attendant bimodule categories from more general Fuchsian groups.
What makes the problem rather difficult from the start is the fact that the most significant Fuchsian/Hecke group is none other than $PSL(2, \mathbb{Z})$ which corresponds to a subfactor of index 1...... No clues there.
The subfactor construction remains as elusive as ever but there has been spinoff, which could end up being far more significant than the original problem.

To conclude, one might have hoped to obtain the holy grail vector by the left orderable group method. But it has been thwarted since the value we are interested in is the critical value ($s = 13$ in the case of $PSL_2(\mathbb{Z})$) and we have shown that there is no element of Bergman space vanishing on the orbit. Thus we know by abstract von Neumann nonsense that there is a trace vector for the critical value of s but we are unable to lay our hands on one!