On the classification of SPT/SET orders with finite internal symmetries

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November 10, 2020

Harvard University, Mathematical Picture Language Seminar

On the classification of topological phases with finite internal symmetries
Liang Kong, Tian Lan, Xiao-Gang Wen, Zhi-Hao Zhang, Hao Zheng arXiv:2003.08898

Many pictures in this PPT were drawn by Zhi-Hao Zhang.

A quantum matter is a state of matter at zero temperature. A quantum phase is the universal class of quantum matters. Quantum phases are distinguished from each other by quantum phase transitions. There are two types of quantum phases: gapped and gapless.

We are interested in a special family of gapped quantum phases called *gapped quantum liquids*. Zeng,Wen:2015 A gapped quantum liquid without symmetry is called a topological order Wen:1989,1990. Gapped quantum liquids with symmetries include gapped spontaneous symmetry breaking orders, symmetry enriched topological (SET) orders and symmetry protected trivial (SPT) orders Gu-Wen:2009,Chen-Gu-Liu-Wen:2013. For mathematicians, both topological orders and SPT orders are special cases of SET orders.

There are many results on the classification of SPT/SET orders. These results are obtained from case-by-case studies and sometimes based on different definitions.

In 1+1D,

1. for bosonic systems, they are classified by triples (G_H, G_ψ, ω), where G_H is the symmetry group of the Hamiltonian, G_ψ is that of the ground state $G_\psi \subset G_H$), and $\omega \in H^2(G_\psi, U(1))$ is a 2-cocycle; Chen, Gu and Wen, Phys. Rev. B 83 (2011) 035107, Schuch, Pérez-García and Cirac, Phys. Rev. B 84 (2011) 165139
2. for fermionic systems, the classification can be obtained from that for bosonic systems via the Jordan-Wigner transformation. Chen, Gu, Wen, Phys. Rev. B 84 (2011) 235128

In 2+1D, there are two approaches towards the classification of SPT/SET orders.

1. based on G-crossed braided fusion 1-categories, works for bosonic systems Barkeshli, Bonderson, Cheng, Wang, Phys. Rev. B 100 (2019) 115147. Important steps were made for fermionic SET orders. Fidkowski, Vishwanath, Metlitski, arXiv:1804.08628.
2. based on the minimal modular extensions of unitary braided fusion 1-categories. Lan, K., Wen, Phys. Rev. B 95 (2017) 235140, Commun. Math. Phys. 351 (2017) 709.

In 3+1D, based on unitary modular 2-categories and σ-model constructions. Lan, K. and Wen, Phys. Rev. X 8 (2018) 021074, Lan, Wen, Phys. Rev. X 9 (2019) 021005, Zhu, Lan and Wen, Phys. Rev. B 100 (2019) 045105

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In 4+1D, examples are known, but classification is unknown.

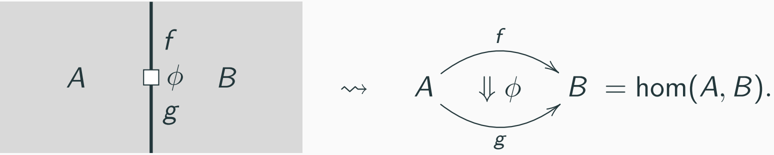
Question: Is there a universal approach towards the classification of topological orders with symmetries in all dimensions?

Two fundamental approaches towards the study of topological orders.

1. Microscopic approach: The "gapped" condition implies that only the ground state matters. This leads to the definition of a gapped quantum liquid as a equivalence class of (ground) states with the equivalence relation defined by local unitary transformations and the stacking of the product states Chen-Gu-Wen:10, Zeng-Wen:15.
   A product state: $\otimes_i |i\rangle \in \mathcal{H}_{tot} = \otimes_i \mathcal{H}_i$, where $|i\rangle \in \mathcal{H}_i$.
2. Macroscopic approach: the collection of all observables in the long wave length limit form a higher category. Kitaev:06, K.-Wen:14, K.-Wen-Zheng:15, Johnson-Freyd:20, K.-Lan-Wen-Zhang-Zheng:20. Other macroscopic formulations. Wen:1990

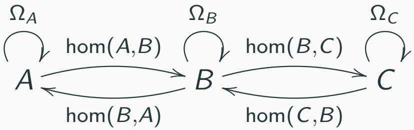
We will only discuss the macroscopic approach.

A topological order is gapped. All correlation functions decay exponentially. It seems that there is no "observable" in the long wave length limit. It turns out that a topological order allows non-trivial (gapped) topological defects.



All these defects form a higher category of n+1D anomaly-free topological orders TO^{n+1}: 0-morphisms $A, B, C, \cdots$ (labels of topological orders) + 1-morphisms are 1-codimensional defects $f, g, \cdots$ + 2-morphisms are 2-codimensional defects ϕ, so on and so forth.

When $A=B$, $\hom(A,A)=\Omega_A$, $1_A\in\hom(A,A)$, $\Omega_A^2=\hom(1_A,1_A)$,

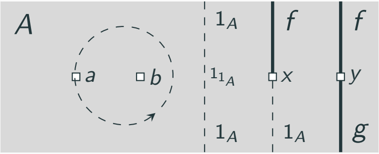


1. Ω_A is unitary fusion n-category of defects of codimension ≥ 1;
2. Ω_A^2 is unitary non-degenerate braided fusion $(n-1)$-category of defects of codimension ≥ 2.

Gaiotto-Johnson-Freyd:2019, Johnson-Freyd:2020

The n+1D topological order A is characterized by Ω_A uniquely up to invertible topological orders. Kitaev:2005, K.Wen:2014, Johnson-Freyd:2020

For a 2+1D topological order A:



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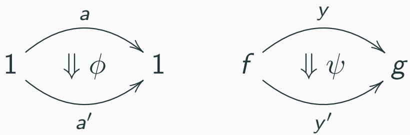

1_A labels the trivial 1d wall; 1_{1_A} = trivial particle, ϕ, ψ are defects in time axis.

1. $a \xrightarrow{\phi} b$ for a 1-category + fusion + braiding = unitary modular tensor 1-category (UMTC) $\mathcal{M}$. nondegeneracy: $(\forall a, c_{b,a} \circ c_{a,b} = \mathrm{id}_{a\otimes b}) \Rightarrow (b = 1_{1_A}^{\oplus n}.)$

Fröhlich:CMP. 47 (1976), Leinaas-Myrheim:1977, Wilczek:1982, Wu:1986, Fröhlich: in "Non-perturbative Quantum Field Theory", G. 't Hooft et al. (eds.)1988, Moore-Seiberg:CMP 123 (1989), Fredenhagen-Rehren-Schroer:CMP 125(2) (1989), Fröhlich-Gabbiani:Rev.Math.Phys. 2(3) 1990, Reshetikhin-Turaev:Inv.Math (1991), Fröhlich-Kerler: Lecture.Notes.in.Mathematics, 1542 (1993), Turaev:Book (1994), Kitaev:Ann.Phys (2006)

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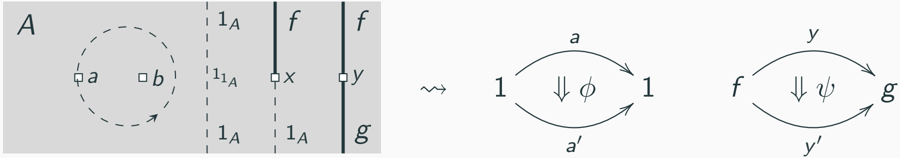

1_A labels the trivial 1d wall; 1_{1_A} = trivial particle, ϕ, ψ are defects in time axis.

2. 1-codimensional defects $1, f, g, h, \cdots$, 2-codimensional defects $x, y, \cdots$, instantons = unitary fusion 2-category $\Sigma\mathcal{M}$, which is the delooping of $\mathcal{M}$.
Douglas-Reutter:2018, Johnson-Freyd:2020, K.-Lan-Wen-Zhang-Zheng:2020. Note that $\mathcal{M} = \Omega\Sigma\mathcal{M}$.

Adding a missing data: chiral central charge c, we obtain $A = (\mathcal{M}, c)$ or $(\Sigma\mathcal{M}, c)$.

It is reasonable that a SPT/SET order can be similarly described by its topological defects up to invertible topological orders.

However, this idea soon runs into problems. We will illustrate the problem in 2+1D SPT orders.

For a 2+1D SPT order with symmetry G, there is no nontrivial topological excitations, but only the trivial ones, i.e. the local excitations that carry symmetry charges (G-representations). They form the category $\mathrm{Rep}(G)$ of G-representations.

- For bosonic symmetries, it is the usual $\mathrm{Rep}(G)$ with symmetric braidings. Here "symmetric" means the double braidings are all trivial.
- For fermionic symmetries, the fermion parity $z \in G$ counts the number (modulo 2) of fermions. Since exchanging two fermions gives a phase factor -1, this provides a fermionic braiding structure on $\mathrm{Rep}(G)$. In order to distinguish two braiding structures, we denote the category $\mathrm{Rep}(G)$ equipped with the fermionic braiding structure by $\mathrm{Rep}(G, z)$.

Both cases can be summarized as a symmetric fusion 1-category $\mathcal{E}$. By Deligne's theorem $\mathcal{E} = \mathrm{Rep}(G), \mathrm{Rep}(G, z)$.

Since a SPT order cannot contain any topological excitations other than symmetry charges, we should expect that there is only one 3D SPT order described by either $\mathrm{Rep}(G)$ or $\mathrm{Rep}(G, z)$.

On the other hand, it was well known that there are many 3D SPT orders with symmetry G, classified by $H^3(G, \mathrm{U}(1))$ in bosonic cases [Chen-Gu-Liu-Wen:13], and by Kitaev's 16-fold way when $G = (\mathbb{Z}_2, z)$ is the fermion parity symmetry [Kitaev:06].

It seems there is an obvious contradiction.

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It seems there is an obvious contradiction.

This contradiction tells us that the description of a SPT order by symmetry charges alone is incomplete.

Question: What are the missing data?

We need some new ideas: gauging the symmetry and boundary-bulk relation.

One way to proceed is based on the idea of gauging the symmetry. Levin and Gu showed that the symmetry can be gauged in different ways and produce different topological orders [Levin-Gu:12]:

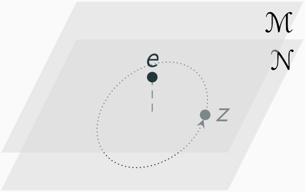
1. The original symmetry transformations are global. By gauging it, we introduce G-gauge fields coupling to the Hamiltonian. As a consequence, the original symmetry becomes a local symmetry of the gauged theory.
2. The new Hamiltonian realizes a 3D topological order because the symmetry is broken. The topological excitations of the gauged theory include both the symmetry charges and the gauge fluxes, and form a UMTC.

Different SPT orders are distinguished by different gauged theories. For $G = \mathbb{Z}_2$, Levin and Gu showed that one gauged theory is the $\mathbb{Z}_2$ topological order, and the second one is the double semion. [Levin-Gu:12]

Later, Lan, K. and Wen introduced a categorical formulation of the gauging process [Lan-K.-Wen:16].

Given a 3D SPT order, we know that the topological excitations are symmetry charges, which form a symmetric fusion category $\mathcal{E} = \mathrm{Rep}(G)$ or $\mathrm{Rep}(G, z)$. Then we can gauge the symmetry by introducing additional topological excitations (gauge fluxes). Together, they form a UMTC $\mathcal{M}$, which describes a 3D topological order (the gauged theory of the SPT order). Therefore we obtain a pair $(\mathcal{M}, \iota_{\mathcal{M}} : \mathcal{E} \hookrightarrow \mathcal{M})$, which is called a modular extension of $\mathcal{E}$.

However, not all modular extensions describe a gauged theory. For example, given a gauged theory $\mathcal{M}$, we can always stack it with a topological order $\mathcal{N}$, then $\mathcal{E} \hookrightarrow \mathcal{M} \hookrightarrow \mathcal{M} \boxtimes \mathcal{N}$ is still a modular extension, but clearly does not describe a gauged theory.



We need a minimal condition. We require that any additional particle y should have nontrivial double braiding with at least one symmetry charge $e \in \mathcal{E}$. This condition excludes the z-particle in above picture.

This requirement is equivalent to say, if a particle y has trivial double braidings with all symmetry charges in $\mathcal{E}$, then y must be a symmetry charge.

In mathematics, the topological excitations in $\mathcal{M}$ which have trivial double braidings with all symmetry charges $\mathcal{E}$ form a subcategory $\mathfrak{Z}_2(\mathcal{E},\mathcal{M})$ of $\mathcal{M}$, called the Müger centralizer. So the minimal condition can be equivalently formulated as $\mathfrak{Z}_2(\mathcal{E},\mathcal{M}) \simeq \mathcal{E}$. A modular extension $(\mathcal{M}, \iota_{\mathcal{M}})$ satisfying this condition is called a minimal modular extension Müger:Adv.Math 150(2) (2000).

We get the characterization of a 2+1D SPT order (up to invertible topological orders):

Theorem [Lan-K.-Wen:16]

A 2+1D SPT order with symmetry given by a symmetric fusion category $\mathcal{E}$ (i.e $\mathrm{Rep}(G)$ or $\mathrm{Rep}(G, z)$) is described by a minimal modular extension $(\mathcal{M}, \iota_{\mathcal{M}} : \mathcal{E} \hookrightarrow \mathcal{M})$ of $\mathcal{E}$, where $\mathcal{M}$ is the UMTC of the topological excitations of the gauged theory.

We denote the set of all minimal modular extensions of $\mathcal{E}$ by $\mathrm{M}_{\mathrm{ex}}(\mathcal{E})$.

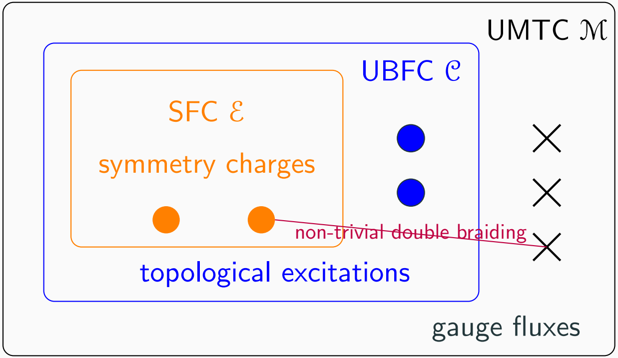
Theorem [Lan-K.-Wen:16]

For any symmetric fusion 1-category $\mathcal{E}$, the set $\mathrm{M_{ex}}(\mathcal{E})$ of minimal modular extensions of $\mathcal{E}$ is a finite abelian group.

- The group multiplication is the stacking of SPT orders (i.e. $\boxtimes_{\mathcal{E}}^{0} \neq \boxtimes_{\mathcal{E}}$).
- The unit of this group is the trivial SPT order $(\mathfrak{Z}_1(\mathcal{E}), \iota_0 : \mathcal{E} \hookrightarrow \mathfrak{Z}_1(\mathcal{E}))$, where $\mathfrak{Z}_1(\mathcal{E})$ is the Drinfeld center of $\mathcal{E}$. In particular, $\mathrm{M_{ex}}(\mathcal{E}) \neq \emptyset$.
- For bosonic symmetries $\mathcal{E} = \mathrm{Rep}(G)$, we have $\mathrm{M_{ex}}(\mathcal{E}) \simeq H^3(G, \mathrm{U}(1))$.
- For the fermion parity symmetry $\mathcal{E} = \mathrm{Rep}(\mathbb{Z}_2, z)$, we have $\mathrm{M_{ex}}(\mathcal{E}) \simeq \mathbb{Z}_{16}$.

This result recovers the usual cohomological classification of bosonic SPT orders and Kitaev's 16-fold way.

Let me briefly mention that the idea of gauging the symmetry directly applies to 2+1D SET orders.



This idea can also be generalized to higher dimenisons.

- Given a higher symmetry defined a unitary symmetry fusion n-category $\mathcal{R}$ (e.g. $n\mathrm{Rep}(G) := \Sigma^{n-1}\mathrm{Rep}(G), n\mathrm{Rep}(G, z), \cdots$, etc. Gaiotto-Johnson-Freyd:2019), gauging the symmetry $\mathcal{R}$ gives a minimal modular extension of $\mathcal{R}$:

$$(\mathcal{M}, \iota_{\mathcal{M}} : \mathcal{R} \hookrightarrow \mathcal{M}),$$

  where $\mathcal{M}$ is a unitary modular n-category K., Lan, Wen, Zhang, Zheng:2020.
  In 4D, the minimal modular extensions of $2\mathrm{Rep}(G)$ associated to each $\omega \in H^4(G, U(1))$ have been constructed explicitly. K., Tian, Shou:2019

- The canonical minimal modular extension is given by

$$(\mathfrak{Z}_1(\mathcal{R}), \iota_0 : \mathcal{R} \hookrightarrow \mathfrak{Z}_1(\mathcal{R})).$$

  It describes the trivial n+2D SPT order with symmetry $\mathcal{R}$.

Although the idea of gauging the symmetry works well, there are still some problems:

1. We can not claim that we have found the missing data because an SPT/SET order is a well defined notion before we gauge the symmetry. So the gauging process is only a way to reveal the missing data instead of a direct description of the missing data. There should be a mathematical description of an SPT/SET order without gauging the symmetry.
2. If a SET order can not be gauged, it is anomalous. In this case, this idea does not work. In an unpublished work, Drinfeld constructed an example of braided fusion 1-category such that minimal modular extension does not exist.

In physics, it was known that an nD SET order can be gauged if and only if its 1-dimension higher bulk is the trivial n+1D SPT order.

This leads to another approach based on the idea of boundary-bulk relation.

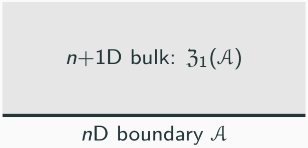
Unique bulk principle [K.-Wen:14]

Each nD topological order has a unique bulk, which is an n+1D topological order.

As a consequence, one can prove the following result.

Boundary-bulk relation [Kitaev-K.:13,K.-Wen:14,K.-Wen-Zheng:15,17]

bulk = the center of the boundary: The braided category of defects of codimension ≥ 2 in an anomaly-free topological order with a gapped boundary is given by the monoidal center of the monoidal category of defects of codimension ≥ 1 in the gapped boundary.



Now we are ready to find a mathematical characterization of 2+1D SPT based on the idea of boundary-bulk relation.

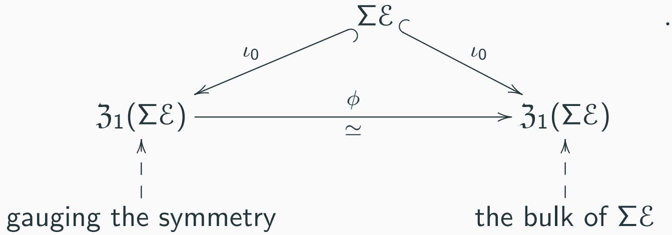
1. Recall that a 2+1D topological order can be described by a UMTC $\mathcal{M}$, or equivalently, by unitary fusion 2-category $\Sigma\mathcal{M}$, which includes all all condensation descendants (including defects of codimension 1). $\Sigma\mathcal{M}$ is called the condensation completion of $\mathcal{M}$. [Douglas-Reutter:18,Gaiotto-Johnson-Freyd:19,Johnson-Freyd:20, K.-Lan-Wen-Zhang-Zheng:20].
2. Similarly, the symmetry charges is given by $\mathcal{E} = \mathrm{Rep}(G), \mathrm{Rep}(G, z)$, or equivalently by $\Sigma\mathcal{E}$, where the symmetry fusion 2-category $\Sigma\mathcal{E}$ consists of all the condensation descendants of the symmetry charges.

The key idea is: the bulk of a 3D SPT order is the trivial 4D SPT order.

Recall that the trivial 4D SPT order with symmetry $\Sigma\mathcal{E}$ is described by a pair

$$(\mathfrak{Z}_1(\Sigma\mathcal{E}), \Sigma\mathcal{E} \xrightarrow{\iota_0} \mathfrak{Z}_1(\Sigma\mathcal{E})).$$

To identify the bulk of a 3D SPT with above pair, we need a braided equivalence $\phi : \mathfrak{Z}_1(\Sigma\mathcal{E}) \to \mathfrak{Z}_1(\Sigma\mathcal{E})$ preserving the symmetry charges, i.e.



This braided equivalence ϕ is the missing data (which lives in the bulk)!

All braided equivalences $\phi : \mathfrak{Z}_1(\Sigma\mathcal{E}) \to \mathfrak{Z}_1(\Sigma\mathcal{E})$ that preserve ι_0 (i.e. the symmetry charges) form a group, denoted by $\mathrm{Aut}^{br}(\mathfrak{Z}_1(\Sigma\mathcal{E}), \iota_0)$.

Consequently, 3D SPT orders with symmetry given by a symmetric fusion 1-category $\mathcal{E}$ (or $\Sigma\mathcal{E}$) are classified by the group $\mathrm{Aut}^{br}(\mathfrak{Z}_1(\Sigma\mathcal{E}), \iota_0)$.

Now we have two classifications of 2d SPT orders with symmetry $\mathcal{E}$.

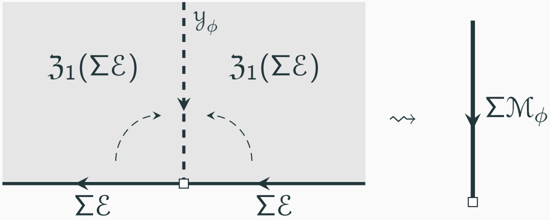
1. The first approach is via gauging the symmetry, and the classification is given by the group $\mathrm{M}_{\mathrm{ex}}(\mathcal{E})$ of minimal modular extensions of $\mathcal{E}$.
2. The second approach is based on the boundary-bulk relation, and the classification is given by the group $\mathrm{Aut}^{br}(\mathfrak{Z}_1(\Sigma\mathcal{E}), \iota_0)$ of braided autoequivalences of $\mathfrak{Z}_1(\Sigma\mathcal{E})$ which preserve the embedding ι_0.

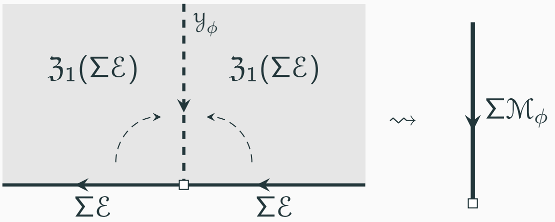
Are these two approaches equivalent?

Mathematically, this amounts to show that $\mathrm{Aut}^{br}(\mathfrak{Z}_1(\Sigma\mathcal{E}), \iota_0) \simeq \mathrm{M}_{\mathrm{ex}}(\mathcal{E})$ as groups. This is a surprising mathematical conjecture which deserves a mathematical proof.

We do not have a proof yet. But we can further clarify the relation between these two approaches.

- Physically, the braided equivalence $\phi : \mathfrak{Z}_1(\Sigma\mathcal{E}) \to \mathfrak{Z}_1(\Sigma\mathcal{E})$ is achieved by tunneling through a 3D invertible domain wall $\mathcal{Y}_\phi$, which is canonically associated to ϕ.
- This invertible domain wall $\mathcal{Y}_\phi$ between two trivial 4D SPT order is clearly a 3D SPT order. This 3D SPT order is the one corresponds to ϕ.
- The gauging process can be recovered by closing the fan.





Mathematically, it means that $\Sigma\mathcal{E} \boxtimes_{\mathfrak{Z}_1(\Sigma\mathcal{E})} \mathcal{Y}_\phi \boxtimes_{\mathfrak{Z}_1(\Sigma\mathcal{E})} \Sigma\mathcal{E} \simeq \Sigma\mathcal{M}_\phi$. This provides a map

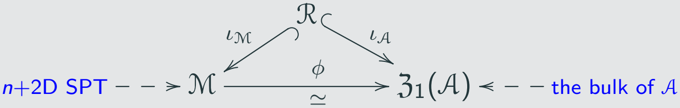
$$\mathrm{Aut}^{br}(\mathfrak{Z}_1(\Sigma\mathcal{E}), \iota_0) \to \mathrm{M}_{\mathrm{ex}}(\mathcal{E})$$
$$\phi \mapsto \mathcal{M}_\phi \simeq \Omega(\Sigma\mathcal{E} \boxtimes_{\mathfrak{Z}_1(\Sigma\mathcal{E})} \mathcal{Y}_\phi \boxtimes_{\mathfrak{Z}_1(\Sigma\mathcal{E})} \Sigma\mathcal{E})$$

The equivalence of these two approaches says that this map should be a group isomorphism. It will be important to find a rigorous proof of this conjecture.

Theorem [K.-Lan-Wen-Zhang-Zheng:2020]

For $n \geq 0$, an n+1D SET order with higher symmetry given by a symmetric fusion n-category $\mathcal{R}$ and with a 't Hooft anomaly (without gravitational anomaly) is characterized by a quintuple $(\mathcal{A}, \iota_{\mathcal{A}}; \mathcal{M}, \iota_{\mathcal{M}}; \phi)$, where

- $(\mathcal{A}, \mathcal{R} \xrightarrow{\iota_{\mathcal{A}}} \mathfrak{Z}_1(\mathcal{A}))$ is a unitary fusion n-category over $\mathcal{R}$ describing all topological excitations (including all condensation descendants).
- $(\mathcal{M}, \iota_{\mathcal{M}})$ is a minimal modular extension of $\mathcal{R}$ (i.e. an n+2D SPT order), which determines the 't Hooft anomaly. When $(\mathcal{M}, \iota_{\mathcal{M}}) = (\mathfrak{Z}_1(\mathcal{R}), \iota_{\mathcal{R}})$, 't Hooft anomaly vanishes.
- $\phi : \mathcal{M} \to \mathfrak{Z}_1(\mathcal{A})$ is a braided equivalence such that the diagram commutes:



In summary, for a potentially anomalous SET order $(\mathcal{A}, \iota_{\mathcal{A}})$,

1. Using the idea of gauging the symmetry, we obtain a categorical description of anomaly-free SPT/SET order $(\mathcal{M}, \iota_{\mathcal{M}})$ in 1-higher dimension;
2. Using the idea of the boundary-bulk relation to identify the bulk $(\mathcal{A}, \iota_{\mathcal{A}})$ of with $(\mathcal{M}, \iota_{\mathcal{M}})$.

This gives a mathematical description of potentially anomalous (gravitational + 't Hooft anomalies) SET orders of all dimensions.

This surprisingly unified framework provide a conceptual understanding of many known SPT/SET classification results in lower dimensions, which were often obtained in a case-by-case manner.

As an example, let us consider the 2D cases. By our general theorem, a 2D SET order is characterized by the following data:

- a symmetric fusion 1-category $\mathcal{R}$ describing the symmetry;
- a unitary fusion 1-cateogry $(\mathcal{A}, \iota_{\mathcal{A}})$ over $\mathcal{R}$ describing the topological excitations;
- a minimal modular extension $(\mathcal{M}, \iota_{\mathcal{M}})$ of $\mathcal{R}$ describing the 't Hooft anomaly (the bulk 2d SPT order);
- a braided equivalence ϕ such that the following diagram commutes:

$$\begin{array}{ccc} & \mathcal{R} & \\ {}^{\iota_{\mathcal{M}}}\swarrow & & \searrow^{\iota_{\mathcal{A}}} \\ \mathcal{M} & \xrightarrow[\simeq]{\phi} & \mathfrak{Z}_1(\mathcal{A}). \end{array}$$

By simply checking quantum dimensions, we can prove that $(\mathcal{R} \hookrightarrow \mathfrak{Z}_1(\mathcal{A}) \to \mathcal{A})$ is an equivalence.

1. In other words, there is no nontrivial topological excitations other than symmetry charges. Hence an anomaly-free 2D SET order is a 2D SPT order.
2. The minimal modular extension $\mathcal{M} \overset{\phi}{\simeq} \mathfrak{Z}_1(\mathcal{A}) \simeq \mathfrak{Z}_1(\mathcal{R})$ is trivial. So there is no 't Hooft anomaly. In other words, a nontrivial 3D SPT order has no gapped symmetry-preserving boundary. Hence the boundaries of a nontrivial 3D SPT order must be gapless or symmetry broken.

These two physical results were well-known and proved previously by other physical methods, and now they follows immediately from our general theory.

So we only need to consider 2D SPT orders, by taking $\mathcal{A} = \mathcal{E}$ and taking the bulk 3D SPT order $(\mathcal{M}, \iota_{\mathcal{M}})$ to be the trivial one $(\mathfrak{Z}_1(\mathcal{R}), \iota_{\mathcal{R}})$.

A 1d SPT order with symmetry $\mathcal{R}$ is uniquely determined by a braided equivalences $\phi : \mathfrak{Z}_1(\mathcal{R}) \to \mathfrak{Z}_1(\mathcal{R})$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
 & \mathcal{R} & \\
{}^{\iota_{\mathcal{R}}}\swarrow & & \searrow^{\iota_{\mathcal{R}}} \\
\mathfrak{Z}_1(\mathcal{R}) & \xrightarrow[\simeq]{\phi} & \mathfrak{Z}_1(\mathcal{R}).
\end{array}
$$

All such braided equivalences form a group $\mathrm{Aut}^{br}(\mathfrak{Z}_1(\mathcal{R}), \iota_{\mathcal{R}})$, which classifies 2D SPT orders.

The group $\mathrm{Aut}^{br}(\mathfrak{Z}_1(\mathcal{R}), \iota_0)$ of braided autoequivalences which preserve the canonical embedding ι_0 is isomorphic to the Picard group $\mathrm{Pic}(\mathcal{R})$ [Davydov-Nikshych:2013].

- For bosonic symmetries $\mathcal{R} = \mathrm{Rep}(G)$, we have $\mathrm{Pic}(\mathcal{R}) \simeq H^2(G, \mathrm{U}(1))$. This result recovers the usual classification of bosonic 1d SPT orders.
- For fermionic symmetries $\mathcal{R} = \mathrm{Rep}(G, z)$, we have [Carnovale:2006]

$$\mathrm{Pic}(\mathrm{Rep}(G,z)) \simeq \begin{cases} H^2(G, \mathrm{U}(1)) \times \mathbb{Z}_2 & \text{if } G = G_b \times \langle z \rangle; \\ H^2(G, \mathrm{U}(1)) & \text{if otherwise.} \end{cases}$$

This result also agrees with the physical classification results.

One can also apply the general theory of 1D SPT orders. This exercise will recover the usual classification of 1D SPT orders as $H^1(G, U(1))$.

In summary, our our classification of SPT/SET orders matches with known results in literature in 1D and 2D.

For $n \geq 3$D SPT/SET orders, our classification goes beyond the usual group cohomology classifications. We believe that our results can serve as a roadmap for mathematicians to join the study.

Summary and outlooks.

- The topological excitations do not characterize a SPT/SET order; there are some missing data. We give two ideas of solving the problem: gauging the symmetry and the boundary-bulk relation. Both can be generalized to higher dimensions.
- We obtain a unified framework for the classification of potentially anomalous SPT/SET orders in all dimensions. In 1D and 2D, our general results agree with existing classification results in physics.
- The equivalence of two approaches in 3D leads to many surprising mathematical conjectures.
- This is not the end of the search for a mathematical framework, but a platform for new and better formulations to emerge.

Thank you!