Stable Character Polynomials for Symmetric Groups

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Partitions

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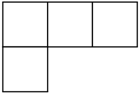

Figure: Diagram of the partition $\lambda = (3, 1)$.

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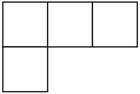

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We write $\lambda[n]$ to denote $(n - |\lambda|, \lambda_1, \ldots, \lambda_r)$. It has size n.

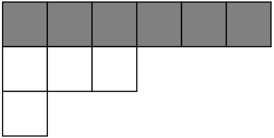

Figure: Diagram of the partition $\lambda[10]$.

Representations of GL_n

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Examples:

- Schur-Weyl duality ($GL_n \curvearrowright (\mathbb{C}^n)^{\otimes d} \curvearrowleft S_d$)
- Highest-weight theory for reductive Lie algebras
- Cohomology of line bundles on the flag manifold (Borel-Weil-Bott theorem)
- Theory of crystals
- Geometric Satake correspondence

Representations of GL_n (cont.)

Irreducible representations $\mathbb{S}^\lambda(\mathbb{C}^n)$ of GL_n correspond to partitions λ of any size with length at most n.

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A vector v has **weight** $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ if

$$\begin{bmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{bmatrix} \cdot v = (x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n}) v$$

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The **Schur function**, s_λ, is the trace of the above matrix acting on $\mathbb{S}^\lambda(\mathbb{C}^n)$:

$$s_\lambda = \sum_\mu \dim(\mu\text{-weight space of } \mathbb{S}^\lambda(\mathbb{C}^n)) x_1^{\mu_1} \cdots x_n^{\mu_n}$$

Example: $\lambda = (1)$, $\mathbb{S}^{(1)}(\mathbb{C}^n) = \mathbb{C}^n$, $s_{(1)} = \sum_i x_i$

Schur Functions

If v_μ, v_ν have weight μ, ν respectively, $v_\mu \otimes v_\nu$ has weight $\mu + \nu$. Similarly,

$$(x_1^{\mu_1} \cdots x_n^{\mu_n})(x_1^{\nu_1} \cdots x_n^{\nu_n}) = x_1^{\mu_1+\nu_1} \cdots x_n^{\mu_n+\nu_n}$$

So if $\mathbb{S}^\rho(\mathbb{C}^n) \otimes \mathbb{S}^\sigma(\mathbb{C}^n) = \bigoplus_\lambda \mathbb{S}^\lambda(\mathbb{C}^n)^{c_{\rho,\sigma}^\lambda}$,

$$s_\rho s_\sigma = \sum_\lambda c_{\rho,\sigma}^\lambda s_\lambda.$$

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This allows the fusion rules (**Littlewood-Richardson coefficients** $c^\lambda_{\rho,\sigma}$) to be found by multiplying polynomials.

$$\begin{aligned} s_{(1)} s_{(1)} &= (\sum_i x_i)^2 \\ &= \sum_{i<j} x_i x_j + \sum_{i \leq j} x_i x_j \\ &= s_{(1,1)} + s_{(2)}. \end{aligned}$$

Borel-Weil-Bott theorem

The **flag manifold** of $GL_n(\mathbb{C})$ consists of all configurations of subspaces of $\mathbb{C}^n$, $0 = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = \mathbb{C}^n$, where V_i is of dimension i. It is a compact manifold with an action of GL_n.

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Theorem (Borel-Weil-Bott)

The cohomology $H^i(L_\mu)$ is zero except possibly for one value of i, where it is an irreducible representation of $GL_n(\mathbb{C})$.

If μ defines a partition (i.e. $\mu_1 \geq \mu_2 \geq \cdots$), then the cohomology $H^0(L_\mu) = \mathbb{S}^\mu(\mathbb{C}^n)$, and $H^i(L_\mu) = 0$ for $i > 0$.

If μ is not a partition, there is an algorithm to determine $H^i(L_\mu)$.

Representations of S_n

There are also a variety of approaches to the representation theory of symmetric groups:

- Young symmetrisers
- The Okounkov-Vershik approach
- Duality with partition algebras ($S_n \curvearrowright (\mathbb{C}^n)^{\otimes d} \curvearrowleft \mathrm{Par}_d(n)$)
- Homology of Springer fibres

Symmetric Groups

Irreducible representations of S_n are **Specht modules** $\mathcal{S}^\lambda$, parametrised by partitions of size n.

For example, $\mathcal{S}^{(n)}$ is the trivial representation, and $\mathcal{S}^{(1^n)}$ is the sign representation.

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The fusion rules are given by **Kronecker coefficients**:

$$\mathcal{S}^\mu \otimes \mathcal{S}^\nu = \bigoplus_\lambda (\mathcal{S}^\lambda)^{k_{\mu,\nu}^\lambda}$$

100-year old problem: give a combinatorial interpretation of $k_{\mu,\nu}^\lambda$

The Specht modules $\mathcal{S}^{\lambda[n]}$ (fixed λ, varying n) exhibit certain stability phenomena.

Theorem (Murnaghan, 1938)

The limit of Kronecker coefficients

$$\tilde{k}^{\lambda}_{\mu,\nu} = \lim_{n\to\infty} k^{\lambda[n]}_{\mu[n],\nu[n]}$$

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exists and is finite.

The $\tilde{k}_{\mu,\nu}^{\lambda}$ are called **reduced Kronecker coefficients**.

This suggests that the "complexity" of $\mathcal{S}^{\lambda[n]}$ is controlled by $|\lambda|$.

Lie Algebra Homology

Let V be a vector space, and $L(V)$ be the free Lie algebra on V. Then $GL(V) \times S_n$ acts on $L(V)^n = L(V) \otimes \mathbb{C}^n$.

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We may compute $H_i(L(V)^n; \mathrm{Sym}(V))$ with the Chevalley-Eilenberg complex, whose chain groups are

$$\bigwedge^i (L(V) \otimes \mathbb{C}^n) \otimes \mathrm{Sym}(V) = \bigoplus_\lambda \mathbb{S}^{\lambda'}(V) \otimes \mathcal{L}^i_\lambda.$$

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The **Littlewood complex** $\mathcal{L}^i_\lambda$ defined above is a complex of modules restricted from $GL_n(\mathbb{C})$: each has a character that is a sum of Schur functions.

Example of $\mathcal{L}_\lambda$

For $\lambda = (1)$, $\mathcal{L}_\lambda$ is the complex

$$0 \to \mathbb{C}^n \to \mathbb{C} \to 0,$$

where the $\mathbb{C}$ is the trivial representation.

The map takes $(z_1, z_2, \ldots, z_n)$ to $\sum_i z_i$.

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- If $n = 0$, then the homology is the trivial representation on the right
- If $n = 1$, then the homology vanishes
- If $n \geq 2$, then the homology consists of mean-zero vectors in $\mathbb{C}^n$ (which equals $\mathcal{S}^{(n-1,1)}$) on the left

Homology of Resolutions

Theorem (R., '20)

Assume that n is large enough that $\lambda[n]$ is a partition. The homology of $\mathcal{L}_\lambda$ is $\mathcal{S}^{\lambda[n]}$ in degree $|\lambda|$ and zero elsewhere.

If n is too small, the homology of $\mathcal{L}_\lambda$ is determined by the Bott algorithm.

This is roughly like a Borel-Weil-Bott theorem for symmetric groups.

More details

The Euler characteristic of $\mathcal{L}_\lambda$ is a symmetric function $s^\dagger_\lambda$ (the irreducible character basis of Orellana and Zabrocki, the stable Specht polynomial of Assaf and Speyer).

When evaluated at the eigenvalues of a permutation matrix, it yields the trace of that permutation matrix acting on $\mathcal{S}^{\lambda[n]}$. Hence

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- We have $s^\dagger_\lambda = s_\lambda +$ lower order terms
- There is a "truncation" procedure to recover $k^\lambda_{\mu,\nu}$ from $\tilde{k}^\tau_{\rho,\sigma}$
- $\mathcal{L}_\lambda$ actually give injective resolutions in Sam and Snowden's category $\mathrm{Rep}(S_\infty)$
- Expressing s_λ in terms of $s^\dagger_\mu$ amounts to understanding restriction from GL_n to S_n.

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We have

$$\begin{aligned} s^\dagger_{(1)} s^\dagger_{(1)} &= (\sum_i x_i)^2 - 2(\sum_i x_i) + 1 \\ &= (\sum_{i<j} x_i x_j - \sum_i x_i + 1) + (\sum_{i \le j} x_i x_j - 2\sum_i x_i) \\ &\quad + (\sum_i x_i - 1) + 1 \\ &= s^\dagger_{(1,1)} + s^\dagger_{(2)} + s^\dagger_{(1)} + s^\dagger_{\emptyset} \end{aligned}$$

Further Directions

This theory answers some questions:

- Assaf and Speyer noted that the character of the free Lie algebra appears in their work with $s^{\dagger}_{\lambda}$, but did not have an explanation.
- Proves a conjecture about symmetric sieving due to Stier, Wellman, and Xu.

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Some unanswered questions remain

- Can this theorem be interpreted "geometrically"?
- Is there a combinatorial formula for $s_\lambda^\dagger$ like there is for s_λ?
- Can this construction be reconciled with other similar constructions?