ANALYSIS ASPECTS IN SUBFACTOR THEORY

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Sorin Popa

UCLA

The M-vN II_1 factors and continuous dimension

Definition

An ∞-dim vN factor M that's **finite** (i.e., it satisfies the finiteness axiom: "$u \in M$, $u^*u = 1$ implies $uu^* = 1$") is called a **II_1 factor**.

- The finiteness axiom is equivalent to the existence of a (unique) trace state τ on M, which is a dimension function on $\mathcal{P}(M)$, i.e., $\tau(p) = \tau(q)$ iff $p \sim q$, and satisfies $\tau(\mathcal{P}(M)) = [0, 1]$ (**continuous dimension**).
- Any MASA $A \subset M$ is diffuse, so $\simeq L^\infty([0, 1])$ when M is $\| \; \|_2$-separable.
- Trace gives rise to Hilbert space $L^2(M, \tau)$ by completing M in norm $\|x\|_2 = \tau(x^*x)^{1/2}$, on which M acts by left multiplication. Any Hilbert M-module ${}_M\mathcal{H}$ is either $L^2M^{\oplus\infty}$ or $\oplus_{i=1}^n L^2(Mp_i)$, for some $p_i \in \mathcal{P}(M)$, which up to isomorphism only depends on $\sum_i \tau(p_i)$. This number is denoted $\dim({}_M\mathcal{H}) \in [0, \infty]$ and is called the **M-vN dimension of ${}_M\mathcal{H}$**.

Amplification by positive real numbers

- Clearly $M^n := \mathbb{M}_n(M)$ is a II_1 factor, $\forall n \geq 1$, whose (unique) normalized trace is given by $\tau((x_{ij})_{i,j}) = \sum_i \tau_M(x_{ii})/n$.

- More generally, for each $t > 0$, one denotes by M^t the "t by t matrix algebra over M", $\mathbb{M}_{t\times t}(M)$, obtained as a "corner" $p\mathbb{M}_n(M)p$, where n is any integer $\geq t$ and $p \in \mathbb{M}_n(M)$ is a projection of (normalized) trace $\tau(p) = t/n$. It is itself a II_1 factor, called the **t-amplification** of M.

- One has $(M^s)^t = M^{st}$. $\mathcal{F}(M) := \{t > 0 \mid M^t \simeq M\}$ is called the **fundamental group of M**.

II_1 factors from geometric data

• II_1 factors arise from a variety of "geometric objects" G, such as groups (more generally groupoids) and their action on spaces (including actions of groups on the Hilbert space, i.e., group representations), followed by "geometric operations" like amplifications, tensor products, crossed products. When such data G is amenable (for instance if the acting group is amenable) then the resulting factor $L(G)$ is amenable thus isomorphic to the *hyperfinite II_1 factor* $(R, \tau) := \overline{\otimes}_n(\mathbb{M}_2(\mathbb{C}), tr)_n$. Notably we have:

• If G is an ICC group then $L(G)$ is the *group II_1 factor* of G.

• $G = (\Gamma \curvearrowright X)$ measure preserving (m.p.) action of a countable group Γ (or more generally a countable groupoid) on the probability space (X, μ) that's free ergodic, gives rise to the *group measure space II_1 factor* $L(G) := L^\infty(X) \rtimes \Gamma$. Moreover, $A = L^\infty(X)$ is maximal abelian in $L(G)$ and its normalizer generates $L(G)$, i.e. it is a *Cartan subalgebra* in $L(G)$.

II_1 factors from randomness

• II_1 factors also arise from "randomness constructions", such as *free product* of tracial vN algebras, $(M, \tau) = (M_0, \tau_0) * (M_1, \tau_1) * .....$

• More generally, given a tracial vN algebra M_0 and a "part of it" $B \subset M_0$, which one knows to be in some relation with elements in another tracial vN algebra, $B \subset N_0$, one can combine the two by imposing "random-relation" (freeness) between $M_0 \ominus B$ and $N_0 \ominus B$, by taking the *amalgamated free product* tracial vN algebra $M = M_0 *_B N_0$.

Symmetries of II_1 factors

- The most natural symmetries of a II_1 factor M are its automorphisms $\theta \in \mathrm{Aut}(M)$. They often come from symmetries of the (geometric) building-data of M. For instance, if G is an ICC group, then any $\delta \in \mathrm{Aut}(G)$ gives rise to $\theta_\delta \in \mathrm{Aut}(L(G))$ in the obvious way.

- Another natural symmetry of a II_1 factor M is any isomorphism $\theta : M \simeq M^t$ for some $t > 0$ (in the fundamental group $\mathcal{F}(M)$)

- Both above symmetries $\theta : M \simeq M^t$ can be encoded as the Hilbert bimodule ${}_M\mathcal{H}_M$, given by $y \cdot \xi \cdot x = \theta^{-1}(y)\xi x$, where $\mathcal{H} = pL^2Me_{11}$.

Such Hilbert bimodules ${}_N\mathcal{H}_M$ are abstractly characterized by the condition $\dim({}_N\mathcal{H}_M) := \dim({}_N\mathcal{H})\dim(\mathcal{H}_M)$ equals 1, where N is a II_1 factor (necessarily equal to an amplification of M by this dimension condition).

Virtual-symmetries and subfactors

• More generally, a *virtual λ-symmetry* of M is a Hilbert $N-M$ bimodule ${}_N\mathcal{H}_M$ with $\dim({}_N\mathcal{H}_M) = \lambda^{-1} < \infty$. This is the same as considering a subfactor embedding $N \hookrightarrow M$ with *Jones index* $[M:N] := \dim({}_N L^2 M)$ $= \dim({}_N L^2 M_M)$ equal to $\lambda^{-1} < \infty$ (or a *virtual isomorphism* of M with another factor, of index λ^{-1}). A natural condition on a virtual-symmetry (subfactor) is that it is *irreducible*, i.e. $N' \cap M = \mathbb{C}$.

• The study of virtual symmetries (or subfactors) of II_1 factors was initiated by Vaughan Jones in 1981, who answered the most fundamental question about these objects, by identifying the set of values of the parameter $[M:N] = \lambda^{-1}$, as being $\{4\cos^2\frac{\pi}{n} \mid n \geq 3\} \cup [4,\infty)$. Moreover, he showed that all these values occur as indices of subfactors of $M = R$. (But for $\lambda^{-1} > 4$ the examples were not irreducible.)

• The next fundamental question about virtual symmetries of a given factor M (respectively of subfactors $N \subset M$) is their classification up to conjugacy by automorphisms of M.

The standard invariant

- Given a subfactor $N \subset M$ of index $[M : N] = \lambda^{-1} < \infty$, *Jones iterated basic construction* gives rise to a tower of factors $M_{-1} = N \subset M = M_0 \subset^{e_1} \subset M_2 \subset^{e_2} \ldots$ with a trace τ and projections $e_i \in M_i$ implementing the τ-preserving expectation of M_{i-1} onto M_{i-2} and satisfying $[e_i, e_j] = 0$ if $|i - j| \geq 2$, $e_i e_{i\pm 1} e_i = \lambda e_i$, $\tau(e_i) = \lambda$, $\forall i$.

The system of inclusions of “higher relative commutants” $\{M_i' \cap M_j\}_{i,j \geq -1}$, together with the trace inherited from $\cup_n M_n$, and the representation of the *Jones λ-sequence of projections* $\{e_i\}_i$, is obviously an isomorphism invariant for the subfactor $N \subset M$ (alternatively, for the virtual symmetry ${}_N\mathcal{H}_M$). It is called the *standard invariant $\mathcal{G}_{N\subset M}$* of $N \subset M$. An important part of the information is contained in a (Cayley-type) weighted pointed bipartite graph $\Gamma_{N\subset M}$ called the *standard graph* of $N \subset M$.

Main (early) questions

• The invariant $\mathcal{G}_{N\subset M}$ was already emphasized in 1982-1983 (essentially by Jones, also Pimsner-Popa). While it took years to understand and "bring to life" these invariants, several questions were considered right away:

(a) Find all $\lambda^{-1} > 4$ for which $\exists$ $N \subset M$ irreducible with $[M:N] = \lambda^{-1}$. Is is possible for $\mathcal{G}_{N\subset M}$ to consist of the Jones projections only, i.e., given any $\lambda^{-1} > 4$, $\exists$ $N \subset M$ with $\mathcal{G}_{N\subset M} = \mathcal{G}^{\lambda} := \{\mathrm{Alg}\{e_k \mid i+2 \leq k \leq j\}\}_{j,i}$, e_i being the Jones λ-sequence of projections? Equivalently, $\forall \lambda^{-1} > 4$, $\exists N \subset M$ with $[M:N] = \lambda^{-1}$ and $\Gamma_{N\subset M} = A_\infty$?

(b) Evident follow up question: characterize abstractly all objects $\mathcal{G}$ that can arise as standard invariants of subfactors, $\mathcal{G} = \mathcal{G}_{N\subset M}$, for some $N \subset M$

(c) Is $\mathcal{G}_{N\subset R}$ a complete invariant? Jones '86: no, $\exists$ many "diagonal subfactors" $N^\sigma \subset R$ with same $\mathcal{G}$. Identify class of $N \subset R$ for which it is.

(d) Find all indices $\lambda^{-1} = [M:N]$ (& objects $\mathcal{G}_{N\subset M}$, graphs $\Gamma_{N\subset M}$) of subfactors of a given M, constructed out of certain data, notably $M = R$.

Realizing $\mathcal{G}^\lambda$ and standard λ-lattices

- P1990: $\forall \lambda^{-1} > 4$, $\exists N \subset M$ such that $[M : N] = \lambda^{-1}$ and $\mathcal{G}_{N\subset M} = \mathcal{G}^\lambda$ (equivalently $\Gamma_{N\subset M} = A_\infty$), i.e, so that $N' \cap M_n = \mathrm{Alg}\{e_i\}_{i=1}^n$, where $N \subset M \subset^{e_1} M_1 \subset^{e_2} \dots$ is the Jones tower. The construction of $N \subset M$ imposed itself deductively, by "randomness" considerations.

- This led me to propose in 1994 the following abstraction for $\mathcal{G}_{N\subset M}$: given any λ^{-1} in $\{4\cos^2\frac{\pi}{n} \mid n \geq 3\} \cup [4, \infty)$, a *standard λ-lattice* $\mathcal{G}$ is a system of finite dimensional C*-algebras $\{A_{ij}\}_{i,j\geq -1}$, with a faithful trace τ and a representation of the Jones λ-sequence of projections $e_1, e_2, \dots$, satisfying a *Jones-Markov axiom* and a *commutation axiom*. And I proved that this is indeed a complete set of axioms: given any such object $\mathcal{G}$, $\exists N \subset M$ with $[M : N] = \lambda^{-1}$ and $\mathcal{G}_{N\subset M} = \mathcal{G}$. Again, the "reconstruction" subfactor is "random" in nature. It depends on an "initial data" Q and is functorial in both $\mathcal{G}$ and Q. Denoted $N^{\mathcal{G}}(Q) \subset M^{\mathcal{G}}(Q)$.

- P-Shlyakhtenko 2000: If $Q = L\mathbb{F}_\infty$ then $N^{\mathcal{G}}(Q) \subset M^{\mathcal{G}}(Q)$ identified as $L\mathbb{F}_\infty$. Thus: given any λ-lattice $\mathcal{G}$, there exists $N \subset M$ of index λ^{-1} such that $\mathcal{G}_{N\subset M} = \mathcal{G}$ and $N \simeq M \simeq L\mathbb{F}_\infty$. Using free probability (Voiculescu).

Standard lattices, planar algebras, virtual groups

• The axiomatization of standard λ-lattices makes them versatile for "reconstruction" purposes. Exploiting the "frugality" of conditions involved led to some obstruction criteria for bipartite graphs to be standard. But overall, it does not allow investigation of concrete examples.

• In 1999 Jones discovered a way of describing the standard invariant as a two dimensional diagrammatic structure of tangles called *planar algebras*. This provides a powerful "calculus tool" to carry out computations for concrete such structures, allowing for instance to classify all standard invariants of subfactors of index ≤ 5.

• Viewing $N \subset M$ with $[M:N] = \lambda^{-1}$ as a virtual λ-symmetry ${}_N\mathcal{H}_M$, with $\dim({}_N\mathcal{H}_M) = \lambda^{-1}$, leads to interpreting $\mathcal{G}_{N\subset M}$ as the 2-tensor category generated by $\mathcal{H}$ under adjointness and relative tensor product (fusion), of Hilbert bimodules ${}_{M_i}\mathcal{K}_{M_j}$, $M_i, M_j \in \{N, M\}$ (finite depth Ocneanu 1987; general case from P 1994). This axiomatisation of $\mathcal{G}_{N\subset M}$, which puts more emphasis on the "multiplication table" (fusion rules) of the virtual symmetry and its adjoint, is what I will call *virtual λ-group*.

Amenability for subfactors and virtual λ-groups

- The standard graph $\Gamma_{N \subset M}$ of a subfactor $N \subset M$ is *amenable* if $\|\Gamma_{N \subset M}\|^2 = [M : N]$. A virtual λ-group $\mathcal{G}$ is *amenable* $\|\Gamma_{\mathcal{G}}\|^2 = \lambda^{-1}$. Equivalent to a key Følner-type condition involving the weights of graphs $\Gamma_{N \subset M}, \Gamma_{\mathcal{G}}$. It is also equivalent to the co-amenability of the symmetric enveloping inclusion of any $N \subset M$ with $\mathcal{G}_{N \subset M} = \mathcal{G}$.

Theorem (Popa 1990-1997)

1. The standard invariant is a complete invariant for hyperfinite subfactors with amenable graph.
2. For $N \subset M$ the following are equivalent:
   (a) $M \simeq R$ and $\Gamma_{N \subset M}$ is amenable;
   (b) the symmetric enveloping II_1 factor of $N \subset M$ is amenable;
   (c) $C^*(M, e_N, M^{op}) \subset \mathcal{B}(L^2 M)$ is simple;
3. If $N \subset M$ has amenable graph then it contains a hyperfinite subfactor $Q \subset R$ as a smooth commuting square embedding (1992 for $\Gamma_{N \subset M}$ ergodic; general case 2018).

Subfactor-amenability is hereditary

Theorem (P 1997)

Let $N \subset M$ be an extremal inclusion of hyperfinite factors with amenable graph, i.e. $\|\Gamma_{N \subset M}\|^2 = [M : N]$. If an inclusion of factors $Q \subset P$ is embeddable into $N \subset M$ with commuting squares, but not necessarily extremal, nor with the same index, then $\mathcal{G}_{Q \subset P}$ is also amenable, i.e. $\|\Gamma_{Q \subset P}\|^2$ equals the minimal index of $Q \subset P$. In particular, if $Q \subset P$ has index > 4 and A_∞-graph (TLJ standard invariant), then it cannot be embedded into $N \subset M$.

Index-rigidity for II_1 factors from geometric data

• We'll say that a II_1 factor M has the *UVC-property* if it has a Cartan subalgebra $A \subset M$ and any MASA $B \subset M$ whose normalizer generates a subfactor of finite index in M (i.e., B is a virtually Cartan subalgebra of M) must be unitary conjugate to A.

• (Popa2001, Ozawa-P 2007) The UVC-property is stable to virtual isomorphism. Moreover, if M is UVC and $N \subset M$ is an irreducible subfactor, then $[M : N]$ is an integer and it has "group-like" standard invariant $\mathcal{G}_{N \subset M}$, of one of the forms C_-, C_+, C_0.

Theorem P-Vaes 2011

Given any free group $\Gamma = \mathbb{F}_n$, $2 \leq n \leq \infty$, and any free ergodic p.m.p. action $\Gamma \curvearrowright X$, the associated II_1 factor $M = L^\infty(X) \rtimes \Gamma$ is UVC. Thus, any irreducible subfactor N of M has integer index and group-like standard invariant $\mathcal{G}_{N \subset M}$. In fact, same is true for many other groups Γ (e.g. any product of non-elementary hyperbolic groups).

• The above statement should hold true for many more group measure space factors:

(a) It has been conjectured that if a group Γ satisfies $\beta_n^{(2)}(\Gamma) \neq 0$ for some $n \geq 1$, then Γ is $\mathcal{C}_s$-rigid, i.e., $M = L^\infty X \rtimes \Gamma$ is UVC for any free ergodic p.m.p. action $\Gamma \curvearrowright X$.

(b) It has also been conjectured that if Γ is an arbitrary non-amenable group and $\Gamma \curvearrowright X = X_0^\Gamma$ is a Bernoulli action, then $M = L^\infty X \rtimes \Gamma$ is UVC. This may even be true for all mixing $\Gamma \curvearrowright X$.

• **Observation**: if $\mathbb{F}_\infty \curvearrowright X$ is a free ergodic profinite action, then $M = L^\infty X \rtimes \mathbb{F}_\infty$ is UVC (by Ozawa-P 2007), so it has highly rigid symmetry structure; but M can also be obtained an inductive limit of an increasing sequence of finite index irreducible embeddings $M_n \hookrightarrow M_{n+1}$ with $M_n \nearrow M$, $M_n \simeq L\mathbb{F}_\infty$, $\forall n$. So at each step M_n has highly ... non-rigid symmetry structure (all virtual λ-groups act on it), but with the resulting limit factor being symmetry-rigid !

Some problems

• Denote by $\mathcal{I}(M)$ the set of all indices of irreducible subfactors of finite index of the II_1 factor M.

• Denote by $\mathbb{E}$ the set of square norms of bipartite graphs. N.B.: this set contains $[\sqrt{5}+2, \infty)$ but is known to only have an increasing sequence of accumulation points converging to $\sqrt{5}+2$ (the first one being 4, which is the accumulation point of $\|A_{n-1}\|^2 = 4\cos^2\frac{\pi}{n}, n \geq 3$).

Index-rigidity onjectures

1° $\mathcal{I}(R) = \mathbb{E}$.

2° If M has a Cartan decomposition then $\mathcal{I}(R) \subset \mathbb{E}$.

More problems

1. Characterize all virtual λ-groups that can "act" on R.
2. Is any $\lambda^{-1} \in \mathcal{I}(R)$ a "TLJ" point ? (i.e., $\exists N \subset R$ with $\mathcal{G}_{N \subset R} = \mathcal{G}^\lambda$?)
3. If $\mathcal{G}$ is non-amenable and can "act" on R, then there exist "many" non-conjugate subfactors of R with same standard invariant $\mathcal{G}$.