Some Inequalities in Locally Compact Quantum Groups

Part of Quantum Fourier Analysis

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Quantum Fourier Analysis

Symmetries	Algebras	Dimensions	Measures
Subfactors	Noncommutative	Finite	Tracial
Fusion Ring	Comm & Noncomm	Finite	Tracial
Groups	Comm & Noncomm	Infinite	Tracial
LCQG	Noncommutative	Infinite	Non-tracial

Locally Compact Quantum Groups: Definition[Kustermans-Vaes 00,03]

A locally compact quantum group $\mathbb{G} = (\mathcal{M}, \Delta, \varphi, \psi)$ consists of

(1) A von Neumann algebra $\mathcal{M}$;

(2) A comultiplication $\Delta : \mathcal{M} \to \mathcal{M}\overline{\otimes}\mathcal{M}$ is a unital normal *-homomorphism satisfying

$$(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta,$$

where $\iota : \mathcal{M} \to \mathcal{M}$ is the identity.

(3) A left Haar weight φ and a right Haar weight ψ on $\mathcal{M}$:

$$(\iota \otimes \varphi)\Delta(x) = \varphi(x)1, \quad (\psi \otimes \iota)\Delta(x) = \psi(x)1;$$

Special Cases

1. $\mathbb{G}$ is a locally compact group if $\mathcal{M}$ is abelian.
2. $\mathbb{G}$ is unimodular if $\varphi = \psi$.
3. $\mathbb{G}$ is a unimodular Kac algebra if $\varphi = \psi$ is tracial. [Kac Vainermann , Enock Schwartz 70s]
4. $\mathbb{G}$ is a compact quantum group if $\varphi = \psi$ is a state. [Woronowicz 92]
5. $\mathbb{G}$ is a discrete quantum group if $\widehat{\mathbb{G}}$ is a compact quantum group. [van Daele 96]
6. Compact Matrix Quantum Groups [Woronowicz 87] : Quantum SU_q, Quantum Permutation Groups [Wang 98] etc.
7. Quantum E(2) Group [Woronowicz 91], Quantum $ax + b$ Group [Woronowicz Zakrzewski 02] , Quantum $az + b$ Group [Woronowicz 01, Soltan 05] etc.
8. Matrix Algebras, Quantum Torus are not LCQGs,

Notations and Setups (Tomita-Takesaki Theory is involved)

We begin with the Gelfand-Naimark-Segal (semi-cycle) construction:

(1) $\mathfrak{N}_\varphi = \{x \in \mathcal{M} : \varphi(x^*x) < \infty\}$

(2) $\mathcal{H}_\varphi = \overline{\mathfrak{N}_\varphi}^{\langle,\rangle}$ is the underlying Hilbert space;

(3) $\Lambda_\varphi : \mathfrak{N}_\varphi \to \mathcal{H}_\varphi$ is the inclusion map.

(4) $\sigma_t^\varphi, \sigma_t^\psi$ are modular automorphisms;

(5) δ is the modular element and $\psi = \varphi_\delta$ (formally $\psi = \varphi(\delta^{1/2} \cdot \delta^{1/2})$);

(6) J_φ is the modular conjugation;

(7) ∇_φ is the modular operator.

Notations and Setups: Multiplicative Unitary

(1) The multiplicative unitary $W \in \mathcal{B}(\mathcal{H}_\varphi \otimes \mathcal{H}_\varphi)$:

$$W^*(\Lambda_\varphi(x) \otimes \Lambda_\varphi(y)) = (\Lambda_\varphi \otimes \Lambda_\varphi)(\Delta(y)x \otimes 1).$$

(2) $\Delta(x) = W^*(1 \otimes x)W$ for any $x \in \mathcal{M}$.

(3) $\widehat{\mathbb{G}}$ the dual locally compact quantum group $(\widehat{\mathcal{M}}, \widehat{\Delta}, \hat{\varphi}, \hat{\psi})$

- $\widehat{\mathcal{M}} = \overline{\{(\omega \otimes \iota)W : \omega \in \mathcal{B}(\mathcal{H}_\varphi)_*\}}^{SOT}$
- $\widehat{\Delta} = \widehat{W}^* \cdot \widehat{W}$, where $\widehat{W} = \Sigma W^* \Sigma$, Σ flips the tensor.
- $\hat{\varphi}$ satisfies $\widehat{\Lambda}((x\varphi \otimes \iota)W) = \Lambda_\varphi(x)$, $x \in \mathfrak{N}_\varphi$.
- $\hat{J}, \widehat{\nabla}$ modular conjugation and operator.
- $\hat{\psi}$ will be given later.

(4) $W \in \mathcal{M} \overline{\otimes} \widehat{\mathcal{M}}$.

(5) If G is a group, $(Wf)(t,s) = f(ts)$ for any $t, s \in G$. $(\mathcal{M} = L^\infty(G))$

It is critical to construct a multiplicative unitary to obtain a quantum group or a hopf algebra !

Notations and Setups: Antipode

(1) S is antipode:

$$S((\iota\otimes\varphi)(\Delta(x^*)(1\otimes y))) = (\iota\otimes\varphi)((1\otimes x^*)\Delta(y))$$
$$S(\iota\otimes\omega)W = (\iota\otimes\omega)W^*$$

(2) Polar decomposition $S = R\tau_{-i/2}$, where R is the unitary antipode, τ is the scaling automorphism.

(3) ψ is taken to be φR

(4) $\widehat{R}(\lambda(\omega)) = \lambda(\omega R)$, where $\lambda(\omega) = (\omega\otimes\iota)W$

(5) $\widehat{\tau}_t(\lambda(\omega)) = \lambda(\omega\tau_t)$

(6) $\hat{S} = \widehat{R}\hat{\tau}_{-i/2}$

(7) $\hat{\psi}$ is taken to be $\hat{\varphi}\widehat{R}$.

- $\mathbb{G}$ is a Kac algebra if τ is trivial and δ is affiliated with the center.

Notations and Setups

(1) An automorphism group ρ_t:

$$\rho_t(\omega)(x) = \omega(\delta^{-it}\tau_{-t}(x)), \quad \omega \in \mathcal{M}_*$$

(2) $\tau_t = \widehat{\nabla}^{it} \cdot \widehat{\nabla}^{-it}$.

(3) $\hat{\tau}_t = \nabla_\varphi^{it} \cdot \nabla_\varphi^{-it}$.

(4) $\sigma_t^\psi = \delta^{it}\sigma_t^\varphi\delta^{-it}$.

(5) $(\widehat{\nabla}^{it} \otimes \nabla_\varphi^{it})W = W(\widehat{\nabla}^{it} \otimes \nabla_\varphi^{it})$.

(6) $\hat{\sigma}_t(\lambda(\omega)) = \lambda(\rho_t(\omega))$.

Noncommutative L^p Space

- Interpolated L^p Space (Izumi's L^p Space) [Terp 82, Izumi 97]
- Spatial L^p Space (Hilsum's L^p Space) [Connes , Hilsum 81]
- Haagerup's L^p Space [Haagerup 79, Terp 81, HJX 08]

Interpolated L^p Space

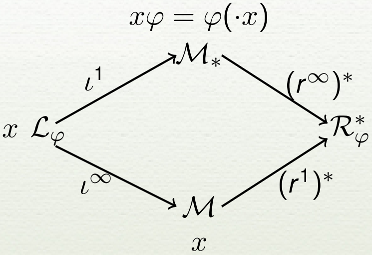

Figure: Embedding with parameter $z = -1/2$

Interpolated L^p Space

By Complex Interpolation Method, we have

(1) $L^1(\mathbb{G}) = \mathcal{M}_*, \ L^\infty(\mathbb{G}) = \mathcal{M};$

(2) $L^1(\mathbb{G}) \cap L^\infty(\mathbb{G}) = \mathcal{L}_\varphi;$

(3) $L^p(\mathbb{G}) = (L^1(\mathbb{G}), \ L^\infty(\mathbb{G}))_{[1/p]};$

(4) $L^2(\mathbb{G}) = \mathcal{H}_\varphi;$

(5) $L^2(\mathbb{G}) \cap L^\infty(\mathbb{G}) = \mathfrak{N}_\varphi;$

(6) $L^1(\mathbb{G}) \cap L^2(\mathbb{G}) = \mathcal{I}_\varphi = \{\omega \in L^1(\mathbb{G}) : |\omega(x^*)| \leq C\|\Lambda_\varphi(x)\|\}$

(7) $\xi_t : L^1(\mathbb{G}) \cap L^t(\mathbb{G}) \to L^t(\mathbb{G});$

(8) $\iota^t : L^\infty(\mathbb{G}) \cap L^t(\mathbb{G}) \to L^t(\mathbb{G});$

Spatial L^p Space

Let ϕ be a normal semifinite faithful weight on the commutant $\mathcal{M}'$ acting on $\mathcal{H}_\varphi$.

(1) Spatial derivative: $d_\varphi = \dfrac{d\varphi}{d\phi}$.

(2) t-Homogeneous operators: $ax \subseteq x\sigma_{it}^{\phi}(a)$, where $a \in \mathcal{M}'$.

(3) Measure: $\displaystyle\int |x|^p d\phi$ for $-\frac{1}{p}$-homogeneous x, $\displaystyle\int \frac{d\omega}{d\phi} d\phi = \omega(1)$;

(4) $L^p(\phi) = \left\{ x : -\dfrac{1}{p}\text{-homogeneous and } \displaystyle\int |x|^p d\phi < \infty \right\}$.

(5) Dense subset: $\left\{ x d_\varphi^{1/p} : \text{ some proper } x \in \mathcal{M} \right\}$;

(6) $\Phi_p : L^p(\mathbb{G}) \to L^p(\phi)$ is the isometric isomorphism satisfying

$$\Phi_p(\xi_p(x\varphi)) = x d_\varphi^{1/p}$$

Haagerup's L^p Space

(1) $\mathcal{M} \rtimes_\sigma^\varphi \mathbb{R} (= \mathcal{N})$ the cross product von Neumann algebra with tracial weight τ. Let θ be the dual action of $\mathbb{R}$ in $\mathcal{N}$.

(2) τ-measurable operators: complete Hausdorff topological space, an algebra w. r. t. strong product and strong sum.

(3) $L_h^p(\mathcal{M}) = \{x : \tau\text{-measurable}, \theta_s(x) = e^{-s/p}x, s \in \mathbb{R}\}$

(4) Measure: $tr(x) = \tau(\chi_{(1,\infty)}(|x|^p))$.

(5) There is an isometric isomorphism between Haagerup's L^p space and Hilsum's L^p space.

We shall identify the three noncommutative L^p spaces properly.

Hölder's Inequality

For any $x \in L^t(\phi), y \in L^s(\phi)$,

$$\|xy\|_{r,\phi} \leq \|x\|_{t,\phi}\|y\|_{s,\phi}, \quad \frac{1}{r} = \frac{1}{t} + \frac{1}{s}.$$

Moreover $\|xy\|_{r,\phi} = \|x\|_{t,\phi}\|y\|_{s,\phi}$ if and only if $\dfrac{|x|^t}{\|x\|_{t,\phi}^t} = \dfrac{|y^*|^s}{\|y\|_{s,\phi}^s}$

Fourier Transform [van Daele 07]

1. Fourier transform: $\mathcal{F}_t : L^t(\phi) \to L^{t'}(\hat{\phi}), \frac{1}{t} + \frac{1}{t'} = 1, 1 \leq t \leq 2$ satisfying

$$\mathcal{F}_t(\Phi_t^{-1}(\xi_t(\omega))) = \hat{\Phi}_{t'}^{-1}(\iota^{t'}(\lambda(\omega))), \quad \omega \in L^1(\mathbb{G}) \cap L^t(\mathbb{G})$$

$$\mathcal{F}_t(x) = \int W(x d_\varphi^{1/t'} \otimes \hat{d}_{\hat{\varphi}}^{1/t'}) d\phi \otimes \iota, \quad x \in L^t(\phi)$$

2. Plancherel's formula: $\|\mathcal{F}_2(x)\|_{2,\hat{\phi}} = \|x\|_{2,\phi}$.
3. $\int \mathcal{F}_t(x) y^* d\phi = \int x \widehat{\mathcal{F}}_t(y)^* d\phi$, where $\widehat{\mathcal{F}}_t$ is the Fourier transform from $L^t(\hat{\phi})$ to $L^{t'}(\phi)$.
4. Hausdorff-Young inequality: [Cooney 10, Caspers 12]

$$\|\mathcal{F}_t(x)\|_{t',\hat{\phi}} \leq \|x\|_{t,\phi}$$

Bi-shifts of Group-like Projection [LWW 17, JLW 18]

(1) Group-like projection B is a projection and

$$\Delta(B)(1 \otimes B) = B \otimes B, \quad B \neq 0$$

(2) Biprojection B is a projection and $\lambda(B\varphi)$ is a multiple of a projection.

(3) Left shift of group-like projection: A projection B_g is a left shift of a group-like projection B if $\varphi(B_g) = \varphi(B)$ and

$$\Delta(B_g)(1 \otimes B) = B_g \otimes B, \quad \Delta(B)(1 \otimes B_g) = R(B_g) \otimes B_g.$$

(4) Bi-shift of group-like projection $x \in L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$ such that

$$x\varphi = (yB_g\varphi) * (\hat{\lambda}(\tilde{B}_h\hat{\varphi})\varphi),$$

where B_g is a left shift of a group-like projection B, $\tilde{B}_h$ is a left shift of the group-like projection $\tilde{B} = \mathcal{R}(\lambda(B\varphi))$.

It acts like translation and modulation of open compact subgroups.

Hausdorff-Young Inequality [LW 17, W 20]

Suppose $\mathbb{G}$ is a locally compact quantum group. The following are equivalent:

(1) $\|\mathcal{F}_t(x)\|_{\frac{t}{t-1},\hat{\phi}} = \|x\|_{t,\phi}$ for some $1 < t < 2$;

(2) $\|\mathcal{F}_t(x)\|_{\frac{t}{t-1},\hat{\phi}} = \|x\|_{t,\phi}$ for all $1 \leq t \leq 2$;

(3) x is a bi-shift of a group-like projection.

This result was only proved for locally compact groups but not for their duals. [Russo 1974, Fournier 1977]

$$C_t = \sup_{0 \neq x \in L^t(\phi)} \frac{\|\mathcal{F}_t(x)\|_{\frac{t}{t-1},\hat{\phi}}}{\|x\|_{t,\phi}} \leq 1.$$

Convolutions

$$(\omega_1 \otimes \omega_2)\Delta, \quad (\omega \otimes \iota)\Delta(a), \quad (\iota \otimes \omega)\Delta(a)$$

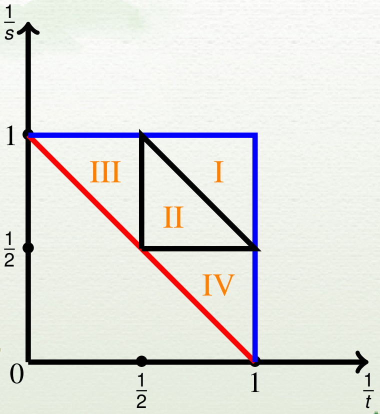

Figure: **The Regions of Young's inequality**

Convolutions

(1) [LWW 17] $\frac{1}{r} + 1 = \frac{1}{t} + \frac{1}{s}$, $1 \leq r, t, s \leq 2$, i.e. Region "I", the convolution is defined as

$$x * \rho_{-i/t'}(y) \in L^r(\phi), \quad \frac{1}{t} + \frac{1}{t'} = 1.$$

(2) Fourier Transform Interchanges Convolution and Product
$\mathcal{F}_r(x * \rho_{-i/t'}(y)) = \mathcal{F}_t(x)\mathcal{F}_s(y)$.

(3) [JLW 18] $\frac{1}{r} + 1 = \frac{1}{t} + \frac{1}{s}$, $1 \leq t, s \leq 2, r \geq 2$, i.e. Region "II", the convolution is defined as

$$\widehat{\mathcal{F}}_{r'}(\mathcal{F}_t(x)\mathcal{F}_s(y)) \in L^r(\phi), \quad \frac{1}{r} + \frac{1}{r'} = 1.$$

Convolutions

Pull back of Involution # (hash):

1. For $1 \leq t \leq 2$, $\mathcal{F}_t(x^{\#}) = \mathcal{F}_t(x)^*, \quad x \in L^t(\phi)$.
2. For $2 \leq t \leq \infty$, $x^{\#} = \widehat{\mathcal{F}}_{t'}(x_0^*), \quad$ where $x = \widehat{\mathcal{F}}_{t'}(x_0)$ for some $x_0 \in L^{t'}(\hat{\phi})$.

Some Facts:

1. $x^{*\#} \neq x^{\#*}$
2. $(xy)^{\#} \neq x^{\#}y^{\#}(\neq y^{\#}x^{\#})$
3. $(x \star y)^{\#} = x^{\#} \star y^{\#}$
4. $(x^{\#})^{\#} = x$
5. $\int x^* y^{\#} d\phi = \int x^{\#} y^* d\phi$

Convolutions [W 20]

$$x \star y = \begin{cases} x * \rho_{-i/t'}(y) & (1/t, 1/s) \text{ in the Region “I"} \\ \widehat{\mathcal{F}}_{r'}(\mathcal{F}_t(x)\mathcal{F}_s(y)) & (1/t, 1/s) \text{ in the Region “II"} \\ x * \rho_{-i/t'}(y^{*\#}) & (1/t, 1/s) \text{ in the Region “III"} \\ x^{*\#} * \rho_{-i/t'}(y) & (1/t, 1/s) \text{ in the Region “IV"} \end{cases}$$

$$\left(\frac{d\omega}{d\phi}\right)^{*\#} * a = (\omega \otimes \iota)\Delta(a), \quad a * \rho_{-i}\left(\frac{d\omega}{d\phi}\right)^{*\#} = (\iota \otimes \omega)\Delta(a).$$

Young's Inequality [LWW 17, JLW 18, W 20]

For $(1/t, 1/s)$ in the Region "I", we have

$$\|x * \rho_{-i/t'}(y)\|_{r,\phi} \leq \|x\|_{t,\phi}\|y\|_{s,\phi}.$$

For $(1/t, 1/s)$ in the Region "II", we have

$$\left\|\widehat{\mathcal{F}}_{r'}(\mathcal{F}_t(x)\mathcal{F}_s(y))\right\|_{r,\phi} \leq \|x\|_{t,\phi}\|y\|_{s,\phi}$$

For $(1/t, 1/s)$ in the Region "III", we have

$$\left\|x * \rho_{-i/t'}(y^{*\#})\right\|_{r,\phi} \leq \|x\|_{t,\phi}\|y\|_{s,\phi}.$$

For $(1/t, 1/s)$ in the Region "IV", we have

$$\left\|x^{*\#} * \rho_{-i/t'}(y)\right\|_{r,\phi} \leq \|x\|_{t,\phi}\|y\|_{s,\phi}.$$

Young's Inequality [LWW 17, W 20]

Let $\mathbb{G}$ be a unimodular locally compact quantum group.
For $(1/t, 1/s)$ in the Region "IV",

$$\left\|\tau_{i/t-i/2}(x) * \tau_{-i/t'}(y)\right\|_{r,\phi} \leq \|x\|_{t,\phi}\|y\|_{s,\phi},$$

For $(1/t, 1/s)$ in the Region "III",

$$\left\|x * \tau_{i/r-i/2}(y)\right\|_{r,\phi} \leq \|x\|_{t,\phi}\|y\|_{s,\phi}.$$

Let $\mathbb{G}$ be a Kac algebra.

$$\left\|x * y\delta^{-1/t'}\right\|_{r,\phi} \leq \|x\|_{t,\phi}\|y\|_{s,\phi}.$$

Young's Inequality [LW 18, W 20]

Suppose $\mathbb{G}$ is a locally compact quantum group. Then the following are equivalent:

(1) $\|x \star y\|_{r,\phi} = \|x\|_{t,\phi}\|y\|_{s,\phi}$ for some $1 < r, t, s < \infty$ such that $\frac{1}{r} + 1 = \frac{1}{t} + \frac{1}{s}$;

(2) $\|x \star y\|_{r,\phi} = \|x\|_{t,\phi}\|y\|_{s,\phi}$ for all $1 < r, t, s < \infty$ such that $\frac{1}{r} + 1 = \frac{1}{t} + \frac{1}{s}$;

(3) x, y are bi-shifts of group-like projections satisfying certain conditions.

Suppose $\mathbb{G}$ is a unimodular Kac algebra, we have

$$\|x * y\|_r \leq \||x| * |y|\|_r^{1/2} \, \||x^*| * |y^*|\|_r^{1/2}.$$

Young's Inequality [LW 18, W 20]

$$C_{t,s} = \sup_{0\neq x\in L^t(\phi), 0\neq y\in L^s(\phi)} \frac{\|x\star y\|_{r,\phi}}{\|x\|_{t,\phi}\|y\|_{s,\phi}} \leq 1$$

(1) $C_{1,1} = C_{1,\infty} = C_{\infty,1} = 1$.

(2) $C_{t,s} = C_{t,r'} = C_{r',s}$, where $\frac{1}{r'} + \frac{1}{t} + \frac{1}{s} = 2$.

(3) $C_{t,s} \leq C_t C_s C_{r'}$ when $(1/t, 1/s)$ is in the Region "II".

Entropic Convolution Inequality [W 20]

For any $\omega \in L^1(\mathbb{G})$, the entropy $H_\phi\left(\frac{d\omega}{d\phi}\right)$ is defined as

$$H_\phi\left(\frac{d\omega}{d\phi}\right) = -\int \frac{d\omega}{d\phi}\left(\ln \frac{d\omega}{d\phi} - \ln d_\varphi\right) d\phi.$$

Suppose that $\omega_2\tau_t = \omega_2$, $0 \leq \theta \leq 1$. Then

$$H_\phi\left(\frac{d\omega_1}{d\phi} * \frac{d\omega_2}{d\phi}\right)$$

$$\geq (1-\theta)H_\phi\left(\frac{d\omega_1}{d\phi}\right) + \theta H_\phi\left(\frac{d\omega_2}{d\phi}\right) + (1-\theta)\omega_2(\ln \delta).$$

The extremizers of the inequality for unimodular Kac algebras are left shifts of group-like projections.

Donoho-Stark Uncertainty Principle [JLW 18]

φ-value of support projection:

$$\mathcal{S}(x) = \varphi(\mathcal{R}(\Phi_t^{-1}(x)^*))$$

where $\mathcal{R}$ is taking the range projection.

Suppose $\mathbb{G}$ is a locally compact quantum group.
Then for any ω in $L^1(\mathbb{G}) \cap L^2(\mathbb{G})$, $1 \leq t \leq 2$, $2 \leq s \leq \infty$, we have

$$\mathcal{S}(\xi_t(\omega))\mathcal{S}(\iota^s(\lambda(\omega))) \geq 1.$$

Moreover for any $x \in L^t(\phi)$

$$\mathcal{S}(x)\mathcal{S}(\mathcal{F}_t(x)) \geq 1.$$

Donoho-Stark uncertainty principle [JLW 18]

The following are equivalent:

1. $\omega \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ is a minimizer.
2. ω is an extremal bi-partial isometry such that $|\omega|\sigma_t^\varphi = |\omega|$, $\hat{\sigma}_t(|\lambda(\omega)|) = |\lambda(\omega)|$, $\forall t \in \mathbb{R}$.
3. ω is a bi-partial isometry, $|\omega|\sigma_t^\varphi = |\omega|$, $\forall t \in \mathbb{R}$, and $\lambda(\omega)$ is in $L^1(\hat{\mathbb{G}})$ such that $\|\hat{\lambda}(\lambda(\omega)\hat{\varphi})\|_\infty = \|\lambda(\omega)\hat{\varphi}\|$.
4. $\omega \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ satisfies that $\mathcal{S}(\omega)\mathcal{S}(\lambda(\omega)) = 1$ and $\hat{\sigma}_t(|\lambda(\omega)|) = |\lambda(\omega)|$.
5. $\omega \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ satisfies that

$$\mathcal{S}(\omega)\mathcal{S}(\lambda(\omega)) = 1, \quad \mathcal{S}(\xi(\omega))\mathcal{S}(\hat{\Lambda}(\lambda(\omega))) = 1.$$

6. ω is a bi-shift of a group-like projection $B \in L^1(\mathbb{G})$.

Hirschman-Beckner uncertainty principle [JLW 18, W 19]

Entropy for L^2 space

$$H_0(\xi) = -\langle (\log |\Phi_2^{-1}(\xi)|^2 - \log d_\varphi) J_\varphi \xi, J_\varphi \xi \rangle.$$

Suppose $\mathbb{G}$ is a locally compact quantum group and $\|\xi\| = 1$.

$$H_0(\xi) + H_0(\mathcal{F}_2(\xi)) \geq 0.$$

The following are equivalent:

(1) ω is a minimizer of Donoho-Stark uncertainty principle;

(2) ω is a minimizer of Hirschman-Beckner uncertainty principle;

(3) ω is a bi-shift of a group-like projection.

Hardy's uncertainty principle [JLW 18]

Suppose $\mathbb{G}$ is a locally compact quantum group with a bi-shift w of a group-like projection. Let $x \in L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$ be such that

$$|x^*| \leq C|w^*|, \quad |\lambda(x\varphi)| \leq C'|\lambda(w\varphi)|,$$

for some $C, C' > 0$.

Then x is a multiple of w.

Sum Set Estimate [LW 17]

Suppose $\mathbb{G}$ is a unimodular Kac algebra with a Haar tracial weight φ. Let p, q be projections in $L^\infty(\mathbb{G})$. Then

$$\max\{\varphi(p), \varphi(q)\} \leq \mathcal{S}(p * q).$$

The following are equivalent:

(1) $\mathcal{S}(p * q) = \varphi(p) < \infty$;

(2) $\varphi(q)^{-1} p * q$ is a projection in $L^1(\mathbb{G})$

(3) $\mathcal{S}(p * (q * R(q)^{*(m)}) * q^{*(j)}) = \varphi(p)$ for some $m \geq 0, j \in \{0, 1\},\ m + j > 0, q^{*(0)}$ means q vanishes.

(4) there exists a biprojection B such that q is a right subshift of B and $p = \mathcal{R}(x * B)$ for some $x > 0$.

Nikoski's Inequality [W 20]

Let $\mathbb{G}$ be a locally compact quantum group. Suppose that $1 \leq s \leq \infty$ and $1 \leq t < \min\{2, s\}$ and $\omega \in L^t(\mathbb{G}) \cap L^s(\mathbb{G})$. Assume that $\widehat{\mathcal{S}}(\xi_t(\omega)) = \mathcal{S}(\mathcal{F}_t(\xi_t(\omega))) < \infty$. Then

$$\|\xi_s(\omega)\|_s \leq \widehat{\mathcal{S}}(\xi(\omega))^{\frac{1}{t}-\frac{1}{s}} \|\xi_t(\omega)\|_t.$$

Moreover, if $t \neq s$, $s \neq \infty$, then $\|\xi_s(\omega)\|_s = \widehat{\mathcal{S}}(\xi(\omega))^{\frac{1}{t}-\frac{1}{s}} \|\xi_t(\omega)\|_t$ if and only if ω is a bi-shift of a group-like projection.

This is proved for locally compact groups in 2019 and earlier by Nikoski for compact groups.

Positivity

(I) **Schur Product Theorem** Let $\mathbb{G}$ be a Kac algebra. Suppose that $x \in L^t(\phi), y \in L^s(\phi)$ such that $x\nabla_\varphi^{1/t'} \geq 0$ and $y\nabla_\varphi^{-1/t'} \geq 0$,

$$x * y\delta^{-1/t'} \geq 0.$$

(II) Let $\mathbb{G}$ be a locally compact quantum group. An element $x \in L^t(\phi)$ is positive definite if

(1) $\mathcal{F}_t(x) \geq 0$ when $1 \leq t \leq 2$.

(2) $\int (y^\# * \rho_{-i/2t}(y))\, x^* d\phi \geq 0$ for any $y \in L_\#^{\frac{2t}{2t-1}}(\phi)$ when $2 \leq t \leq \infty$.

More

(1) Positive definite [Daws Salmi 13, Runde Viselter 14]

(2) Amenability [Many]

(3) Idempotent states [Salmi, Pal, Franz, Skalski etc]

(4) Sharp Hausdorff-Young inequality (Lie groups) [Klein Russo 78, Cowling Martini Muller Parcet 19]

(5) Sidon sets (compact quantum groups) [Wang 16]

(6) ....

Questions

(1) Find the norm of Fourier transform for locally compact quantum groups.
(2) Find the best constant C_t for proper locally compact quantum groups.
(3) Is it true that $C_t = \hat{C}_t$, where $\hat{C}_t$ is the best constant for the dual?
(4) Find the best constant $C_{t,s}$ for proper locally compact quantum groups.
(5) Is it true that $C_{t,s} = \hat{C}_{t,s}$, where $\hat{C}_{t,s}$ is the best constant for the dual?
(6) Is it true that $C_{t,s} = C_t C_s C_{r'}$?
(7) Find the reverse Young inequality.
(8) Prove the fractional Young inequality.

Questions

(9) Find the formulation of the Brascamp-Lieb inequalities which generalize Young's inequality.

(10) Find the right formulation of sumset estimation for locally compact quantum groups.

(11) Are the minimizers of entropy convolution inequality left shift of group-like projection?

(12) Does the amenability imply the co-amenability of the dual?

(13) Suppose $x \in L^t(\phi)$ is positive definite, where $t > 2$. Is there $x_0 > 0$ such that $\widehat{\mathcal{F}}_{t'}(x_0) = x$?

(14) Fourier multipliers on locally compact quantum groups? https://www.researchgate.net/project/Fourier-multipliers-on-quantum-groups-and-noncommutative-Lp-spaces

(15) Perturbation theory?

(16) Relation between Fourier transform and (quantum) differentiation?

Thank You for Attention!