The information in a wave

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Alice and Bob

Cartoon by John Richardson in Physics World, March 1998

Quantum mechanics is probabilistic!

The information carried by a classical wave

Suppose that Alice encodes and sends information by an undulatory signal, what information can Bob get by the wave packet in a given region at later time?

By a wave (or wave packet), we mean a real solution of the Klein-Gordon equation

$$(\Box + m^2)\Phi = 0 \ ,$$

with compactly supported, smooth Cauchy data $\Phi|_{x^0=0}$, $\Phi'|_{x^0=0}$.

Classical field theory describes Φ by the stress-energy tensor $T_{\mu\nu}$, that provides the energy-momentum density of Φ at any time.

But, how to define the information, or entropy, carried by Φ in a given region at a given time?

We give a classical answer to such a classical question by Operator Algebras and Quantum Field Theory

Energy conditions

In classical physics, particularly general relativity, there are various mathematical conditions that express the physical idea that energy is positive.

Energy conditions are often used proofs of various important theorems about black holes, such as the no hair theorem or the laws of black hole thermodynamics, singularity theorems, etc.

For example the Null Energy Condition states that

$$T_{ab}k^a k^b \geq 0$$

for every future-pointing null vector field k. Here T is the stress-energy tensor

(Non)-positivity of local energy in QFT

In Quantum Field Theory local energy cannot be positive: If $f \geq 0$ is a smooth function with compact support then

$$T_{00}(f) \equiv \int f(x) T_{00}(x) dx \ngeq 0$$

(Epstein, Glaser, Jaffe '65).

Proof. $T_{00}(f)$ is local, if $T_{00}(f) \geq 0$ then

$$||T_{00}(f)^{1/2}\mathrm{vac}||^2 = (\mathrm{vac}, T_{00}(f)\mathrm{vac}) = 0 \implies T_{00}(f) = 0$$

by Reeh-Schlieder theorem

Quantum energy local lower bounds

In Quantum Field Theory local energy is however lower bounded. For example in Conformal Chiral QFT

$$T_{00}(f) \geq const_f$$

for any f is a non-negative smooth function with compact support

$$T_{00}(f) \not\geq 0$$

(Fewster, Hollands; Weiner).

Indeed, the Fourier modes of T are the Virasoro algebra generators, so we have a projective representation of $\mathrm{Diff}(S^1)$, hence $T(f)$ is shifted by a constant.

Thermal equilibrium states

A primary role in thermodynamics is played by the equilibrium distribution.

Gibbs states:

Finite quantum system: $\mathfrak{A}$ matrix algebra with Hamiltonian H and evolution $\tau_t = \mathrm{Ad}e^{itH}$. Equilibrium state φ at inverse temperature β is given by the Gibbs property

$$\varphi(X) = \frac{\mathrm{Tr}(e^{-\beta H}X)}{\mathrm{Tr}(e^{-\beta H})}$$

What are the equilibrium states at infinite volume where there is no trace, no inner Hamiltonian?

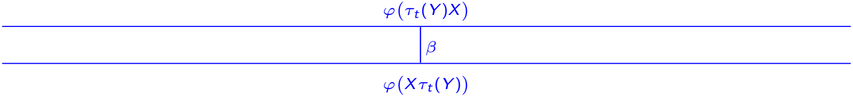
KMS equilibrium condition (Haag-Hugenoltz-Winnink)

Infinite volume. $\mathfrak{A}$ a C^*-algebra, τ a one-par. automorphism group of $\mathfrak{A}$. A state φ of $\mathfrak{A}$ is KMS at inverse temperature $\beta > 0$ if for $X, Y \in \mathfrak{A}$ $\exists$ function F_{XY} s.t.

(a) $F_{XY}(t) = \varphi\big(X\tau_t(Y)\big)$

(b) $F_{XY}(t + i\beta) = \varphi\big(\tau_t(Y)X\big)$

F_{XY} bounded analytic on $S_\beta = \{0 < \Im z < \beta\}$, continuous on $\bar{S}_\beta$



KMS states generalise Gibbs states, equilibrium condition for infinite systems

Tomita-Takesaki modular theory

$\mathcal{M}$ be a von Neumann algebra on $\mathcal{H}$, $\varphi = (\Omega, \cdot\Omega)$ normal faithful state on $\mathcal{M}$. Embed $\mathcal{M}$ linearly into $\mathcal{H}$ (Ω cyclic)

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow[\text{isometric}]{X \mapsto X^*} & \mathcal{M} \\ {\scriptstyle X \to X\Omega}\Big\downarrow & & \Big\downarrow{\scriptstyle X \to X\Omega} \\ \mathcal{H} & \xrightarrow[\text{non isometric}]{S_0 : X\Omega \mapsto X^*\Omega} & \mathcal{H} \end{array}$$

$S = \bar{S}_0$, $\Delta = S^* S > 0$ positive selfadjoint

$$t \in \mathbb{R} \mapsto \sigma_t^{\varphi} \in \mathrm{Aut}(\mathcal{M})$$

$$\sigma_t^{\varphi}(X) = \Delta^{it} X \Delta^{-it}$$

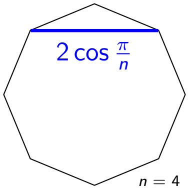
intrinsic dynamics associated with φ (*modular automorphisms*).

Jones index

Factors (von Neumann algebras with trivial center) are “very infinite-dimensional” objects. For an inclusion of factors $\mathcal{N} \subset \mathcal{M}$ the Jones index $[\mathcal{M} : \mathcal{N}]$ measures the relative size of $\mathcal{N}$ in $\mathcal{M}$. Surprisingly, the index values are quantised:

$$[\mathcal{M} : \mathcal{N}] = 4\cos^2\left(\frac{\pi}{n}\right) \ , \quad n = 3, 4, \ldots \quad \text{or} \quad [\mathcal{M} : \mathcal{N}] \geq 4$$

Jones index appears in many places in math and in physics.



Quantum Field Theory

In QFT we have a quantum system with infinitely many degrees of freedom. The system is relativistic and there is particle creation and annihilation.

No mathematically rigorous QFT model with interaction still exists in 3+1 dimensions!

On the other hand, subfactor methods are quite effective in lower dimension (Kawahigashi, L. classification of $c < 1$ chiral CFT).

Haag local QFT:

$$O \text{ spacetime regions} \mapsto \text{von Neumann algebras } \mathcal{A}(O)$$

to each region O one considers the observables in O.

Local QFT nets

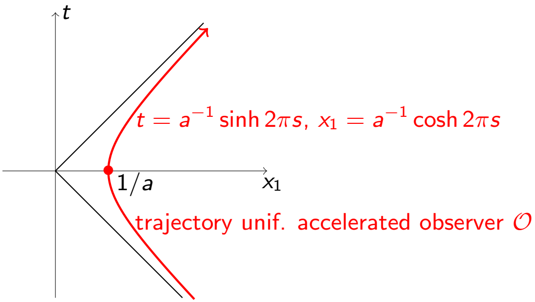
Local net $\mathcal{A}$ on spacetime M: map $O \subset M \mapsto \mathcal{A}(O) \subset B(\mathcal{H})$ s.t.

- *Isotony*, $O_1 \subset O_2 \implies \mathcal{A}(O_1) \subset \mathcal{A}(O_2)$
- *Locality*, O_1, O_2 spacelike $\implies [\mathcal{A}(O_1), \mathcal{A}(O_2)] = \{0\}$
- *Poincaré covariance* (conformal, diff covariant) .
- *Positive energy and vacuum vector*.

$O \mapsto \mathcal{A}(O)$: "Noncommutative chart" in QFT

Bisognano-Wichmann theorem, Sewell's comment

Rindler spacetime (wedge $x_1 > |x_0|$), vacuum modular group



a uniform acceleration of $\mathcal{O}$
s/a proper time of $\mathcal{O}$
$\beta = 2\pi/a$ inverse KMS temperature of $\mathcal{O}$

Hawking-Unruh effect!
Time is geodesic, quantum gravitational effect!

Entropy of finite systems

$X = \{x_1, \dots x_n\}$ a set of events. If x_i occurs with probability p_i, its information is $-\log p_i$

$$\text{Shannon entropy}: S(P) = -\sum p_i \log p_i \ .$$

If $Q = \{q_1, \dots q_n\}$ other probability distribution (state)

$$\text{Relative entropy}: S(P\|Q) = \sum p_i(\log p_i - \log q_i)$$

mean value in the state P of the difference between the information carried by the state P and the state Q.

Noncommutative entropy: $\varphi = -\mathrm{Tr}(\rho_\varphi \cdot)$ state on a matrix algebra

$$\text{von Neumann entropy}: S(\varphi) = -\mathrm{Tr}(\rho_\varphi \log \rho_\varphi)$$

Umegaki's relative entropy

$$S(\varphi\|\psi) =: \mathrm{Tr}\big(\rho_\varphi(\log \rho_\varphi - \log \rho_\psi)\big)$$

Araki's relative entropy

An infinite quantum system is described by a von Neumann algebra $\mathcal{M}$ typically not of type I so Tr does not exists; however Araki's relative entropy between two faithful normal states φ and ψ on $\mathcal{M}$ is defined in general by

$$S(\varphi|\psi) \equiv -(\eta, \log \Delta_{\xi,\eta}\, \eta)$$

where ξ, η are cyclic vector representatives of φ, ψ and $\Delta_{\xi,\eta}$ is the relative modular operator associated with ξ, η.

$$S(\varphi|\psi) \geq 0$$

positivity of the relative entropy

Relative entropy is one of the key concepts. We take the view that relative entropy is a primary concept and all entropy notions are derived concepts

Entropy of quantum channels

$\mathcal{N}, \mathcal{M}$ vN algebras, $\alpha : \mathcal{N} \to \mathcal{M}$ completely positive, normal unital map (quantum operation), ω faithful normal state of $\mathcal{M}$; set

$$\mathrm{H}_\omega(\alpha) \equiv \sup_{(\omega_i)} \sum_i S(\omega|\omega_i) - S(\omega \cdot \alpha|\omega_i \cdot \alpha)$$

supremum over all ω_i with $\sum_i \omega_i = \omega$.

The entropy $H(\alpha)$ of α is defined by

$$\mathrm{H}(\alpha) = \inf_\omega \mathrm{H}_\omega(\alpha)$$

infimum over all "full" states ω for α. Clearly $\mathrm{H}(\alpha) \geq 0$ because $\mathrm{H}_\omega(\alpha) \geq 0$ by the monotonicity of the relative entropy .

α is a quantum channel if its entropy $\mathrm{H}(\alpha)$ is finite.

Generalisation of Stinespring dilation

Let $\alpha : \mathcal{N} \to \mathcal{M}$ be a normal, completely positive unital map between the vN algebras $\mathcal{N}$, $\mathcal{M}$. A pair (ρ, v) $\rho : \mathcal{N} \to \mathcal{M}$ a homomorphism, $v \in \mathcal{M}$ an isometry s.t.

$$\alpha(n) = v^* \rho(n) v \ , \quad n \in \mathcal{N} \ .$$

(ρ, v) is *minimal* if the left support of $\rho(\mathcal{N}) v \mathcal{H}$ is equal to 1.

Let $\alpha : \mathcal{N} \to \mathcal{M}$ be a normal, CP unital map with $\mathcal{N}$, $\mathcal{M}$ properly infinite. There exists a minimal dilation pair (ρ, v) for α.
If (ρ_1, v_1) is another minimal pair, $\exists!$ unitary $u \in \mathcal{M}$ such that

$$u\rho(n) = \rho_1(n) u \, , \quad v_1 = uv, \qquad n \in \mathcal{N}$$

We have

$$H(\alpha) = \log Ind(\rho) \qquad \text{(minimal index)}$$

Bimodules and CP maps

Let $\alpha : \mathcal{N} \to \mathcal{M}$ be a completely positive, normal, unital map and ω a faithful normal state of $\mathcal{M}$

$\exists!$ $\mathcal{N} - \mathcal{M}$ bimodule $\mathcal{H}_\alpha$, with a cyclic vector $\xi_\alpha \in \mathcal{H}$ and left and right actions ℓ_α and r_α, such that

$$(\xi_\alpha, \ell_\alpha(n)\xi_\alpha) = \omega_{\rm out}(n)\,, \quad (\xi_\alpha, r_\alpha(m)\xi_\alpha) = \omega_{\rm in}(m)\,,$$

with $\omega_{\rm in} \equiv \omega$, $\omega_{\rm out} \equiv \omega_{\rm in} \cdot \alpha$. Converse is true.

CP map $\alpha \longleftrightarrow$ cyclic bimodule $\mathcal{H}_\alpha$

We have

$$H(\alpha) = \log Ind(\mathcal{H}_\alpha) \qquad \text{(Jones' index)}$$

The modular group of a quantum channel

$\mathcal{H}$ an $\mathcal{N}-\mathcal{M}$-bimodule with finite Jones' index $\mathrm{Ind}(\mathcal{H})$

Given faithful, normal, states φ, ψ on $\mathcal{N}$ and $\mathcal{M}$, I define the modular operator $\Delta_{\mathcal{H}}(\varphi|\psi)$ of $\mathcal{H}$ with respect to φ, ψ as

$$\Delta_{\mathcal{H}}(\varphi|\psi) \equiv d(\varphi \cdot \ell^{-1})/d(\psi \cdot r^{-1} \cdot \varepsilon)\ ,$$

$d(\cdot)/d(\cdot)$ Connes' spatial derivative, $\varepsilon : \ell(\mathcal{N})' \to r(\mathcal{M})$ is the minimal conditional expectation

$\log \Delta_{\mathcal{H}}(\varphi|\psi)$ is called the modular Hamiltonian of the bimodule $\mathcal{H}$, or of the quantum channel α if $\mathcal{H}$ is associated with α.

The physical Hamiltonian at inverse temperature $\beta > 0$ is given by

$$K_{\mathcal{H}}(\varphi_1|\varphi_2) = -\beta^{-1} log \Delta_{\mathcal{H}}(\varphi_1|\varphi_2) + \beta^{-1} \log d$$

Properties of the physical Hamiltonian

- $U_{\mathcal{H}}(t) = e^{itK_{\mathcal{H}}}$ *implements the dynamics:*

$$U_{\mathcal{H}}(t)(\varphi|\psi)\ell(n)U_{\mathcal{H}}(-t)(\varphi|\psi) = \ell\big(\sigma_t^{\varphi}(n)\big)$$

$$U_{\mathcal{H}}(t)(\varphi|\psi)r(m)U_{\mathcal{H}}(-t)(\varphi|\psi) = r\big(\sigma_t^{\psi}(m)\big)$$

- *additivity for composition:* (Connes's bimodule tensor product)

$$U_{\mathcal{H}}(t)(\varphi_1|\varphi_2) \otimes U_{\mathcal{K}}(t)(\varphi_2|\varphi_3) = U_{\mathcal{H}\otimes\mathcal{K}}(t)(\varphi_1|\varphi_3)$$

- *charge symmetry:*

$$U_{\bar{\mathcal{H}}}(t)(\varphi_2|\varphi_1) = \overline{U_{\mathcal{H}}(t)}$$

- *additivity for disjoint systems:*

If $T : \mathcal{H} \to \mathcal{H}'$ is a bimodule intertwiner, then

$$TU_{\mathcal{H}}(t)(\varphi_1|\varphi_2) = U_{\mathcal{H}'}(t)(\varphi_1|\varphi_2)T$$

Landauer's bound for infinite systems

Landauer's principle: *any logically irreversible manipulation of information, such as the erasure of a bit or the merging of two computation paths, must be accompanied by a corresponding entropy increase in non-information bearing degrees of freedom of the information processing apparatus or its environment* (cf. C. Bennet)

Let $\alpha : \mathcal{N} \to \mathcal{M}$ be a quantum channel between quantum systems $\mathcal{N}$, $\mathcal{M}$. based on the thermodynamical relation $dF = dE - TdS$, the free energy is

$$F_\alpha \equiv \big\langle K_{\mathcal{H}}(\varphi_1|\varphi_2) + \beta^{-1} log\,\Delta_{\mathcal{H}}(\varphi_1|\varphi_2)\big\rangle = \beta^{-1}\log d \geq 0$$

If α is irreversible, then

$$F_\alpha \geq \frac{1}{2} kT \log 2$$

The *original lower bound* for the incremental free energy is

$$F_\alpha \geq kT \log 2$$

it remains true for finite-dim. systems $\mathcal{N}$, $\mathcal{M}$.

Energy conditions in QFT

In QFT, Bousso, Fisher, Liechenauer, and Wall proposed the Quantum Null Energy Condition, QNEC: For null direction deformations

$$\langle T_{uu} \rangle \geq \frac{1}{2\pi} S''_A(\lambda) ,$$

here T stress-energy tensor, S_A entanglement entropy relative to the deformed region A, S''_A second derivative of S_A w.r.t. the deformation parameter λ.

Physical arguments give the QNEC from the inequality

$$S''(\lambda) \geq 0$$

with $S(\lambda)$ Araki's relative entropy of every state w.r.t. the vacuum.

QNEC

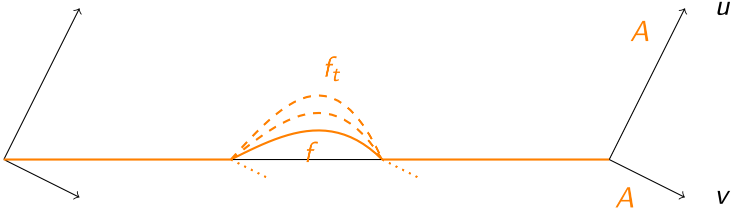


Figure: The function f is the boundary of the deformed region on the null horizon. The entire deformed region is its causal envelop A.

Positivity of the second derivative of the *relative entropy* appears unexpectedly: $S''(\lambda) \geq 0$

See Ceyhan and Faulkner '19.

First and second quantisation

First quantisation: map

$$O \subset \mathbb{R}^d \mapsto H(O) \text{ real linear space of } \mathcal{H}$$

local, covariant, etc.

Second quantisation: map

$$O \subset \mathbb{R}^d \mapsto \mathcal{A}(O) \text{ v.N. algebra on } e^{\mathcal{H}}$$

$\mathcal{A}(O) = \mathcal{A}(H(O))$

In our case $H(O)$ is generated by the waves with Cauchy data in B (O double cone with time-zero basis B)

"First quantization is a mystery, but second quantization is a functor" – Edward Nelson.

Standard subspaces

$\mathcal{H}$ complex Hilbert space and $H \subset \mathcal{H}$ a closed, real linear subspace. *Symplectic complement*:

$$H' = \{\xi \in H : \Im(\xi, \eta) = 0 \ \forall \eta \in H\}$$

H is *cyclic* if $\overline{H + iH} = \mathcal{H}$ and *separating* if $H \cap iH = \{0\}$.

A standard subspace H of $\mathcal{H}$ is a closed, real linear subspace of $\mathcal{H}$ which is both cyclic and separating. H is standard iff H' is standard.

H standard subspace $\to$ anti-linear operator $S : D(S) \subset \mathcal{H} \to \mathcal{H}$,

$$S : \xi + i\eta \to \xi - i\eta, \ \ \xi, \eta \in H$$

$S^2 = 1|_{D(S)}$. S is closed and densely defined, indeed

$$S_H^* = S_{H'}$$

Modular theory for standard subspaces

Set $S = J\Delta^{1/2}$, polar decomposition of $S = S_H$.

Then J is an anti-unitary involution, $\Delta > 0$ is non-singular and $J\Delta J = \Delta^{-1}$.

$$\Delta^{it} H = H, \quad JH = H'$$

(one particle Tomita-Takesaki theorem).

Eackmann, Osterwalder; Rieffel, van Daele; Leyland, Roberts, Testard

Entropy of a vector relative to a real linear subspace

Our analysis relies on the concept of entropy S_k of a vector k in a Hilbert space $\mathcal{H}$ with respect to a real linear subspace H of $\mathcal{H}$. Our formula for the entropy of k with respect to H is

$$S_k = \Im(k, P_H i \log \Delta\, k)$$

Here P_H is the crucial cutting projection $P_H : H + H' \to H$

$$P_H : h + h' \mapsto h$$

in terms of J and Δ,

$$P_H = \Delta^{-1/2}(\Delta^{-1/2} - \Delta^{1/2})^{-1} + J(\Delta^{-1/2} - \Delta^{1/2})^{-1}$$

Entropy of a wave

Waves' time-independent symplectic form

$$\frac{1}{2}\int_{x^0=t}(\Phi'\Psi - \Psi'\Phi)dx \ ,$$

The symplectic form is the imaginary part of complex Hilbert space scalar product (that depends on the mass).

Waves with Cauchy data supported in the half-space $x^1 \geq 0$ form a real linear subspace $H(W)$ (W wedge).

The entropy S_Φ of Φ is the entropy of the vector Φ w.r.t. $H(W)$

By the Bisognano-Wichmann theorem, we have the modular group of $H(W)$

Let Φ be a real Klein-Gordon wave and $H = H(W)$.

The entropy $S_\Phi(\lambda)$ of Φ w.r.t. the wedge region W_λ is the entropy of the vector Φ w.r.t. the standard subspace $H(W_\lambda)$.

$$S_\Phi(\lambda) = 2\pi \int_{x^0=\lambda,\, x^1\geq\lambda} (x^1 - \lambda)\, T_{00}(x)\, dx$$

then

$$S''_\Phi(\lambda) = 2\pi \int_{x^0=\lambda,\, x^1=\lambda} \langle v, Tv\rangle dx \geq 0 \ ,$$

where v is the light-like vector $v = (1, 1, 0 \dots, 0)$.

Here the energy density is

$$T_{00} = \frac{1}{2}\big(\Phi'^2 + |\nabla\Phi|^2 + m^2\Phi^2\big)$$

Entropy of coherent sectors

H real linear subspace of $\mathcal{H}$ $\to$ von Neumann algebra on $e^{\mathcal{H}}$

$$\mathcal{A}(H) = \{V(h) : h \in H\}''$$

Given a wave $\Phi \in \mathcal{H}$ consider the automorphism of $\mathcal{A}(H)$

$$\beta_\Phi = \mathrm{Ad}V(\Phi)^*|_{\mathcal{A}(H)} \ .$$

The relative vacuum relative entropy on $\mathcal{A}(H)$ is given by

$$S(\omega \cdot \beta_\Phi \| \omega) = S_\Phi$$

the entropy of the vector Φ w.r.t. H.

QNEC for coherent states

$\mathcal{A}(W)$ von Neumann algebra of the free, neutral scalar field on the Rindler spacetime. Given a wave Φ consider the automorphism

$$\beta_\Phi = \mathrm{Ad}V(\Phi)^*|_{\mathcal{A}(W)}$$

and the entropy of β_Φ relative to the subwedge $W_\lambda = W + (\lambda, \lambda, 0, \dots, 0)$ holds true.

The entropy of β_Φ with respect to the vacuum on W_λ is given by

$$S_\Phi(\lambda) \equiv S(\varphi_\Phi|_{\mathcal{A}(W_\lambda)} \| \varphi|_{\mathcal{A}(W_\lambda)})$$

In particular, the second derivative of $S''_\Phi(\lambda)$ gives the QNEC inequality for coherent states and constant null translations

$$S''_\Phi(\lambda) \geq 0$$

(F. Ciolli, G. Ruzzi, R. L.)

Entropy of localised states: $U(1)$-current model

Case of $U(1)$-current j: ℓ real function in $S(\mathbb{R})$ and $\lambda \in \mathbb{R}$. We have

$$S(\lambda) = \pi \int_{\lambda}^{+\infty} (x-\lambda)\ell^2(x)\mathrm{d}x \ ,$$

$S(\lambda)$ vacuum relative entropy of excited state by $j \mapsto j + \ell$, so

$$S'(\lambda) = -\pi \int_{\lambda}^{+\infty} \ell^2(x)\mathrm{d}x \leq 0 \ ,$$

$$S''(\lambda) = \pi \ell^2(\lambda) \geq 0$$

positivity of S''

The vacuum energy density is $E(\lambda) = \frac{1}{2}\ell^2(\lambda)$ so we have here the QNEC:

$$E(\lambda) = \frac{1}{2\pi} S''(\lambda) \geq 0$$

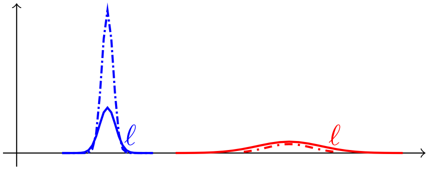


Figure: Two distributions, blue and red, for the same charge $q = \int \ell$. The dashed lines plot the corresponding entropy density rate $S''(t)$: blue high entropy, red low entropy.

Further work in CFT by S. Hollands and L. Panebianco

Classical and QFT waves

Entropy of a vector —(one-particle space / *classical field theory*)→ Entropy of a wave

Entropy of a vector ↓ (Fock space) Entropy of a coherent state

Entropy of a wave ↓ (2^{nd} quantisation) Entropy of Klein-Gordon QFT

Entropy of a coherent state —(local algebras / *quantum field theory*)→ Entropy of Klein-Gordon QFT

The entropy of a vector Φ with respect to a real linear subspace $H(O)$ has a double physical interpretation: classically, it measures the information carried by a wave packet in the spacetime region O; from the quantum point of view, it gives the vacuum relative entropy, on the algebra $\mathcal{A}(O)$ associated with $H(O)$, of the coherent state induced by Φ on the Fock space.

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