Resonances for open quantum maps

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joint work with Long Jin (YMSC Tsinghua)

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- Much studied issue: existence of spectral gap (do waves decay exponentially?)
- Known for a while under a pressure condition
- Using a fractal uncertainty principle (FUP), D–Zahl ’16 and Bourgain–D ’18 proved the existence of spectral gap for all convex co-compact hyperbolic surfaces
- The setting of open quantum maps makes the FUP approach easier to describe
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Overview of open quantum maps

- **Resonances**: complex characteristic frequencies of decaying waves in systems where energy is allowed to escape (e.g. obstacle scattering)

- **Open quantum chaos** studies the distribution of resonances, e.g. spectral gaps and fractal Weyl laws, with applications going as far as computer networks: Ermann–Frahm–Shepelyansky Rev. Mod. Phys. ’15:

![Diagram](attachment:image.png)

Eigenvalues for the Google Matrix of the Linux kernel and Weyl asymptotics
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- **Open quantum maps**: popular models in open quantum chaos
  See reviews by Nonnenmacher ’11 (math), Novaes ’13 (physics)
- Proposed experiments: Hannay–Keating–Ozorio de Almeida ’94, Brun–Schack ’99
- Attractive model for numerical experimentation:
Open baker’s maps

Open baker’s maps $\kappa = \kappa_{M,A}$ are determined by
- an integer $M \geq 3$, the base
- a set $A \subset \{0, \ldots, M - 1\}$, the alphabet
- we always assume $1 < |A| < M$

$\kappa$ is a canonical relation on $(0,1)_x \times (0,1)_\xi$:

$$\kappa : (x, \xi) \mapsto \left(Mx - a, \frac{\xi + a}{M}\right)$$

if $x \in \left(\frac{a}{M}, \frac{a + 1}{M}\right)$, $a \in A$

Basic model for a hyperbolic transformation with ‘holes’ through which one can escape
Cantor sets

For $k \in \mathbb{N}$, the domain and range of $\kappa^k$ are

$$\Gamma_k^- := \text{Domain}(\kappa^k) = \{(x, \xi) : \lfloor M^k \cdot x \rfloor \in C_k\}$$

$$\Gamma_k^+ := \text{Range}(\kappa^k) = \{(x, \xi) : \lfloor M^k \cdot \xi \rfloor \in C_k\}$$

where $C_k \subset \{0, \ldots, M^k - 1\}$ is a discrete Cantor set:

$$C_k = C_k(M, A) = \left\{ \sum_{r=0}^{k-1} a_r M^r : a_0, \ldots, a_{k-1} \in A \right\}$$
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The limiting Cantor set

$$C_\infty := \bigcap_k \bigcup_{c \in C_k} \left[ \frac{c}{M^k}, \frac{c+1}{M^k} \right] \subset [0, 1]$$

has Hausdorff dimension

$$\delta := \frac{\log |A|}{\log M} \in (0, 1)$$

Topological pressure: $P(s) = \delta - s$, $s \in \mathbb{R}$
Discrete microlocal analysis

Let $\ell^2_N := \ell^2(\mathbb{Z}_N)$, $\mathbb{Z}_N = \{0, \ldots, N - 1\}$, $N \gg 1$. Fourier transform:

$$\mathcal{F}_N : \ell^2_N \to \ell^2_N, \quad \mathcal{F}_N u(j) = \frac{1}{\sqrt{N}} \sum_\ell e^{-2\pi ij\ell/N} u(\ell)$$

Quantization of observables on the torus $T^2 = S^1_x \times S^1_\xi$, $S^1 = \mathbb{R}/\mathbb{Z}$:

$$a \in C^\infty(T^2) \quad \mapsto \quad \text{Op}_N(a) : \ell^2_N \to \ell^2_N$$

Op$_N$(a) can localize in both position $x$ and frequency $\xi$

Properties

- $a = a(x) \quad \Longrightarrow \quad \text{Op}_N(a) = a_N, \quad a_N(j) = a(j/N)$
- $a = a(\xi) \quad \Longrightarrow \quad \text{Op}_N(a) = \mathcal{F}_N^* a_N \mathcal{F}_N$
- $[\text{Op}_N(a), \text{Op}_N(b)] = -\frac{i}{2\pi N} \text{Op}_N(\{a, b\}) + O(N^{-2}) \ell^2_N \to \ell^2_N$
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Open quantum baker’s maps

Example: $M = 3$, $\mathcal{A} = \{0, 2\}$. We put $N := M^k$ and

$$B_N = \mathcal{F}_N^* \begin{pmatrix} \chi_{N/3} \mathcal{F}_{N/3} \chi_{N/3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \chi_{N/3} \mathcal{F}_{N/3} \chi_{N/3} \end{pmatrix} : \ell^2_N \rightarrow \ell^2_N$$

where we fix $\chi \in C_0^\infty((0, 1); [0, 1])$, $\chi_N(j) = \chi(j/N)$

Why is $B_N$ a quantization of $\kappa_{M,\mathcal{A}}$? It satisfies Egorov’s theorem:

$$B_N \text{ Op}_N(a) = \text{ Op}_N(b) B_N + O(N^{-1})_{\ell^2_N \rightarrow \ell^2_N}$$

if $a(x, \xi) = b(y, \eta)$ when $\kappa_{M,\mathcal{A}}(x, \xi) = (y, \eta)$, $\xi, y \in \text{supp } \chi$
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- Resonances = eigenvalues of $B_N$

$$\text{Spec}(B_N) \subset D(0, 1)$$

- Similar procedure works for any $M, \mathcal{A}$
Numerical example: \( M = 5, \mathcal{A} = \{1, 3\} \)

\[ \text{Spec}(B_N) \text{ for } k = 2, \ N = M^k \]
Numerical example: $M = 5, \mathcal{A} = \{1, 3\}$

$\text{Spec}(B_N)$ for $k = 3, N = M^k$
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$\text{Spec}(B_N)$ for $k = 4$, $N = M^k$
Numerical example: $M = 5$, $\mathcal{A} = \{1, 3\}$

$\text{Spec}(B_N)$ for $k = 5$, $N = M^k$
Results: spectral gaps

Define the spectral radius of $B_N$:

$$R_N := \max \{|\lambda|: \lambda \in \text{Spec}(B_N)\}, \quad N := M^k$$

**Theorem 1 [D–Jin ’16]**

There exists (explicitly computable!)

$$\beta = \beta(M, A) > \max \left(0, \frac{1}{2} - \delta\right)$$

such that $B_N$ has an asymptotic spectral gap of size $\beta$:

$$\limsup_{N \to \infty} R_N \leq M^{-\beta} < 1 \quad (1)$$

The convention $M^{-\beta} = e^{-\beta \log M}$ is due to $\kappa$ having expansion rate $M$.

The bound (1) with $\beta = -P(1/2) = \frac{1}{2} - \delta$ is the pressure gap, valid under the pressure condition $\delta < \frac{1}{2}$.
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Numerical example: $M = 5$, $A = \{1, 3\}$, $N = M^5$

For some cases the gap of Theorem 1 approximates the spectral radius well.
Numerical example: $M = 5$, $\mathcal{A} = \{1, 2\}$, $N = M^5$

...and for some cases, this upper bound is far from sharp
Previous work

Nonnenmacher–Zworski ’07, Walsh quantization of open quantum baker’s maps which uses the Fourier transform on $\otimes^k\mathbb{Z}_M$ instead of $\mathbb{Z}_N$: gap for $M = 3$, $A = \{0, 2\}$, but no gap for $M = 4$, $A = \{0, 2\}$

General hyperbolic systems:

- Patterson ’76, Sullivan ’79, Ikawa ’88, Gaspard–Rice ’89, Nonnenmacher–Zworski ’09: pressure gap $\beta = -P(\frac{1}{2})$ for $P(\frac{1}{2}) < 0$

- Naud ’05, Petkov–Stoyanov ’10, Stoyanov ’11, ’12, Bourgain–Gamburd–Sarnak ’11, Oh–Winter ’16: improved gap $\beta = -P(\frac{1}{2}) + \varepsilon$ for some systems with $P(\frac{1}{2}) \leq 0$, where $\varepsilon > 0$ depends on the system in an unspecified way. Build on Dolgopyat ’98

- D–Zahl ’16, Bourgain–D ’18: gap $\beta > 0$ for all convex co-compact hyperbolic surfaces with $P(\frac{1}{2}) = 0$. Uses fractal uncertainty principle
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- D–Zahl ’16, Bourgain–D ’18: gap $\beta > 0$ for all convex co-compact hyperbolic surfaces with $P(\frac{1}{2}) = 0$. Uses fractal uncertainty principle
Reduction to fractal uncertainty principle

Let \((B_N - \lambda)u = 0, \|u\|_{\ell^2_N} = 1, |\lambda| \geq c > 0\)

Iterate Egorov’s theorem \(\rho k\) times, where \(N = M^k, 0 < 1 - \rho \ll 1\)

\[
B_N^k \text{Op}_N(a)u = \text{Op}_N(b)B_N^k u + \mathcal{O}(N^{-\infty})
\]

if \(a(x, \xi) = b(y, \eta) + \text{L.O.T.}\) when \(\varphi^k(x, \xi) = (y, \eta)\)

This is still possible since the resulting symbols vary on the scale \(N^{-1}\)

Recall \(\Gamma_k^- = \text{Domain}(\varphi^k), \Gamma_k^+ = \text{Range}(\varphi^k)\)
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Recall \(\Gamma^-_k = \text{Domain}(\varkappa^k), \quad \Gamma^+_k = \text{Range}(\varkappa^k)\)

- \(a \equiv 1, \quad b = 1_{\Gamma^+_k} \implies u = \text{Op}_N(1_{\Gamma^+_k})u + O(N^{-\infty})\)
- \(b \equiv 1, \quad a = 1_{\Gamma^-_k} \implies \|\text{Op}_N(1_{\Gamma^-_k})u\| \geq |\lambda|^k\)
- Contradiction if \(|\lambda| \geq M^{-\beta + \epsilon}\) and the fractal uncertainty principle holds with exponent \(\beta:\)

\[
\|\text{Op}_N(1_{\Gamma^-_k})\text{Op}_N(1_{\Gamma^+_k})\|_{\ell^2_N \to \ell^2_N} \leq CN^{-\beta}
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Want to prove the fractal uncertainty principle

$$\|Op_N(1_{\Gamma_k^-})Op_N(1_{\Gamma_k^+})\|_{\ell^2_N \rightarrow \ell^2_N} \leq CN^{-\beta}$$

Using the relation of $\Gamma_k^\pm$ with the Cantor set $C_k \subset \mathbb{Z}_N$, rewrite this as

$$\|1_{C_k}F_N1_{C_k}\|_{\ell^2_N \rightarrow \ell^2_N} \leq CN^{-\beta} \quad (2)$$

(2) ⇒ no function can be localized on $C_k$ in both position and frequency

Volume bound: $N = M^k, \quad |C_k| = |A|^k = N^\delta, \quad \|F_N\|_{\ell^1_N \rightarrow \ell^\infty_N} \leq N^{-1/2}$

⇒ (2) with $\beta = \frac{1}{2} - \delta$, recovering the pressure gap

To prove Theorem 1, we need to improve over $\beta = 0$ and the volume bound
Want to prove the fractal uncertainty principle

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Proof of fractal uncertainty principle

**Theorem 2 [D–Jin ’16]**

We have \( \|1_{C_k} \mathcal{F}_N 1_{C_k}\|_{\ell^2_N \to \ell^2_N} \leq N^{-\beta} \) for some

\[
\beta = \beta(M, A) > \max\left(0, \frac{1}{2} - \delta\right)
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- **Submultiplicativity:** if \( r_k := \|1_{C_k} \mathcal{F}_N 1_{C_k}\|_{\ell^2_N \to \ell^2_N} \) then \( r_k + \ell \leq r_k \cdot r_\ell \)
- Thus enough to show that \( r_k < \min(1, N^{\delta-1/2}) \) for some \( k \)
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\beta = \beta(M, A) > \max \left(0, \frac{1}{2} - \delta \right)
\]

- **Submultiplicativity**: if \( r_k := \|1_{C_k}F_N1_{C_k}\|_{\ell^2_N \to \ell^2_N} \) then \( r_{k+\ell} \leq r_k \cdot r_\ell \)
- Thus enough to show that \( r_k < \min(1, N^{\delta - 1/2}) \) for some \( k \)
- \( r_k < 1 \): if not, then find nonzero \( u = 1_{C_k}u, F_Nu = 0 \) on \( \mathbb{Z}_N \setminus C_k \)

By cyclic shift, may assume that \( M - 1 \notin A \). The polynomial
\[
p(z) = \sum_j u(j)z^j
\]
has degree at most \( \max C_k \leq (M - 1)M^{k-1} \) and at least
\[
|\mathbb{Z}_N \setminus C_k| \geq M^k - (M - 1)^k
\]
routes. Contradiction for large \( k \)
Proof of fractal uncertainty principle

**Theorem 2 [D–Jin ’16]**

We have \( \|1_{C_k} \mathcal{F}_N 1_{C_k}\|_{\ell^2_N \to \ell^2_N} \leq N^{-\beta} \) for some

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\beta = \beta(M, \mathcal{A}) > \max \left(0, \frac{1}{2} - \delta\right)
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- **Submultiplicativity:** if \( r_k := \|1_{C_k} \mathcal{F}_N 1_{C_k}\|_{\ell^2_N \to \ell^2_N} \) then \( r_k \leq r_k \leq r_k \cdot r_\ell \)
- Thus enough to show that \( r_k < \min(1, N^{\delta - 1/2}) \) for some \( k \)
- \( r_k < N^{\delta - 1/2} = |C_k|/\sqrt{N} \): if not, then

\[
\|1_{C_k} \mathcal{F}_N 1_{C_k}\|_{\ell^2_N \to \ell^2_N} = \frac{|C_k|}{\sqrt{N}} = \|1_{C_k} \mathcal{F}_N 1_{C_k}\|_{\text{HS}}
\]

Then \( 1_{C_k} \mathcal{F}_N 1_{C_k} \) has rank 1, so all \( 2 \times 2 \) minors are zero. Contradiction when \( |\mathcal{A}| > 1, \ k = 2 \)
Spectral gaps

More on fractal uncertainty exponents

X axis: $\delta$; Y axis: FUP exponent $\beta$ (numerics); all alphabets with $M \leq 10$

Solid line: $\beta = \max(0, \frac{1}{2} - \delta)$, dashed line: $\beta = -\frac{P(1)}{2} = \frac{1-\delta}{2}$
More on fractal uncertainty exponents

Bounds on $\beta$ as $M \to \infty$:

$\delta \leq 1/2$:

$$\beta - \left(\frac{1}{2} - \delta\right) \gtrsim \frac{1}{M^8 \log M}$$

$\delta \approx 1/2$: using additive energy,

$$\beta \gtrsim \frac{1}{\log M}$$

$\delta \geq 1/2$:

$$\beta \gtrsim \exp \left(- M^{1-\delta} + o(1) \right)$$

- Examples of alphabets (arithmetic progressions) with $\delta \leq 1/2$ and
  $$\beta - \left(\frac{1}{2} - \delta\right) \lesssim \frac{M^{2\delta - 1}}{\log M}$$

- Examples of special alphabets with $\beta = \frac{1-\delta}{2}$
More on fractal uncertainty exponents

Bounds on $\beta$ as $M \to \infty$:

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- Examples of special alphabets with $\beta = \frac{1-\delta}{2}$
Special alphabets with $\beta = \frac{1-\delta}{2}$

We call $\mathcal{A}$ a special alphabet, if

$$\text{for all } j, \ell \in \mathcal{A}, \ j \neq \ell, \ \text{we have } \mathcal{F}_M(1_\mathcal{A})(j - \ell) = 0 \quad (3)$$

Such $\mathcal{A}$ have $\beta = \frac{1-\delta}{2} = -\frac{P(1)}{2}$, which is the largest possible value of $\beta$ and all nonzero singular values of $1_{C_k} \mathcal{F}_N 1_{C_k}$ are equal to $N^{-\beta}$.
Special alphabets with $\beta = \frac{1-\delta}{2}$

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Example: $M = 6$, $\mathcal{A} = \{1, 4\}$, $N = M^5$
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Example: $M = 8, \mathcal{A} = \{2, 4\}, N = M^4$
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Example: $M = 8, \ A = \{1, 2, 5, 6\}, \ N = M^4$
Special alphabets with $\beta = \frac{1-\delta}{2}$

We call $\mathcal{A}$ a **special alphabet**, if

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Such $\mathcal{A}$ have $\beta = \frac{1-\delta}{2} = -\frac{P(1)}{2}$, which is the largest possible value of $\beta$ and all nonzero singular values of $1_{C_k}\mathcal{F}_N 1_{C_k}$ are equal to $N^{-\beta}$

**Conjecture 1 (band structure)**

Assume $(M, \mathcal{A})$ satisfies (3). Then there exists $\mu > \frac{1-\delta}{2}$ such that:

- For any $\varepsilon > 0$ and $N$ large, there is a second gap
  $$\text{Spec}(B_N) \cap \{ M^{-\mu} \leq |\lambda| \leq M^{-\frac{1-\delta}{2}} - \varepsilon \} = \emptyset$$

- Eigenvalues in the first band satisfy exact fractal Weyl law:
  $$|\text{Spec}(B_N) \cap \{|\lambda| \geq M^{-\mu}\}| = |\mathcal{A}|^k = N^{\delta}$$

Conjecture 1 is confirmed by numerics
Results: resonance counting

We count eigenvalues of $B_N$ in annuli:

$$\#(N, \nu) = \left| \text{Spec}(B_N) \cap \{ |\lambda| \geq M^{-\nu} \} \right|$$

Theorem 3 [D–Jin ’16]

For each $\varepsilon > 0$ and $\nu > 0$ we have the fractal Weyl upper bound

$$\#(N, \nu) \leq C_{\nu, \varepsilon} N^{m(\delta, \nu) + \varepsilon}, \quad m(\delta, \nu) = \min(\delta, 2\nu + 2\delta - 1)$$

Note: $m = \delta$ for $\nu \geq \frac{1-\delta}{2} = -\frac{P(1)}{2}$, $m < 0$ for $\nu < \frac{1}{2} - \delta = -P\left(\frac{1}{2}\right)$
Resonance counting

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We count eigenvalues of $B_N$ in annuli:

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- Sjöstrand ’90, Guillopé–Lin–Zworski ’04, Sjöstrand–Zworski ’07, Nonnenmacher–Sjöstrand–Zworski ’11, ’14, Datchev–D ’13: $\#(N, \nu) \leq C_\nu N^\delta$ for more general hyperbolic situations
- Lu–Sridhar–Zworski ’03: concentration of decay rates near $\nu = -P(1)/2$. Jakobson–Naud ’12 conjectured gap of this size
- Naud ’14, Jakobson–Naud ’14: $\#(N, \nu) \leq C_\nu N^{m(\nu)}$, $m(\nu) < \delta$ for $\nu < \frac{1}{2} - \delta$ for convex co-compact hyperbolic surfaces
- D ’15: Theorem 3 for convex co-compact hyperbolic manifolds
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$$\#(N, \nu) \leq C_{\nu, \varepsilon} N^{m(\delta, \nu) + \varepsilon}, \quad m(\delta, \nu) = \min(\delta, 2\nu + 2\delta - 1)$$

No matching lower bounds are known, except

**Nonnenmacher–Zworski ’07:** Exact fractal Weyl law for Walsh quantization

**Conjecture 2 (fractal Weyl law)**

For each $\nu > \frac{1-\delta}{2}$, we have $\#(N, \nu) \geq c_\nu N^\delta > 0$

Conjecture 2 is also supported by numerics
Resonance counting

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**Ideas of the proof**

- Recall that for $(B_N - \lambda)u = 0$, $\|u\| = 1$, $|\lambda| \geq M^{-\nu}$,
  
  $$u = \text{Op}_N(1_{\Gamma_+^k})u + \mathcal{O}(N^{-\infty}), \quad \|\text{Op}_N(1_{\Gamma_-^k})u\| \geq N^{-\nu}$$

- The first statement $\Rightarrow$ $\#(N, \nu) \lesssim \text{Rank} (\text{Op}_N(1_{\Gamma_+^k})) = N^\delta$

- Both statements together $\Rightarrow$ $\#(N, \nu) \lesssim N^{2\nu + 2\delta - 1}$
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Numerical example: $M = 6$, $\mathcal{A} = \{1, 2, 3, 4\}$
Resonance counting

Numerical example: $M = 6, \mathcal{A} = \{1, 2, 3, 4\}$

$k = 3$
Numerical example: $M = 6$, $A = \{1, 2, 3, 4\}$
Numerical example: $M = 6, \mathcal{A} = \{1, 2, 3, 4\}$

Plot of $\log \#(M^k, \nu) / \log M$ as a function of $k$
Numerical example: $M = 6, \mathcal{A} = \{1, 2, 3, 4\}$

Linear fits for the growth exponent of $\#(N, \nu)$ and the bound of Theorem 3
Summary

- We obtain results on spectral gap
- We use fractal uncertainty principle, the fine structure of the associated Cantor sets, and simple tools from harmonic analysis, algebra, combinatorics, and number theory
- We also show a fractal Weyl upper bound
- We discover that the studied systems form a rich class with a variety of different types of behavior
Thank you for your attention!
Results: dependence on cutoff

Recall that the definition of $B_N = B_{N,\chi}$ involved a cutoff function

$$\chi \in C_0^\infty((0, 1); [0, 1])$$

e.g. for $M = 3$, $\mathcal{A} = \{0, 2\}$

$$B_N = \mathcal{F}_N^* \begin{pmatrix} \chi_{N/3} \mathcal{F}_{N/3} \chi_{N/3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \chi_{N/3} \mathcal{F}_{N/3} \chi_{N/3} \end{pmatrix}$$

**Theorem 4 [D–Jin ’16]**

Assume that $\chi_1, \chi_2 \in C_0^\infty((0, 1); [0, 1])$ and $\chi_1 = \chi_2$ near the Cantor set $C_\infty \subset [0, 1]$. Then for each $\nu$, eigenvalues of $B_{N,\chi_1}$ in \{\(|\lambda| \geq M^{-\nu}\)\} are $O(N^{-\infty})$ quasimodes of $B_{N,\chi_2}$. 
If $0, M - 1 \not\in \mathcal{A}$ it is natural to take $\chi = 1$ near $C_\infty$.
However we cannot take $\chi \equiv 1$:

$M = 5$, $\mathcal{A} = \{1, 3\}$, $N = M^5$, $\chi_1 = \chi_2 = 1$ near $C_\infty$