Quantum Fourier Analysis

https://www.researchgate.net/project/Quantum-Fourier-analysis

Presented by **JINSONG WU**

Joint with Chunlan Jiang, Zhengwei Liu, Simeng Wang

Institute for Advanced Study in Mathematics
Harbin Institute of Technology

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East China Normal University
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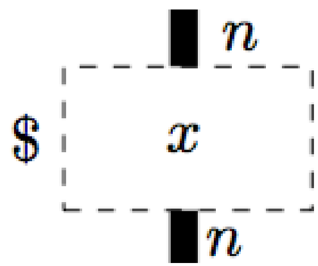
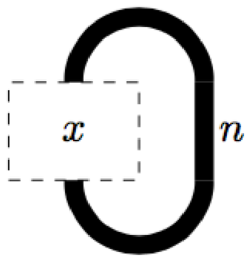
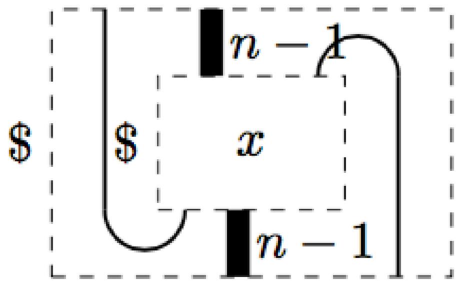
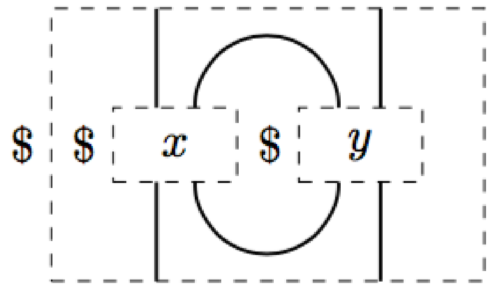
$\mathbb{R}$	REAL LINE
G	LOCALLY COMPACT ABELIAN GROUP
$\mathbb{K}$	KAC ALGEBRA
$\mathbb{G}$	LOCALLY COMPACT QUANTUM GROUP
$\mathscr{P}$	SUBFACTOR PLANAR ALGEBRA
$\mathscr{C}$	MODULAR TENSOR CATEGORY

	Element	Measure
$\mathbb{R}$	Function $f(s)$	Lebesgue Measure ds
G	Function $f(g)$	Haar Measure dg
$\mathbb{K}$ $\mathbb{G}$	Operator x	Haar Weight φ
$\mathscr{P}$	Box 	Trace 
$\mathscr{C}$	Homomorphism	Trace

	Fourier Transform
$\mathbb{R}$	$\int f(s)e^{-2\pi its}ds$
G	L_f
$\mathbb{K}\,\mathbb{G}$	$(\varphi\otimes\iota)((x\otimes I)W)$
$\mathscr{P}$	
$\mathscr{C}$	S-Matrix

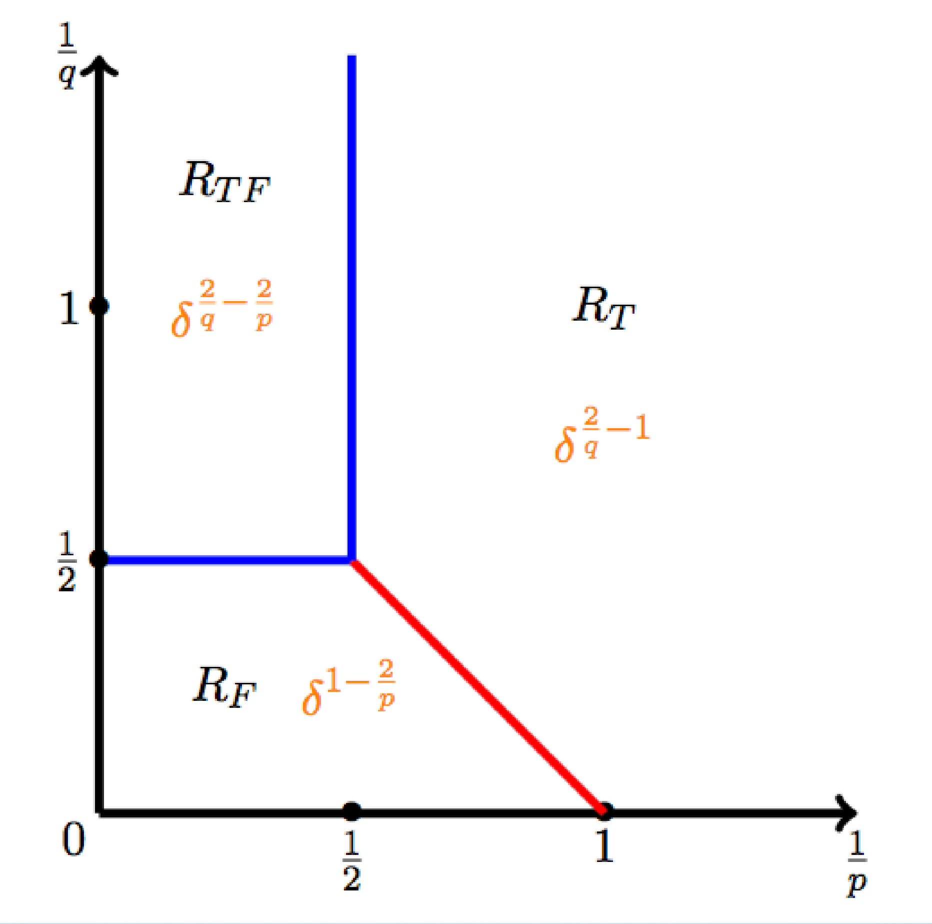
	Convolution
$\mathbb{R}$	$\int f_1(s) f_2(t-s)ds$
G	$\int f_1(g) f_2(g^{-1}h)dg$
$\mathbb{K}\,\mathbb{G}$	$((x\varphi)S^{-1}\otimes\iota)(\Delta(y))$
$\mathscr{P}$	
$\mathscr{C}$	Tensor product

$\mathbb{R}$	Standard Gaussian Function		Gaussian Function
G	Subgroup	Coset	Translation & modulation
$\mathbb{K}\quad\mathbb{G}$	Group-like projection	Left (right) shift	Bi-shift
$\mathscr{P}$	biprojection	Left (right) shift	Bi-shift
$\mathscr{C}$	Fusion subcategory	Left (right) shift	Bi-shift

	Hausdorff-Young inequality
$\mathbb{R}$	Babenko 1961, Beckner 1975, Christ 2014
G	Kunze 1958, Russo 1974, Fournier 1977, Terp 1981
$\mathbb{K}$ $\mathbb{G}$	Cooney 2012
$\mathscr{P}$	Jiang–Liu–W 2016, 2018
$\mathscr{C}$	Liu–W 2018

The Norm of Fourier Transform for Subfactors

$$K(1/q,1/p)^{-1}\|x\|_p \leq \|\mathfrak{F}_s(x)\|_q \leq K(1/p,1/q)\|x\|_p.$$



Regions	Extremizers
$1/p + 1/q > 1, 1/p > 1/2$	trace-one projections
$1/p + 1/q = 1, 1/2 < 1/p < 1$	bi-shifts of biprojections
$1/p = 1, 1/q = 0$	extremal elements
$1/p = 1/2, 1/q = 1/2$	$\mathscr{P}_{2,\pm}$
$1/p + 1/q < 1, 0 < 1/q < 1/2$	Fourier transform of trace-one projections
$1/q = 0, 0 \leq 1/p < 1$	extremal unitary elements
$1/q = 1/2, 0 \leq 1/p < 1/2$	unitary elements
$1/q > 1/2, 1/p = 1/2$	Fourier transform of unitary elements
$1/q > 1/2, 1/p < 1/2$	biunitary elements if exist

	Young's inequality
$\mathbb{R}$	Young 1912, Babenko 1961, Beckner 1975
G	Klein-Russo 1978
$\mathbb{K}$ $\mathbb{G}$	Liu-W-Wang 2017
$\mathscr{P}$	Jiang-Liu-W 2016, 2018
$\mathscr{C}$	Liu-W 2018

The Norm of Convolution for Subfactors

$r \geq 1$

$$\|x*y\|_r \leq \begin{cases} \delta^{1+\frac{2}{r}-(\frac{2}{p}+\frac{2}{q})}\|x\|_p\|y\|_q & p,q\geq 1, \frac{1}{p}+\frac{1}{q}\geq 1, \frac{1}{p}+\frac{1}{q}<1+\frac{1}{r} \text{ or } \\ & \frac{1}{p}+\frac{1}{q}<1, \\ \delta^{\frac{2}{r}-\frac{2}{p}}\|x\|_p\|y\|_q & 0<q<1, r<p, \\ \delta^{\frac{2}{r}-\frac{2}{q}}\|x\|_p\|y\|_q & 0<p<1, r<q, \\ \delta^{-1}\|x\|_p\|y\|_q & \text{otherwise} \end{cases}$$

$0 < r < 1$

$$\|x*y\|_r \leq \begin{cases} \delta^{1+\frac{2}{r}-(\frac{2}{p}+\frac{2}{q})}\|x\|_p\|y\|_q & p,q\geq 1 \\ \delta^{\frac{2}{r}-\frac{2}{p}}\|x\|_p\|y\|_q & 0<q<1, 1<p, \\ \delta^{\frac{2}{r}-\frac{2}{q}}\|x\|_p\|y\|_q & 0<p<1, 1<q, \\ \delta^{\frac{2}{r}-3}\|x\|_p\|y\|_q & 0<p,q<1 \end{cases}$$

Uncertainty Principles

	Donoho-Stark Uncertainty Principle	Hirschman-Beckner uncertainty principle	Hardy's uncertainty principle	Renyi uncertainty principle
$\mathbb{R}$		Hirschman 1957 Beckner 1975	Hardy 1933	
G	Donoho-Stark 1989 Smith 1990 Ozaydm-Przebinda 2004	Ozaydm-Przebinda 2004	Jiang-Liu-W 2016 Liu-W 2017	Gilbert-Rzeszotnik 2010 Madiman-Xu 2016
$\mathbb{K}$	Crann-Kalantar 2014 Liu-W 2017	Crann-Kalantar 2014 Liu-W 2017	Liu-W 2017	
$\mathbb{G}$	Jiang-Liu-W 2018	Jiang-Liu-W 2018	Jiang-Liu-W 2018	
$\mathscr{P}$	Jiang-Liu-W 2016	Jiang-Liu-W 2016	Jiang-Liu-W 2016	Liu-W 2018
$\mathscr{C}$	Liu-W 2018	Liu-W 2018	Liu-W 2018	Liu-W 2018

	Sum-set Theorem
G	Tao–Vu 2006
$\mathbb{K}$	Liu–W 2017
$\mathscr{P}$	Jiang–Liu–W 2018
$\mathscr{C}$	Liu–W 2018

WE NEED IT TO CHARACTERIZE THE EXTREMIZERS OF YOUNG'S INEQUALITY !

Work in Progress ………………

SUGGESTED BY QUANHUA XU!

	Brascamp-lieb inequality
$\mathbb{R}$	Brascamp–Lieb 1976 Lieb 1990 Ball 1989 Barthe 1998 Carlen–Lieb–Loss 2004 Bennett–Carbery–Christ–Tao 2008
$\mathscr{P}$	

Partial Results

$$\left\| \prod_{1\le j\le m} x_j \right\|_1 \le \begin{cases} \prod_{1\le j\le m} \|x_j\|_{p_j}, & \sum_{1\le j\le m} \frac{1}{p_j} \ge 1, \\ \delta^{2-2\sum_{1\le j\le m}\frac{1}{p_j}} \prod_{1\le j\le m} \|x_j\|_{p_j}, & \sum_{1\le j\le m} \frac{1}{p_j} < 1. \end{cases}$$

$$\left\| \mathfrak{F}_s \left(\prod_{1\le j\le m} x_j \right) \right\|_1 = \begin{cases} \delta \prod_{1\le j\le m} \|x_j\|_{p_j} & \sum_{1\le j\le m} \frac{1}{p_j} \ge \frac{1}{2} \\ \delta^{2-2\sum_{1\le j\le m}\frac{1}{p_j}} \prod_{1\le j\le m} \|x_j\|_{p_j} & \sum_{1\le j\le m} \frac{1}{p_j} < \frac{1}{2} \end{cases}$$

$$\|x_2 \mathfrak{F}_s(x_1)\|_1 \le \begin{cases} \delta^{1-\frac{2}{p_1}} \|x_1\|_{p_1} \|x_2\|_{p_2} & 1 \le p_1 \le 2, 0 < p_2 < p_1 \ \ or \ \ p_1 \ge 2, 0 < p_2 < 2 \\ \delta^{1-\frac{2}{p_2}} \|x_1\|_{p_1} \|x_2\|_{p_2} & 1 \le p_1 \le 2, p_2 \ge p_1 \ \ or \ \ p_1 < 1, p_2 > 1 \\ \delta^{2-\frac{2}{p_1}-\frac{2}{p_2}} \|x_1\|_{p_1} \|x_2\|_{p_2} & p_1 \ge 2, p_2 \ge 2 \\ \delta^{-1} \|x_1\|_{p_1} \|x_2\|_{p_2} & 0 < p_1 < 1, 0 < p_2 < 1. \end{cases}$$

Partial Results

$$\|(x_1 * x_2)x_3\|_1 \leq \begin{cases} \delta^{3-2(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3})}\|x_1\|_{p_1}\|x_2\|_{p_2}\|x_3\|_{p_3} & p_1, p_2 \geq 1, \frac{1}{p_1}+\frac{1}{p_2} \geq 1, \frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3} < 2 \ \textit{or} \\ & \frac{1}{p_1}+\frac{1}{p_2} < 1, p_3 \geq 1 \\ \delta^{1-\frac{2}{p_1}-\frac{2}{p_2}}\|x_1\|_{p_1}\|x_2\|_{p_2}\|x_3\|_{p_3} & 0 < p_3 < 1, \frac{1}{p_1}+\frac{1}{p_2} < 1, \\ \delta^{1-\frac{2}{p_1}-\frac{2}{p_3}}\|x_1\|_{p_1}\|x_2\|_{p_2}\|x_3\|_{p_3} & 0 < p_2 < 1, \frac{1}{p_1}+\frac{1}{p_3} < 1, \\ \delta^{1-\frac{2}{p_2}-\frac{2}{p_3}}\|x_1\|_{p_1}\|x_2\|_{p_2}\|x_3\|_{p_3} & 0 < p_1 < 1, \frac{1}{p_2}+\frac{1}{p_3} < 1, \\ \delta^{-1}\|x_1\|_{p_1}\|x_2\|_{p_2}\|x_3\|_{p_3} & \textit{otherwise} \end{cases}$$

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QUESTIONS

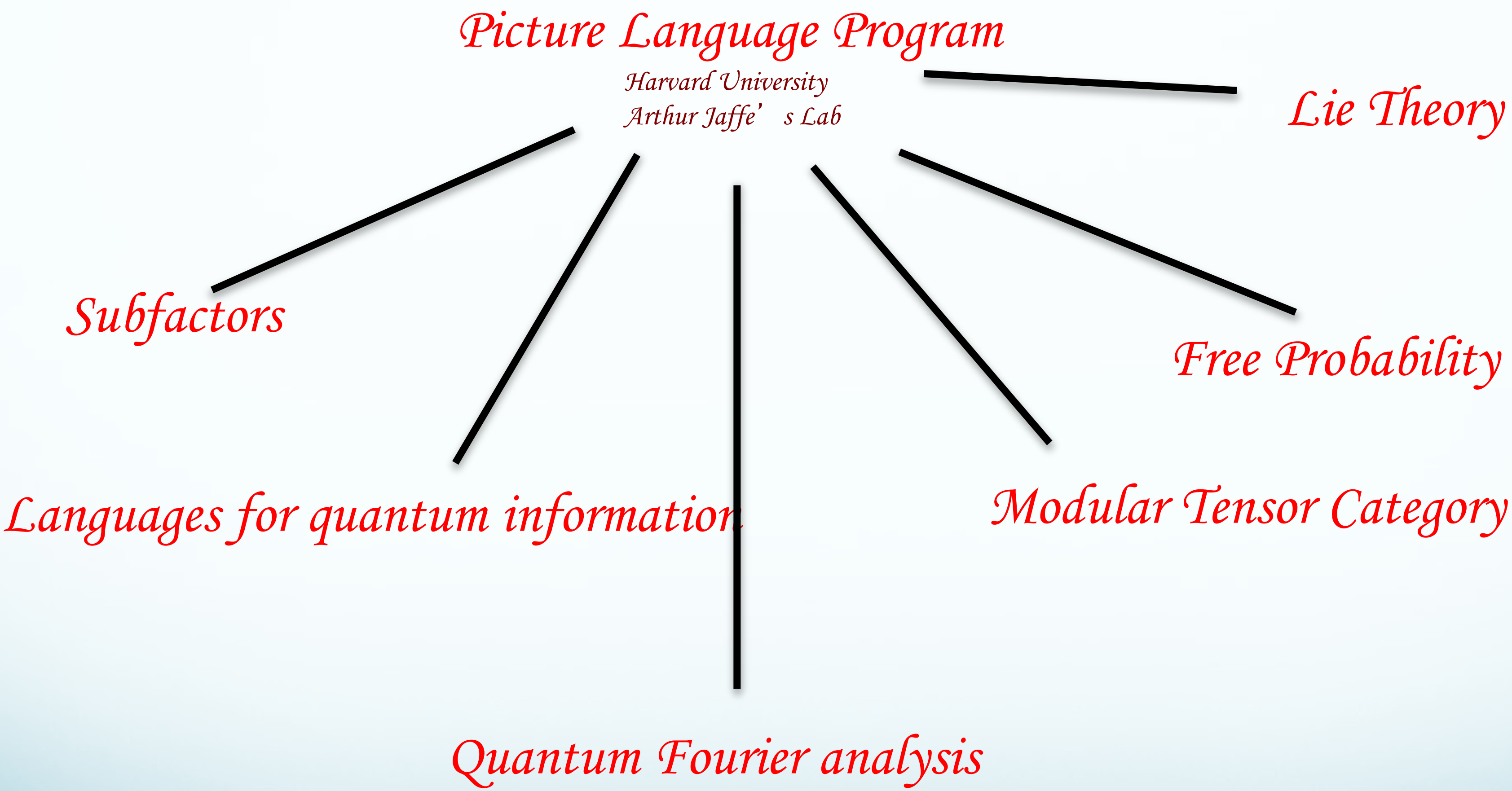
1	*Is there any way to generalize subfactors and locally compact quantum groups?*
2	*Do we have "Sum-set Theorem" for locally compact quantum groups?*
3	*Can we characterize the extremizers of Hausdorff-Young inequality for locally compact quantum groups?*
4	*Can we characterize the extremizers of Young's inequality for locally compact quantum groups?*
5	*Can we characterize the norm of Fourier transform for compact (discrete) quantum groups?*
6	*Can we formulate the Brascamp-Lieb inequality for locally compact quantum groups?*

QUESTIONS

7	*Can we characterize the minimizers of uncertainty principles for n-box spaces?*
8	*Can we describe the extremizers of Hausdorff-Young inequality and Young’ s inequality for n-box spaces?*
9	*Can we formulate the reverse Brascamp-Lieb inequality for subfactors?*
10	*Can we find the Brascamp-Lieb constants for the inequalities?*
11	*If an element is almost a biprojection, is it close to a biprojection?*
12	*Can we give a sharp Hausdorff-Young inequality for subfactors depending on the distance to the element?*

QUESTIONS

13	*Can we construct bi-shifts of biprojections for given subfactors?*
14	*Do we have Hausdorff-Young inequality for infinite index subfactors?*
15	*Do we have Young’ s inequality for infinite index subfactors?*
16	*Do we have uncertainty principles for infinite index subfactors?*
17	*Can we establish additive combinatorics for modular tensor categories?*
18	*How can we apply the analysis results to quantum information?*


Picture Language Program
Harvard University
Arthur Jaffe's Lab
Lie Theory
Subfactors
Free Probability
Languages for quantum information
Modular Tensor Category
Quantum Fourier analysis

THANK YOU !