Euclidean Quantum Field Theory: Axioms and Automorphic Forms

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1 Euclidean Quantum Fields: the theory of very smooth partition functions

Questions by Don Zagier, 40 years ago

- What is quantum field theory?
- What is a field?
- Why are physicists interested in automorphic forms, in particular in modular forms?

Mathematicians need short, memorable answers.

Here is an attempt for the euclidean case, presented at a leisurely pace.

Any (specimen of) euclidean quantum field theory is determined by its partition function.

First, fix a dimension d.

Let $\mathcal{R}$ be the set of compact d-dimensional manifolds M with Riemannian metric g (up to isomorphisms).

Partition functions are maps

$$Z : \mathcal{R} \to \mathbb{R}_+.$$

One example is

$$Z = det(\Delta_g + m^2)^{-1/2},$$

where Δ_g is the Laplacian for the metric g and m is a real number. This defines the quantum field theory of a free scalar of mass m.

An example for $d = 2$ is the (2,5) minimal model. When M has the topology of a torus, hermitean metric $g = \exp(\chi) dz d\bar{z}$ and modulus τ as a complex curve one has

$$Z(M) = \exp\left(-\frac{11}{120\pi} \int \chi K \right) (|f(\tau)|^2 + |g(\tau)|^2),$$

where f, g are the Rogers-Ramanujan modular functions and

$$K = -\partial\bar{\partial}\chi$$

is the Gauss curvature. Modular forms come into play, since

$$det'(\Delta_g) = V \exp\left(-\frac{1}{24\pi} \int \chi K \right) Im(\tau) |\eta(\tau)|^4.$$

Partition functions must satisfy three axioms: They must

- be multiplicative on disjoint unions,
- be smooth,
- have finite bound for derivatives.

To get some feeling for the impact of the axioms, we immediately consider the case $d = 1$, though in this case there are no proper fields, just 'cosmological constants'.

The axioms need some polishing. In particular one has to decide which metrics are allowed (continuous and piecewise smooth may work). Moreover, smoothness will be required with respect to both affine and (newly defined) boundary derivatives. The third axiom refers to the latter. Consideration of affine derivatives may turn out to be unnecessary.

To define smoothness for functions on infinite dimensional spaces we need the following, fairly standard notation. For any set S let $[S]$ be the real vector space with basis $[s]$ for any $s \in S$. To define derivatives we will need sets such that any element s can be scaled by positive real ϵ, but note that $[\epsilon s]$ is another basis element and not equal to $\epsilon[s]$.

1.1 The first axiom

$$Z(M_1 \sqcup M_2) = Z(M_1)Z(M_2).$$

For $d = 1$ this axiom implies that Z is determined by its values on circles of circumference L.

1.2 The second axiom

For $d = 1$ the second axiom states that Z is C^∞. There are two ways to generalize smoothness to higher d. One is based on the fact that the space of metrics on a compact manifold is locally affine, the second one deals with changes in small bounded neighbourhoods of M. The second way is more general, because it allows changes of the topology. To be on the safe side, we require that Z is smooth with respect to both.

Derivatives of any order will be defined at once.
For functions f on $\mathbb{R}$ the standard second derivative can be written in terms of $f(x-\epsilon)-2f(x)+f(x+\epsilon)$, higher derivatives in terms of finite sums $\sum_i a_i f(x+\epsilon v_i)$ with suitable a_i, v_i. This can be generalized to infinite dimensions.

Let $\mathcal{A}$ be an affine space and $\mathcal{F}$ a set of functions with domain $\mathcal{A}$ and range $\mathbb{R}$. Let V be the set of vectors of $\mathcal{A}$. The elements of $[V]$ have the form $A = \sum_i a_i [v_i]$ (finite sum, real a_i). We define $\mathcal{F}$ to be smooth if for any non-zero A there is a real number k so that for any $f \in \mathcal{F}$ the limit

$$D(A)f(x) = \lim_{\epsilon \to 0} \epsilon^{-k} \sum_i a_i f(x + \epsilon v_i)$$

exists and is generically non-zero. The number k is the order of $D(A)$. An affine derivative of $\mathcal{F}$ is a finite sum of such derivatives, not necessarily of the same order, modulo sums that annihilate all elements of $\mathcal{F}$.

When $\mathcal{A}$ has finite dimension and $\mathcal{F}$ is the set of all C^∞ functions the vector space of affine derivatives is just the ordinary one.

Part A of the second axiom is that the set given by Z and its affine derivatives with respect to the metric is smooth.

For part B we consider derivatives for partition functions based on local changes of elements of $\mathcal{R}$. A natural operation is cutting and glueing. Let $\mathcal{B}$ be the set of manifolds with Riemannian metric that have the unit sphere in R^d as boundary. Such manifolds can be scaled by $\epsilon \in \mathbb{R}_+$ so that the boundary becomes the sphere with radius ϵ. Given a frame (vielbein) E at a point $x \in M$ one can cut out a small sphere of radius ϵ in M and glue in the rescaled b. The result will be called $b\epsilon M(x)$. Let $B \in [\mathcal{B}]$ so that $B = \sum_i a_i[b_i]$. Local derivatives $D(B)$ of order $k \in \mathbb{R}$ are defined as before, as limits

$$D(B,x,E)Z = \lim_{\epsilon \to 0} \epsilon^{-k} \sum_i a_i Z(b_i \epsilon M(x)).$$

More generally, for any $n \in \mathbb{N}$ let $\mathcal{B}^n$ be the set of Riemannian manifolds b with n numbered boundary components, each provided with a metric isomorphism to the unit sphere in R^d. After scaling by sufficiently small ϵ, b can be glued in at n distinct points $x_1, \dots, x_n$ of M. Linear combinations yield multilocal derivatives $D(B, x_1, \dots, x_r)$. We call n the boundary number of the derivative.

Part B of the second axiom requires that Z is smooth with respect to derivatives of arbitrary boundary number.

For $d = 1$, affine derivatives and $\mathcal{B}^1$ derivatives have the same effect. In contrast, let $b \in \mathcal{B}^2$ be given by a circle with two intervals removed. Such a b can be glued in $M_1 \sqcup M_2$ so that it connects the two components. It yields a derivative $D([b])$ of order 0 so that $D([b])Z(L_1, L_2) = Z(L_1 + L_2)$.

1.3 The third axiom

Disjoint union yields a natural map $\mathcal{B}^m \times \mathcal{B}^n \to \mathcal{B}^{m+n}$, which lifts to a bilinear map $[\mathcal{B}^m] \times [\mathcal{B}^n] \to [\mathcal{B}^{m+n}]$. Thus certain derivatives can be written in terms of derivatives of lesser boundary number. The third axiom states that there is a natural number N (the bound of Z) so that any derivative of Z of boundary number greater than N can be written bilinearly in terms of derivatives of lesser boundary number.

For $d = 1$ and $N = 1$ this means in particular that $Z(L_1 + L_2)$ can be written as a bilinear expression in the derivatives of Z at L_1 and at L_2. This means that Z satisfies a linear ODE with constant coefficients. Moreover it is easy to see that for $d = 1$ any partition function of finite bound has bound 1. Thus the three axioms fit well together. One obtains a stratified set of possible partition functions, with a finite number of parameters for each stratum, as expected for higher d.

2 The fields

The B-type derivatives of the partition function are fields of the theory. Their order yields the scaling dimensions of the fields. For $B \in [\mathcal{B}^n]$, $D(B)Z$ is the corresponding n-point function.

By construction the fields form a vector space that is filtered by the scaling dimension. One expects that the corresponding graded quotients are finite dimensional. This excludes the free scalar field theory for $d = 2$, which has infinitely many fields of scaling dimension 0.

Free scalar fields for $d > 2$ have another important feature. The standard field ϕ that satisfies the equation $(\Delta_g + m^2)\phi = 0$ changes sign under an involution symmetry of the theory. Only even fields can be constructed as derivatives of Z. The $\mathcal{B}^2$ field $\phi(x_1)\phi(x_2)$ is even, but can only be factorized by introducing odd supplementary fields.

There are more examples of this phenomenon. Partition functions that satisfy the axioms form a semigroup under multiplication. Generically all fields of the factors should be recovarable as fields of the product, but this is not always the case. Consider the n-th power of a partition function. All fields of Z^n are invariant under permutations of the factors. In this case it is natural to recover the fields of the factors in terms of supplementary fields in non-trivial representations of the permutation group. If Z has bound 1, this recovers bound 1 for the product theory.

In the analogous case of algebraic quantum field theory on Minkowski space it has been argued that this situation is generic. In tne Euclidean case one also would like to prove that one can construct a bound 1 theory with a symmetry group G for which all fields invariant under G are derivatives of Z.

2.1 The energy-momentum tensor

Einstein descibed the energy-momentum tensor as affine derivative with respect to the metric. Let $v = (h_{\mu\nu})$. Then

$$D([v] - [0])Z = -\frac{1}{2} \int h_{\mu\nu}(x) T^{\mu\nu}(x) Z dV,$$

with the volume measure dV given by the metric g. In terms of boundary derivatives, let b_1, b_2 be unit balls with metrics g^1, g^2 and corresponding volume elements dV^1, dV^2. Then $D([b_1] - [b_2])$ is a derivative of order d and an interchange of limits yields

$$D([b_1] - [b_2])(x) = -\frac{1}{2} T^{ij}(x) \int (g^1_{ij} dV^1 - g^2_{ij} dV^2),$$

where the indices i, j refer to the frame.

Derivatives of boundary number 1 using unit balls only are defined to yield the fields in the vacuum sector of Z.

2.2 Unitary and non-unitary theoories

For affine derivatives continuity yields $D([v]) = 1$ for any v. Similarly, for a unit ball b with any metric one expects $D([b]) = 1$. Unitary theories are presumably characterized by the property that $D([b]) = 1$ for any b. In other theories, $D([b])$ may have negative scaling dimension. The simplest case is the (2,5) minimal model. When b is any torus with a disk removed, $D([b])$ has order -2/5. All such $D([b])$ are proportional. The rest of the talk concerns this field.

3 Conformal field theory for $d = 2$

The (2,5) minimal model is a conformal theory with $d = 2$. Such theories are characterized by the fourth axiom

$$g_{\mu\nu}T^{\mu\nu}dV = -\frac{c}{12\pi}K,$$

where K is the Gauss curvature of the metric and the number c is called the central charge. From now on we only will consider theories of this kind.

Consider two metrics related by a Weyl transformation,

$$g^2 = \exp(\chi)g^1.$$

Weyl transformations are generated by $g_{\mu\nu}T^{\mu\nu}$, so their effect on Z can be calculated. One obtains

$$\log(Z(g^2)/Z(g^1)) = \frac{c}{48\pi}\int \chi(K^1 + K^2),$$

where K^1, K^2 are the Gauss curvature forms of g^1, g^2.

When the genus of M is zero, then any two metrics are related by an isomorphism and a Weyl transformation, so Z is fixed up to a factor, which can be chosen according to convenience.

By definition, the elements of $\mathcal{R}$ are defined up to isomorphism. Thus a change of the metric by reparametrization must not change Z. For infinitesimal reparametrizations this is expressed by the continuity equation

$$D_\mu T^{\mu\nu} = 0,$$

where D_μ is the covariant derivative on M with respect to the Levi-Civita connection for g.

Let $g = e^\rho dz d\bar{z}$. It is convenient to express $T^{\mu\nu}$ in terms of the complexified tangent space with basis $\partial_z, \partial_{\bar{z}}$. Then $T_{\underline{zz}}$ is given by the curvature and $T_{\bar{z}\bar{z}}$ is the complex conjugate of T_{zz}. One calculates that

$$T(z) = 2\pi T_{zz} + \frac{c}{12}(\partial_z^2 \log\rho - \frac{1}{2}(\partial_z\rho)^2)$$

is invariant under Weyl transformations. Moreover the continuity equation just states that $T(z)$ is holomorphic.

Whereas T_{zz} transforms homogeneously under complex coordinate change, $T(z)$ transforms as a projective connection. Invariance under diffeomorphisms implies that $T(z)T(w)Z$ has a Laurent expansion of the Virasoro form

$$T(z)T(w) = \frac{c}{2}(z-w)^{-4}+(z-w)^{-2}(T(z)+T(w))+N(TT)(w)+O(z-w).$$

In general, arbitrary holomorphic fields Φ, Ψ of Z have similar Laurent expansions, with a constant term denoted by $N(\Phi\Psi)$. The operation N defines an algebra on the vector space of holomorphic fields that is neither commutative nor associative. When the Laurent expansion is written in terms of the Fourier components $\oint z^n \Phi(z) ds$ one obtains a W-algebra (Zamolodchikov), graded by the rotations of z. The vertex operator algebra community calls the grade 0 part Zhu's algebra, which is a bit preposterous. On the other hand Zhu appears to be the first person who conceptionalized the fact that the algebra given by N becomes commutative and associative when one mods ou the ideal given by the image of ∂_z, in other words the derivative fields. The resulting algebra will be called Zhu's algebra without further specification.

When Zhu's algebra is finite dimensional as a vector space (in particular nil), Z can be calculated explicitly in terms of ordinary differential equations with regular singularities. The simplest non-trivial case is the two-dimensional algebra spanned by 1 and the class $\mathcal{T}$ of T. The relation $\mathcal{T}^2 = 0$ translates into $N(TT) = \alpha\partial^2 T$. Invariance of the Virasoro Laurent expansion under coordinate change yields $c = -22/5$ and $\alpha = 3/10$.

For simplicity let M be hyperelliptic and let the metric be given by the flat metric of the complex plane, except for circles around ∞ and the ramification points z_i. Inside of those circles we also choose the appropriate flat metric. Differentiation of Z with respect to z_i is given by $\oint dz T(z) Z$, where the integral is along a small circle around the corresponding ramification point. Differentiation of $T(z)Z$ with respect to the z_i is given by $\oint dw T(w)T(z)Z$ and so on. When Zhu's algebra is finite, the n-point functiions for large n can be expressed linearly in terms of n-point functiions with smaller n, so that one obtains linear holomorphic ODEs for Z. Thus Z factorizes locally into a finite sum of products of holomorphic and anti-holomorphic functions of the z_i.

The bound on n arises as follows. The Virasoro Laurent expansion specifies all singularities of the n-point functions, so the latter are given by sections of holomorphic line bundles. Due to the relation $N(TT) = \alpha\partial^2 T$ their restriction to the partial diagonals is given by $(n-1)$-point functions. For sufficiently large n this restriction determines the n-point function entirely. For genus g of M this happens when n is the Fibonacci number F_{2g}. Thus one obtains second order ODEs for $g = 1$ and fifth order ODEs for $g = 2$ with respect to any of the ramification points.

The solutions form a finite dimensional vector space and the monodromy group for the z_i equation is independent of the location of the other ramification points. For $g = 1$ one finds a classical hypergeometric ODE with algebraic solutions, contained in the famous list of Schwarz. In terms of the complex modulus the solutions are the Rogers-Ramanujan functions, as stated in the beginning of this talk.

At the locus of nodal curves the Frobenius indices are 0 and -2/5. When a zero cycle in holonomy is contracted and M becomes a union $M_1 \sqcup M_2$ with one marked point in each component, the -2/5 component yields the one-point function of the corresponding $D([b_1])$ on M_2 or conversely. When another cycle is contracted, this yields a manifold M of genus $g-1$ with two marked points. The -2/5 component yields the 2-point function of the corresponding $D([b_i])$ $i=1,2$ on M. They also satisfy ODEs obtained by reduction of the original system. When one starts at $g=2$ the fifth order equation reduces to an order 3 equation of the field $D([b])$ of scaling dimension -2/5. The ODEs determine the n-point functions of the fields up to proportionality constants. Finiteness of the bound implies that there are no multiplicities, so the corresponding partition function is unique, up to normallization by a cosmological constant.

4 Conclusion

For conformal field theories in 2 dimensions, the axioms given above are easy to work with and immediately yield interesting mathematical results. They are much easier to remember than the vertex operator algebra axioms of Borcherds. When Zhu's algebra is not finite dimensional one needs more powerful methods, but they certainly are within reach. The next cases to be considered are massive perturbations of conformal theories for $d = 2$ and conformally invariant theories for $d = 3$. There are many known results and it may be interesting to integrate them in the frame described above.

Apart from people mentioned already the ideas sketched above go back to work of G. Segal, D. Friedan, A. Raina and to a stimulating note by S. Hawking, who suggested that there are no fundamental uncharged scalar fields, since they might tunnel through small black holes anywhere, resulting just in a cosmological constant.