Time–energy uncertainty relation and quantum error correction for noisy quantum metrology

Philippe Faist  Freie Universität Berlin

with: Mischa P Woods, Victor V Albert, Joe Renes, Jens Eisert, John Preskill

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There are fundamental measurement precision limits imposed by quantum mechanics.
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Observables can be incompatible.

**Heisenberg uncertainty principle**

\[ \Delta x \cdot \Delta p \geq \frac{\hbar}{2} \]

A measurement disturbs the quantum state \( \rho \)

\[ \rho \rightarrow \frac{P \rho P}{\text{tr}(P \rho)} \]
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A measurement disturbs the quantum state \( \rho \)

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**Time is not an observable.**

Time-Energy uncertainty principle

\[ \Delta E \cdot \Delta t \geq \frac{\hbar}{2} \]

e.g. time it takes for any observable to change by at least one standard deviation
quantum state $\rho(t)$

time evolution

$\partial_t \rho = -i[H, \rho]$
How accurately can a measurement reveal the value of $t$?
one-parameter family of states $\rho(t_0)$

sense with observable $T$

Braunstein/Caves PRL 1994; Holevo 2011; ...
one-parameter family of states $\rho(t_0)$

$\rho(t) = \text{quantum state}$
$= \text{positive semidefinite operator}$

$T = \text{observable} = \text{Hermitian operator}$

$\rightarrow \text{Estimate: } t_{\text{est.}} = \text{tr}(\rho T) \equiv \langle T \rangle_{\rho}$

Braunstein/Caves PRL 1994; Holevo 2011; ...
one-parameter family of states $\rho(t_0)$

sense with observable $T$

Which observable $T$ gives the best estimate for $t$ around $t_0$?

$$\min \left\langle (T - t_0)^2 \right\rangle = ?$$

$$\left\langle T \right\rangle_{\rho(t_0 + dt)} = t_0 + dt + O(dt^2)$$

$\rho(t) = $ quantum state
= positive semidefinite operator

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Which observable $T$ gives the best estimate for $t$ around $t_0$?

$\min \langle (T - t_0)^2 \rangle = ?$

quadratic optimization problem

$\langle T \rangle_{\rho(t_0 + dt)} = t_0 + dt + O(dt^2)$

$\min \langle (T - t_0)^2 \rangle = \frac{1}{F(\rho(t_0), \partial_t \rho(t_0))}$

Cramér–Rao bound

$F(\rho, D) = \text{tr}(\rho R^2)$

where $R$ solves $(\rho R + R \rho)/2 = D$

optimal sensing observable $T - t_0 \propto R$

$\rho(t) = \text{quantum state}$

$= \text{positive semidefinite operator}$

$T = \text{observable} = \text{Hermitian operator}$

$\rightarrow$ Estimate: $t_{\text{est.}} = \text{tr}(\rho T) \equiv \langle T \rangle_{\rho}$
one-parameter family of states

\[ \rho(t_0) \]

sense with observable \( T \)

Which observable gives the best estimate for around \( t \)?

\[ \rho(t) = \text{quantum state} \]
\[ T = \text{observable} = \text{Hermitian operator} \]

Estimate: \[ t_{\text{est.}} = \text{tr}(\rho T) \equiv \langle T \rangle_\rho \]

The Quantum Fisher Information \( F(\rho, \partial_t \rho) \) gives the accuracy to which \( t \) can be sensed locally around \( t_0 \).

\[ \min \langle (T - t_0)^2 \rangle = \frac{1}{F(\rho, \partial_t \rho)} \]

Optimal sensing observable \( T - t_0 \propto R \)

Braunstein/Caves PRL 1994; Holevo 2011; ...
Example: unitary pure state evolution

$$|\psi_0\rangle \quad e^{-iHt} |\psi_0\rangle$$

$$t_0$$

$$\min (\Delta T)^2 = \frac{1}{F(\rho, \partial_t \rho)}$$

$$F(\rho, D) = \text{tr}(\rho R^2)$$

where $R$ solves

$$(\rho R + R\rho)/2 = D$$
Example: unitary pure state evolution

\[ |\psi_0\rangle \xrightarrow{e^{-iHt}} |\psi_t\rangle \]

\[ \frac{\partial_t \psi}{\partial_t \psi^2} = (\partial_t \psi)\psi + \psi(\partial_t \psi) \rightarrow R = 2\partial_t \psi \]

\[ F(\psi, \partial_t \psi) = \text{tr}(\psi(2\partial_t \psi)^2) \rightarrow 4(\langle H^2 \rangle - \langle H \rangle^2) \]

\[ \min (\Delta T)^2 = \frac{1}{F(\rho, \partial_t \rho)} \]

\[ F(\rho, D) = \text{tr}(\rho R^2) \]

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Example: unitary pure state evolution

\[ |\psi_0\rangle \quad \text{e}^{-iHt} |\psi_0\rangle \]

where

\[ t_0 \]

\[ \partial_t \psi = \partial_t \psi^2 = (\partial_t \psi)\psi + \psi(\partial_t \psi) \quad \rightarrow \quad R = 2\partial_t \psi \]

\[ \rightarrow \quad F(\psi, \partial_t \psi) = \text{tr}(\psi(2\partial_t \psi)^2) \quad \ldots = 4\left(\langle H^2 \rangle - \langle H \rangle^2 \right) \]

Optimal accuracy for sensing \( t \):

\[ \Delta T = \frac{1}{\sqrt{4(\langle H^2 \rangle - \langle H \rangle^2)}} \]

Optimal sensing observable:

\[ T \propto \partial_t \psi \]

\[ \min (\Delta T)^2 = \frac{1}{F(\rho, \partial_t \rho)} \]

\[ F(\rho, D) = \text{tr}(\rho R^2) \]

where \( R \) solves

\[ (\rho R + R\rho)/2 = D \]
Example: unitary pure state evolution

\[ e^{-iHt} |\psi_0\rangle \]

where \( t_0 \)

\[ \partial_t \psi = \partial_t \psi^2 = (\partial_t \psi)\psi + \psi(\partial_t \psi) \quad \Rightarrow \quad R = 2\partial_t \psi \]

\[ \Rightarrow \quad F(\psi, \partial_t \psi) = \text{tr}(\psi(2\partial_t \psi)^2) \quad \ldots \quad = 4\langle H^2 \rangle - \langle H \rangle^2 \]

Optimal accuracy for sensing \( t \):

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\[ F(\rho, D) = \text{tr}(\rho R^2) \]

where \( R \) solves \((\rho R + R \rho)/2 = D \)

For a pure state evolving according to some Hamiltonian, the optimal sensitivity is given by the **variance of the Hamiltonian**.
Pure state accuracy

$$\Delta T = \frac{1}{\sqrt{4(\langle H^2 \rangle - \langle H \rangle^2)}}$$

$n$ spin-$1/2$ particles
Pure state accuracy

\[ \Delta T = \frac{1}{\sqrt{4(\langle H^2 \rangle - \langle H \rangle^2)}} \]

\( n \) spin-1/2 particles

\[ H = \frac{\omega \sigma^{(1)}_Z}{2} + \cdots + \frac{\omega \sigma^{(n)}_Z}{2} \]

\[ |\psi\rangle = \frac{|\uparrow \cdots \uparrow\rangle + |\downarrow \cdots \downarrow\rangle}{\sqrt{2}} \]

Heisenberg scaling

\[ F = 4\langle H^2 \rangle = n^2 \omega^2 \rightarrow \Delta T \approx \frac{1}{n} \]

Giovannetti, Lloyd, Maccone PRL 2006
The quantum sensitivity advantage disappears in the presence of noise.

What are the ultimate sensitivity limits in the presence of noise?
How much sensitivity is there left?
Alice

$t \approx t_0$

$\mathcal{N}$

Bob
Information about energy gained by the Environment (Eve) should trade off with Bob’s sensitivity to time.
$|\psi(0)\rangle$  \[ t = 0 \]

\[ t \approx t_0 \]

sense with $T$

\[ t \rightarrow \text{time } t \]
A new parameter $\eta$ is generated by $T$. The evolution of the state $\psi$ is described by the equations:

- $\eta \psi = i[T, \psi]$
- $\partial_t \psi = -i[H, \psi]$

The state $\psi$ evolves with respect to $H$ and $T$. The expectation value of $H$ at time $t$ is given by:

$\langle H \rangle_{\psi(t_0, \eta_0 + d\eta)} = \eta_0 + d\eta$
We identify a parameter $\eta$ that represents energy, which is complementary to time $t$ locally at $|\psi\rangle$. 

\[
\langle H \rangle_{\psi(t_0, \eta_0 + d\eta)} \approx \eta_0 + d\eta
\]

\[
\partial_\eta \psi = i[T, \psi]
\]

\[
\partial_t \psi = -i[H, \psi]
\]

\[
\bar{H} = H - \langle H \rangle
\]

\[
\propto \{H, \psi\}
\]
Alice
$t \approx t_0$

complementary channel

Bob
$F_{Bob,t} = F(\mathcal{N}(\psi), \partial_t \mathcal{N}(\psi))$

Eve
$F_{Eve,\eta} = F(\hat{\mathcal{N}}(\psi), \partial_\eta \hat{\mathcal{N}}(\psi))$
Alice $t \approx t_0$

complementary channel

Bob

$F_{\text{Bob},t} = F(\mathcal{N}(\psi), \partial_t \mathcal{N}(\psi))$

Eve

$F_{\text{Eve},\eta} = F(\hat{\mathcal{N}}(\psi), \partial_\eta \hat{\mathcal{N}}(\psi))$

Main Result

$$\frac{F_{\text{Bob},t}}{F_{\text{Alice},t}} + \frac{F_{\text{Eve},\eta}}{F_{\text{Alice},\eta}} = 1$$

Fisher information trade-off
Main Result

\[
\frac{F_{Bob,t}}{F_{Alice,t}} + \frac{F_{Eve,\eta}}{F_{Alice,\eta}} = 1
\]

Bob’s sensitivity to time

Eve’s sensitivity to energy

Fisher information trade-off

Normalization makes units consistent

Bob

\[ F_{Bob,t} = F(\mathcal{N}(\psi), \partial_t \mathcal{N}(\psi)) \]

Alice

Bob’s sensitivity to time

Eve

\[ F_{Eve,\eta} = F(\mathcal{N}(\psi), \partial_\eta \mathcal{N}(\psi)) \]

complementary channel

normalization makes units consistent
Useful bounds on the Fisher information of noisy states

Example: single qubit

Proof sketch

Necessary and sufficient conditions for zero sensitivity loss

“metrological codes”

Fisher information uncertainty relation for any two parameters

Example: Ising spin chain with amplitude-damping noise
Useful bounds on the Fisher information of noisy states

Necessary and sufficient conditions for zero sensitivity loss

“metrological codes”

Example: single qubit

Proof sketch

Example: Ising spin chain with amplitude-damping noise

Proof sketch

\[ \frac{F_{Bob,t}}{F_{Alice,t}} + \frac{F_{Eve,\eta}}{F_{Alice,\eta}} = 1 \]
Example: 1 qubit

\[ H = \frac{\omega}{2} \sigma_z \]

\[ |\psi\rangle = \frac{1}{\sqrt{2}} [ |\uparrow\rangle + |\downarrow\rangle ] = |+\rangle \]

\[ 4\langle H^2 \rangle = \omega^2 \]
Example: 1 qubit

\[ H = \frac{\omega}{2} \sigma_z \]

\[ |\psi\rangle = \frac{1}{\sqrt{2}} [|\uparrow\rangle + |\downarrow\rangle] = |+\rangle \]

\[ 4\langle H^2 \rangle = \omega^2 \]

Evolve for \( t_0 \):

\[ \psi_t = \frac{1}{2} \begin{bmatrix} 1 & e^{-i\omega t} \\ e^{i\omega t} & 1 \end{bmatrix} \]
Example: 1 qubit

$$H = \frac{\omega}{2} \sigma_z$$

$$|\psi\rangle = \frac{1}{\sqrt{2}} [|\uparrow\rangle + |\downarrow\rangle] = |+\rangle$$

$$4\langle H^2 \rangle = \omega^2$$

Evolve for $$t_0$$:

$$\psi_t = \frac{1}{2} \begin{bmatrix} 1 & e^{-i\omega t} \\ e^{i\omega t} & 1 \end{bmatrix}$$

Apply dephasing along the X axis

$$\rho_{\text{Bob}} = \cos^2\left(\frac{\omega t}{2}\right) |+\rangle \langle +X| + \sin^2\left(\frac{\omega t}{2}\right) |-\rangle \langle -X-|$$
Example: 1 qubit

$$H = \frac{\omega}{2} \sigma_z$$

$$\psi = \frac{1}{\sqrt{2}} [|\uparrow\rangle + |\downarrow\rangle] = |+\rangle$$

$$4\langle H^2 \rangle = \omega^2$$

Evolve for $$t_0$$:

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Apply dephasing along the X axis

$$\rho_{\text{Bob}} = \cos^2\left(\frac{\omega t}{2}\right)|+X+| + \sin^2\left(\frac{\omega t}{2}\right)|-X-|$$

Eve's sensitivity to energy

$$F_{\text{Eve,} \eta} \propto F(\hat{N}(\psi); \hat{N}(\{\overline{H}, \psi\})) = 0 \quad \Rightarrow \text{zero loss of sensitivity for Bob}$$
Example: 1 qubit

\[ H = \frac{\omega}{2} \sigma_z \]

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Evolve for \( t_0 \):

\[ \psi_t = \frac{1}{2} \begin{bmatrix} 1 & e^{-i\omega t} \\ e^{i\omega t} & 1 \end{bmatrix} \]

Apply dephasing along the X axis

\[ \rho_{Bob} = \cos^2\left(\frac{\omega t}{2}\right) |+X+| + \sin^2\left(\frac{\omega t}{2}\right) |-X-| \]

Eve’s sensitivity to energy

\[ F_{Eve,\eta} \propto F(\hat{N}^*(\psi); \hat{N}((\overline{H}, \psi))) = 0 \]

\[ \propto \sigma_z \]

\[ \rightarrow \text{zero loss of sensitivity for Bob} \]

Direct calculation:

\[ F_{Bob,t} = \omega^2 = F_{Alice,t} \]
Main proof ingredients

\[
\frac{F_{\text{Bob}, t}}{F_{\text{Alice}, t}} + \frac{F_{\text{Eve}, \eta}}{F_{\text{Alice}, \eta}} = 1
\]
Main proof ingredients

Variational characterization of the Fisher information

\[ F(\rho; D) = \frac{1}{4} \max_{S=S^+} [\text{tr}(DS) - \text{tr}(\rho S^2)] \]

\[ = \min \left\{ \text{tr}(LL^+) : \rho^{1/2}L + L^+\rho^{1/2} = D \right\} \]

Macieszczak

1312.1356;

Holevo 2011;

...
Main proof ingredients

Variational characterization of the Fisher information

\[ \frac{1}{4} F(\rho; D) = \max_{S=S^t} [\text{tr}(DS) - \text{tr}(\rho S^2)] \]

\[ = \min \left\{ \text{tr}(LL^+): \rho^{1/2}L + L^+\rho^{1/2} = D \right\} \]

Connection to Fidelity / Bures metric

\[ F(\rho(t_0); \partial_t \rho(t_0)) = -8 \left. \frac{d^2}{dt^2} \right|_{t_0} F(\rho(t_0), \rho(t)) \]

Fidelity of quantum states
Main proof ingredients

Variational characterization of the Fisher information

\[ \frac{1}{4} F(\rho; D) = \max_{S=S^+} [\text{tr}(DS) - \text{tr}(\rho S^2)] \]

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Connection to Fidelity / Bures metric

\[ F(\rho(t_0); \partial_t \rho(t_0)) = -8 \frac{d^2}{dt^2} \bigg|_{t_0} F(\rho(t_0), \rho(t)) \]

Uhlmann’s theorem:

\[ F(\mathcal{N}(\psi), \mathcal{N}(\psi')) = \max_{W_E} |\langle \psi | V^+(1 \otimes W_E)V | \psi' \rangle| \]

Fidelity of quantum states

optim. over \( W_E \)

\leftrightarrow \text{optim. over } S
Necessary and sufficient conditions for zerosensitivity loss

"metrological codes"

Fisher information

uncertainty relation for any two parameters

Useful bounds on the Fisher information of noisy states

Example: single qubit

Proof sketch

Example: Ising spin chain with amplitude-damping noise

\[
\frac{F_{\text{Bob},t}}{F_{\text{Alice},t}} + \frac{F_{\text{Eve},\eta}}{F_{\text{Alice},\eta}} = 1
\]
e.g., decohere in fixed basis → simpler bound for amplitude damping noise

\[ F_{\text{Eve'}}(\eta) \leq F_{\text{Eve}}(\eta) \]
e.g., decohere in fixed basis
→ simpler bound for amplitude damping noise

\[ F_{\text{Eve'}}(\eta) \leq F_{\text{Eve}}(\eta) \]

\[ \frac{F_{\text{Bob}}(t)}{F_{\text{Alice}}(t)} \leq 1 - \frac{F_{\text{Eve'}}(\eta)}{F_{\text{Alice}}(t)} \]

We find bounds on the quantum Fisher information of mixed states that might be simpler to compute (e.g., because the state is diagonal).
Example: keep only high probability error events

Hilbert space of Eve’ is smaller than that of Eve → quantum Fisher information (QFI) is easier to compute.
Example: keep only high probability error events

$E_k: \ k = 1, \ldots, m$

$m' \ll m$

Hilbert space of Eve' is smaller than that of Eve → quantum Fisher information (QFI) is easier to compute.

Example: dephasing for amplitude-damping noise

QFI is easier to compute for a diagonal state
Useful bounds on the Fisher information of noisy states

Necessary and sufficient conditions for zero sensitivity loss

“metrological codes”

Example: single qubit

Proof sketch

Example: Ising spin chain with amplitude-damping noise

Fisher information uncertainty relation for any two parameters

\[
\frac{F_{Bob,t}}{F_{Alice,t}} + \frac{F_{Eve,\eta}}{F_{Alice,\eta}} = 1
\]
uncertainty relation for any two generators $A, B$:

\[
\begin{align*}
\partial_a \psi &= -i [A, \psi] \\
\partial_b \psi &= -i [B, \psi]
\end{align*}
\]
uncertainty relation for any two generators $A, B$:

$$\partial_a \psi = -i[A, \psi]$$
$$\partial_b \psi = -i[B, \psi]$$

Fisher information tradeoff for any two parameters

$$\frac{F_{\text{Bob}, a}}{F_{\text{Alice}, a}} + \frac{F_{\text{Eve}, b}}{F_{\text{Alice}, b}} \leq 1 + 2 \sqrt{1 - \frac{\langle i[A, B] \rangle^2}{4 \sigma_A^2 \sigma_B^2}}$$

$$\sigma_X^2 = \langle X^2 \rangle - \langle X \rangle^2$$
uncertainty relation for any two generators $A, B$:

$$\partial_a \psi = -i[A, \psi]$$
$$\partial_b \psi = -i[B, \psi]$$

Fisher information tradeoff for any two parameters:

$$\frac{F_{\text{Bob}, a}}{F_{\text{Alice}, a}} + \frac{F_{\text{Eve}, b}}{F_{\text{Alice}, b}} \leq 1 + 2 \sqrt{1 - \frac{\langle [A, B] \rangle^2}{4 \sigma_A^2 \sigma_B^2}}$$

$$\sigma_X^2 = \langle X^2 \rangle - \langle X \rangle^2$$

= 0 if $A, B$ saturate the Robertson uncertainty relation

$$\sigma_A \sigma_B \geq \frac{1}{2} |\langle [A, B] \rangle|$$

(e.g., energy $H$ & time $T$)
uncertainty relation for any two generators $A, B$:
\[ \partial_a \psi = -i[A, \psi] \]
\[ \partial_b \psi = -i[B, \psi] \]

Fisher information tradeoff for any two parameters
\[ \frac{F_{\text{Bob}, a}}{F_{\text{Alice}, a}} + \frac{F_{\text{Eve}, b}}{F_{\text{Alice}, b}} \leq 1 + \frac{1}{2} \sqrt{1 - \frac{|\langle i[A, B] \rangle|^2}{4 \sigma_A^2 \sigma_B^2}} \]

\[ \sigma_X^2 = \langle X^2 \rangle - \langle X \rangle^2 \]

Fisher information trade-off relation that depends on incompatibility of $A, B$
\[ \sigma_A \sigma_B \geq \frac{1}{2} |\langle i[A, B] \rangle| \]
(e.g., energy $H$ & time $T$)
Useful bounds on the Fisher information of noisy states

Proof sketch

Example: single qubit

Fisher information uncertainty relation for any two parameters

$\frac{F_{\text{Bob},t}}{F_{\text{Alice},t}} + \frac{F_{\text{Eve},\eta}}{F_{\text{Alice},\eta}} = 1$

Necessary and sufficient conditions for zero sensitivity loss

“metrological codes”

Example: Ising spin chain with amplitude-damping noise
Quantum Error Correction protects quantum states from noise.
Quantum Error Correction protects quantum states from noise.

Can Quantum Error Correction protect clocks from noise?
Quantum Error Correction protects quantum states from noise.

Can Quantum Error Correction protect clocks from noise?

The code must be time-covariant.

Hayden et al., 1709.04471
Quantum Error Correction protects quantum states from noise.

Can Quantum Error Correction protect clocks from noise? The code must be time-covariant.

Covariant codes that can correct local errors don’t exist! (Eastin–Knill)

Hayden et al., 1709.04471

Eastin & Knill
PRL 2009
What if we recover $\rho$ only approximately?
What if we recover $\rho$ only approximately?

$\epsilon \geq \frac{\Delta H_L}{2n \max_i \Delta H_i}$

Logical Hamiltonian

Number of subsystems

Hamiltonian of $i$-th subsystem

$\rho' \approx \epsilon \rho$

PhF et al. PRX 2020
Woods & Alhambra Quantum 2020
Kubica et al. 2004.11893
Zhou et al. 2005.11918
Yang et al. 2007.09154
Our trade-off relation transposes this inequality to the quantum Fisher information.

What if we recover only approximately?

$\epsilon > \frac{n}{2} \max |\Delta H_i|$

number of subsystems

logical Hamiltonian of $i$-th subsystem

$\frac{F_{\text{obt}}}{F_{\text{Alice}}} + \frac{F_{\text{Even}}}{F_{\text{Alice}}} = 1$

$\rho' \approx \rho$

Woods & Alhambra Quantum 2020

PhF et al. PRX 2020

Kubica et al. 2004.11893

Zhou et al. 2005.11918

Yang et al. 2007.09154
Quantum error correction can help if the Hamiltonian is not aligned with the noise.

Achieve same sensitivity scaling as without noise (Heisenberg scaling) \[ \iff \]

The signal Hamiltonian is not parallel to the noise (Hamiltonian not in Kraus span)

"is it the signal or is it noise?"

Demkowicz-Dobrzański+ N. Comm. 2011; Escher+ N. Phys 2011; Kessler+ / Arrad+ / Dür+ PRL 2014; Zhou+ N. Comm. 2018; Layden+ PRL 2019; ...

noise operators
Quantum error correction can help if the Hamiltonian is not aligned with the noise.

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Do we need full error correction or will a simpler scheme suffice?
\[
\frac{F_{Bob,t}}{F_{Alice,t}} + \frac{F_{Eve,\eta}}{F_{Alice,\eta}} = 1
\]
If the second term vanishes, the noisy clock has the same sensitivity as the noiseless one.

\[
\frac{F_{\text{Bob},t}}{F_{\text{Alice},t}} + \frac{F_{\text{Eve},\eta}}{F_{\text{Alice},\eta}} = 1
\]

\[
F_{\text{Bob},t} = F_{\text{Alice},t}
\]
If the second term vanishes, the noisy clock has the same sensitivity as the noiseless one.

\[ F_{Bob,t} + F_{Eve,\eta} = 1 \]

\[ F_{Eve,\eta} = 0 \]

\[ \text{tr} \left[ Z \Pi E_k^* E_k \Pi \right] = 0 \]

"code qubit" spanned by

\[ |\psi\rangle = |+\rangle \text{ and } H|\psi\rangle \]

\[ \Pi = \text{projector on that space} \]

\[ Z = \text{logical Pauli-Z operator} \]
If the second term vanishes, the noisy clock has the same sensitivity as the noiseless one.

\[
\frac{F_{\text{Bob},t}}{F_{\text{Alice},t}} + \frac{F_{\text{Eve},\eta}}{F_{\text{Alice},\eta}} = 1
\]

\( F_{\text{Eve},\eta} = 0 \)

\[ \Leftrightarrow \quad \text{tr}\left[ Z \prod E_k^\dagger E_k \Pi \right] = 0 \]

"code qubit" spanned by
\( |\psi\rangle = |+\rangle \) and \( H |\psi\rangle \)

\( \Pi = \) projector on that space
\( Z = \) logical Pauli–Z operator

Corresponding quantum error correction conditions

\[ \Pi E_k^\dagger E_k \Pi \propto \Pi \]

Knill & Laflamme
PRA1997
If the second term vanishes, the noisy clock has the same sensitivity as the noiseless one.

\[
\frac{F_{Bob,t}}{F_{Alice,t}} + \frac{F_{Eve,\eta}}{F_{Alice,\eta}} = 1
\]

\[F_{Eve,\eta} = 0\]

\[\leftrightarrow \text{ tr} \left( Z \cap E_k^+ E_k \Pi \right) = 0\]

“code qubit” spanned by \(|\psi\rangle = |+\rangle\) and \(H|\psi\rangle\)

\(\Pi\) = projector on that space

\(Z\) = logical Pauli–Z operator

“metrological codes” are more general than error–correcting codes.

Corresponding quantum error correction conditions

\[\Pi E_k^+ E_k \Pi \propto \Pi\]

Knill & Laflamme

PRA1997
Our earlier example was a metrological code.

\[ H = \frac{\omega}{2} \sigma_z \]

\[ |\psi\rangle = \frac{1}{\sqrt{2}} [|\uparrow\rangle + |\downarrow\rangle] = |\rangle \]

dehasing along X

\[ \rho_{\text{Bob}} = \cos^2 \left( \frac{\omega t}{2} \right) |+\rangle \langle +| + \sin^2 \left( \frac{\omega t}{2} \right) |-\rangle \langle -| \]

\[ F_{\text{Eve, } \eta} \propto F(\hat{N}(\psi); \hat{N}(\{|H, \psi\}\})) = 0 \]

\[ \propto \sigma_z \]

\[ \rightarrow \text{zero loss of sensitivity for Bob} \]
Our earlier example was a metrological code.

\[ H = \frac{\omega}{2} \sigma_z \quad |\psi\rangle = \frac{1}{\sqrt{2}}[|\uparrow\rangle + |\downarrow\rangle] = |+\rangle \]

dehasing along X

\[ \rho_{Bob} = \cos^2\left(\frac{\omega t}{2}\right)|+\rangle\langle+| + \sin^2\left(\frac{\omega t}{2}\right)|-\rangle\langle-| \]

\[ F_{Eve,\eta} \propto F(\hat{N}(\psi), \hat{N}([\hat{H}, \psi])) = 0 \quad \propto \sigma_z \]

→ zero loss of sensitivity for Bob

metrological code conditions

\[ \text{tr}\left[Z \cap E_{k'}^+ E_k \cap 1, X \right] = 0 \quad \checkmark \]
Example: Many-body system with Ising-type interactions

ZZ interactions (can also include XX & YY)

cf. e.g. Ouyang IEEE TIT 2022;...
Example: Many-body system with Ising-type interactions

\[ |\psi\rangle = \frac{1}{\sqrt{2}} [|0 \ldots 0\rangle + |c^n\rangle] \]

\[ c^n = \text{configuration that violates many interaction terms} \]

\[ \text{tr}[Z \cap E_k^+, E_k \cap] \neq 0 \]

prone to sensitivity loss

ZZ interactions (can also include XX & YY)

cf. e.g. Ouyang IEEE TIT 2022;...
Example: Many-body system with Ising-type interactions

\[ |\psi\rangle = \frac{1}{2} \left[ |0 \cdots 0\rangle + |1 \cdots 1\rangle \right. \]
\[ + \left. |c^n\rangle + |\bar{c}^n\rangle \right] \]

\( c^n \) = configuration that violates many interaction terms

\[ \text{tr} \left[ Z \cap E_k^\dagger E_k \Pi \right] = 0 \]

no sensitivity loss in case of a single localized error

ZZ interactions (can also include XX & YY)

cf. e.g. Ouyang IEEE TIT 2022;...
Useful bounds on the Fisher information of noisy states

Example: single qubit

Proof sketch

\[ \frac{F_{\text{Bob},t}}{F_{\text{Alice},t}} + \frac{F_{\text{Eve},\eta}}{F_{\text{Alice},\eta}} = 1 \]

Fisher information uncertainty relation for any two parameters

Necessary and sufficient conditions for zero sensitivity loss

“metrological codes”

Example: Ising spin chain with amplitude-damping noise
\[ H = \sum \sigma_z^{(j)} \sigma_z^{(j+1)} \]

1D spin chain

ferromagnet–antiferromagnet state

\[ |\psi\rangle = \frac{1}{\sqrt{2}} (|00 \ldots 00\rangle + |01 \ldots 01\rangle) \]

spin–coherent (clock) state

\[ |\psi\rangle = |+\rangle^\otimes n \]

Bob\((t)\)

\[ n = 50 \text{ qubits} \]
$H = \sum \sigma_Z^{(j)} \sigma_Z^{(j+1)}$

1D spin chain

ferromagnet-antiferromagnet state

$|\psi\rangle = \frac{1}{\sqrt{2}} (|00 \ldots 00\rangle + |01 \ldots 01\rangle)$

“metrological code state”

$|\psi\rangle = \frac{1}{2} (|00 \ldots 00\rangle + |11 \ldots 11\rangle$

$+ |01 \ldots 01\rangle + |10 \ldots 10\rangle)$

spin-coherent (clock) state

$|\psi\rangle = |+\rangle^\otimes n$

$n = 50$ qubits

Amplitude damping parameter $\rho$

$F_{Bob}(t)$
Useful bounds on the Fisher information of noisy states

Example: single qubit

Proof sketch

Fisher information uncertainty relation for any two parameters

\[ \frac{F_{\text{Bob},t}}{F_{\text{Alice},t}} + \frac{F_{\text{Eve},\eta}}{F_{\text{Alice},\eta}} = 1 \]

Necessary and sufficient conditions for zero sensitivity loss

“metrological codes”

Example: Ising spin chain with amplitude-damping noise
Good clock state = one that hides energy from Eve

Fisher information counterpart to entropic uncertainty relations  
Coles+ PRL 2019

Also applicable to continuous Markovian noise, under some additional assumptions

Extension for multi-parameter metrology?

strongly interacting many-body probes might offer better sensitivity

NV centers  
Zhou+ PRX 2020

lattice atomic clocks  
Goban+ Nat. 2018

What are the implications of having only a finite number of samples?
Thank you for your attention!
Example: 1 qubit

\[ H = \frac{\omega}{2} \sigma_z \]

\[ |\psi\rangle = \frac{1}{\sqrt{2}}[|\uparrow\rangle + |\downarrow\rangle] \]

\[ 4\langle H^2 \rangle = \omega^2 \]

Evolve for \( t_0 \):

\[ \psi_t = \frac{1}{2} \begin{bmatrix} 1 & e^{-i\omega t} \\ e^{i\omega t} & 1 \end{bmatrix} \]

Apply partial dephasing

\[ \mathcal{D}_p[\rho] = \begin{bmatrix} \rho_{00} & (1 - p)\rho_{01} \\ (1 - p)\rho_{10} & \rho_{11} \end{bmatrix} \]

Eve:

\[ \hat{\mathcal{D}}_p[\rho] = \begin{bmatrix} (1 - \frac{p}{2}) \text{tr}(\rho) & \sqrt{\frac{p}{2}(1 - \frac{p}{2})} \text{tr}(\sigma_z \rho) \\ \sqrt{\frac{p}{2}(1 - \frac{p}{2})} \text{tr}(\sigma_z \rho) & \frac{p}{2} \text{tr}(\rho) \end{bmatrix} \]

sensitivity loss:

\[ F(\hat{\mathcal{N}}(\psi); \hat{\mathcal{N}}(|\overline{H}, \psi\rangle)) = \omega^2 [1 - (1 - p)^2] \]

Direct calculation:

\[ F_{\text{Bob}, t} = \omega^2 (1 - p)^2 \]

\[ F_{\text{Alice}, t} = \omega^2 \]
Seek observable $T$ with minimal variance at $\rho(t_0)$, such that:

$$\langle T \rangle_{\rho(t_0 + dt)} = t_0 + dt + O(dt^2)$$

Cramér–Rao bound

$$\langle (T - t_0)^2 \rangle \geq \frac{1}{F(\rho; \partial_t \rho)}$$

Fisher information defined here via optimal $T$

Braunstein/Caves PRL 1994; Holevo 2011; ...