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W*-rigidity paradigms for embeddings of II_1 factors

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II_1 factors

Fundamental work of Murray – von Neumann, 1943

- Weakly closed $*$-subalgebras $M \subset B(H)$. Nowadays called **von Neumann algebras**.
- **Bicommutant theorem:** $M' = \{T \in B(H) \mid ST = TS$ for all $S \in M\}$ and $M = M''$.
- **Factor:** $\mathcal{Z}(M) = M \cap M' = \mathbb{C}1$.
- Factors come in **different types** with $B(H)$ being of type I.
- **II_1 factors:** factors with a tracial state $\tau : M \to \mathbb{C}1$.

⇝ Continuous dimension $\tau(p) \in [0,1]$.

⇝ One of the most fascinating mathematical structures, sometimes extremely rich in symmetries, sometimes extremely rigid.

Group factors and group measure space factors

Geometric / group theoretic / ergodic / Lie theoretic **initial data** $\rightsquigarrow$ **II_1 factor**.

Group II_1 factors

Let G be a countable group with infinite conjugacy classes (icc).

- Left regular representation: $\lambda : G \to \mathcal{U}(\ell^2(G)) : \lambda_g \delta_h = \delta_{gh}$.
- $\mathbb{C}[G] = \mathrm{span}\{\lambda_g \mid g \in G\}$ and $L(G) = \text{weak closure } \mathbb{C}[G]$.

Group measure space II_1 factors

Let G be an infinite group and $G \curvearrowright (X, \mu)$ a free, ergodic, pmp action.

- $A = L^\infty(X)$ and $M = A \rtimes G$.
- Generated by $A \subset M$ and $L(G) \subset M$ with $u_g^* F(\cdot) u_g = F(g\cdot)$.

Isomorphism and embedding problems for II_1 factors

Many-to-one paradigm

⤳ Large classes of very distinct initial data may give isomorphic II_1 factors.

One-to one / W*-rigidity paradigm

⤳ Large classes of initial data such that $\mathcal{S} \ni S \mapsto M(S) \in \mathbf{II_1}$ is one-to-one.

- Up to isomorphism of II_1 factors.
- Up to stable isomorphism $N \cong M^t$.

  Recall: for $0 < t < 1$, take $p \in M$ with $\tau(p) = t$ and put $M^t = pMp$.
  In general, $M^t = p(M_n(\mathbb{C}) \otimes M)p$.
- Up to virtual isomorphism, up to embeddability, ..., see later.

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Many-to-one results: hyperfiniteness

Murray-von Neumann: a II_1 factor M is **hyperfinite** if there exists an increasing sequence $A_n \subset M$ of finite dimensional $*$-subalgebras with $\bigcup_n A_n$ weakly dense in M.

Canonical construction: $R = M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes \cdots$.

Theorem (Murray-von Neumann, 1943)

- All hyperfinite II_1 factors are isomorphic!
- Every II_1 factor M contains copies of $R \hookrightarrow M$.

"The possibility exists that any factor in the case II_1 is isomorphic to a sub-ring of any other such factor."

Notation: $N \hookrightarrow M$ if there exists an embedding of N into M.

Also, $N \hookrightarrow_s M$ if there exists a $t > 0$ with $N \hookrightarrow M^t$.

Many-to-one results: amenability

- A group G is called **amenable** if there exists a translation invariant mean on G, i.e. $m(g\mathcal{U}) = m(\mathcal{U})$.
- A von Neumann algebra $M \subset B(H)$ is called **amenable** if there exists a (not necessarily normal) conditional expectation $P : B(H) \to M$.
- A group II_1 factor $L(G)$ is amenable if and only if G is amenable.
- If M is an amenable II_1 factor and $N \hookrightarrow_s M$, then N is amenable.

$\leadsto$ If Λ is nonamenable and Γ is amenable, then $L(\Lambda) \not\hookrightarrow_s L(\Gamma)$.

Theorem (Connes, 1976)

All amenable II_1 factors are isomorphic!

Thus, for all amenable icc groups G, we have $L(G) \cong R$. Thus, if $N \hookrightarrow_s R$, then $N \cong R$!

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First non-embeddability results

Connes-Jones, 1983

If Λ has Kazhdan's property (T) and if Γ has the Haagerup property, then $L(\Lambda) \not\hookrightarrow_s L(\Gamma)$.

For example, $L(\mathrm{SL}(3,\mathbb{Z})) \not\hookrightarrow_s L(\mathbb{F}_\infty)$.

⇝ Intrinsic definition of property (T) for II_1 factors.

Cowling-Haagerup, 1988

If Γ_n is a lattice in $\mathrm{Sp}(1,n)$, then $L(\Gamma_n) \not\hookrightarrow_s L(\Gamma_m)$ for $n > m$.

⇝ The Cowling-Haagerup constant is decreasing under (stable) embeddings.

⇝ More rigid objects do not embed in less rigid objects.

First non-embeddability results

Countable-to-one results

- (Popa, 1986) On icc property (T) groups, $\Gamma \mapsto L(\Gamma)$ is countable-to-one.
- (Ozawa, 2002) There is no separable II_1 factor N such that $M \hookrightarrow N$ for all separable II_1 factors.

  $\rightsquigarrow$ On separable II_1 factors, the relation $\hookrightarrow$ has no upper bound.

Open problems.

- **Conjecture.** If $n > m$, then $L(\mathrm{PSL}(n,\mathbb{Z})) \not\hookrightarrow_s L(\mathrm{PSL}(m,\mathbb{Z}))$.
- Does $L(\mathbb{F}_2) \hookrightarrow M$ for any nonamenable II_1 factor M ?
  (von Neumann – Day problem for II_1 factors)
- Which II_1 factors M embed into $L(\mathbb{F}_2)$?

Embeddability of Bernoulli crossed products

Theorem (Popa-V, 2021)

Let $\Gamma = \mathbb{F}_n$ be a free group and (A_0, τ) amenable (e.g. abelian).

We build: $M(A_0, \tau) = (A_0, \tau)^{\otimes\Gamma} \rtimes (\Gamma \times \Gamma)$.

Then, $M(B_0, \tau) \hookrightarrow_s M(A_0, \tau)$ if and only if the initial data embed: $(B_0, \tau) \hookrightarrow (A_0, \tau)$.

- With $A_0 = \mathbb{C}^2$ and $\tau(x, y) = ax + (1 - a)y$: mutually non embeddable.
- With $A_0 = L^\infty([0, a] \cup \{1\})$ and $\tau =$ Lebesgue on $[0, a]$ and atom $1 - a$ at 1, we get $M_a \hookrightarrow_s M_b$ iff $a \leq b$.
- With $A_0 = R \oplus R$ and varying τ : all mutually embeddable, but not stably isomorphic.

⤳ We now exploit/generalize this much further.

Fundamental group of a II_1 factor

Murray-von Neumann: the fundamental group $\mathcal{F}(M)$ is the subgroup of $\mathbb{R}_+^*$ given by $\mathcal{F}(M) = \{t > 0 \mid M^t \cong M\}$.

- (Murray-von Neumann, 1943) $\mathcal{F}(R) = \mathbb{R}_+^*$.
- (Connes, 1980) If Γ is icc with property (T), then $\mathcal{F}(L(\Gamma))$ is countable.
- (Popa, 2001) With $M = L^\infty(\mathbb{T}^2) \rtimes \mathrm{SL}(2,\mathbb{Z})$, we have $\mathcal{F}(M) = \{1\}$.
- (Popa, 2003) Any countable subgroup of $\mathbb{R}_+^*$ as fundamental group.
- (Popa-V, 2008) Many uncountable (Borel) subgroups of $\mathbb{R}_+^*$ as fundamental group.
- (Popa-V, 2011) For any $M = L^\infty(X) \rtimes \mathbb{F}_2$, we have $\mathcal{F}(M) = \{1\}$.

$\rightsquigarrow$ **Wide open :** intrinsic description of possible fundamental groups !

One-sided fundamental group

Notation: $\mathcal{F}_s(M) = \{t > 0 \mid M \hookrightarrow M^t\}$. **Always:** $\mathbb{N} \subset \mathcal{F}_s(M)$. **Put** $\mathcal{F} = \mathcal{F}_s(M)$.

- Compose embeddings: $\mathcal{F} \cdot \mathcal{F} \subset \mathcal{F}$.
- Direct sum of embeddings: $\mathcal{F} + \mathcal{F} \subset \mathcal{F}$.
- Infinite direct sums of embeddings: if $\mathcal{F} \cap (0,1) \neq \emptyset$, then $\mathcal{F} = \mathbb{R}^*_+$.

$\rightsquigarrow$ Only known computations give $\mathcal{F}_s(M) = \mathbb{N}$ or $\mathcal{F}_s(M) = \mathbb{R}^*_+$.

Theorem (Popa-V, 2021)

Let $\mathbb{N} \subset \mathcal{F} \subset [1, +\infty)$ with $\mathcal{F}$ countable, $\mathcal{F} + \mathcal{F} \subset \mathcal{F}$ and $\mathcal{F} \cdot \mathcal{F} \subset \mathcal{F}$.

Let M be one of our Bernoulli crossed products. Define $P = \left(*_{s \in \mathcal{F}} M^{1/s}\right) * L(\mathbb{F}_\infty)$.

Then, $\mathcal{F}_s(M) = \mathcal{F}$. **Example:** $\mathbb{N} + \sqrt{p}\,\mathbb{N}$.

Wilder embeddability results

We have seen: "base space" $(A_0, \tau) \rightsquigarrow$ Bernoulli crossed product $M(A_0, \tau)$.

Now: augmentation functor assigning to any infinite group Γ an icc group H_Γ.

Theorem (Popa-V, 2021)

Let Γ be an infinite group.

- Put $G_\Gamma = \mathbb{F}_{1+|\Gamma|}$ freely generated by a_0 and $(a_g)_{g\in\Gamma}$.
- Define $\pi : G_\Gamma \to \mathbb{Z} * \Gamma$ by $\pi(a_0) = 1 \in \mathbb{Z}$ and $\pi(a_g) = g \in \Gamma$.
- Put $N_\Gamma = \pi^{-1}(\Gamma)$.
- Define $H_\Gamma = (\mathbb{Z}/2\mathbb{Z})^{(J)} \rtimes (G_\Gamma \times G_\Gamma)$ with $J = (G_\Gamma \times G_\Gamma)/\Delta(N_\Gamma)$.

We have $L(H_\Lambda) \hookrightarrow_s L(H_\Gamma)$ if and only if $\Lambda \hookrightarrow \Gamma$

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Partially ordered sets of II_1 factors

We already constructed $(M_t)_{t\in\mathbb{R}}$ with $M_t \hookrightarrow_s M_r$ iff $t \leq r$. **Now:** $(M_i)_{i\in I}$ for $(I, \leq)$.

- A subset $I_0 \subset I$ of a partially ordered set is called **sup-dense** if every element of I is the supremum of a subset of I_0.

  Example: $\mathbb{Q} \subset \mathbb{R}$ is sup-dense.

- We say that $(I, \leq)$ is **separable** if $(I, \leq)$ admits a countable sup-dense subset.

Theorem (Popa-V, 2021)

For any separable partial order $(I, \leq)$, we construct a concrete family of separable II_1 factors $(M_i)_{i\in I}$ such that $M_i \hookrightarrow_s M_j$ if and only if $i \leq j$ if and only if $M_i \hookrightarrow M_j$.

With ω_1 the first uncountable ordinal, also $(\omega_1, \leq)$ can be realized as a chain of separable II_1 factors w.r.t. $\hookrightarrow_s$ and $\hookrightarrow$.

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II_1 factors without nontrivial self-embeddings

- Stable self-embeddings: $M \hookrightarrow M^d$ with $d > 0$.
- We say that $\theta : M \hookrightarrow M^d$ is **trivial** if $d \in \mathbb{N}$ and θ is unitarily conjugate to $M \to M_d(\mathbb{C}) \otimes M : a \mapsto 1 \otimes a$.
- We consider now: II_1 factors M for which all stable self-embeddings are trivial.

  $\rightsquigarrow$ They have trivial fundamental group, trivial outer automorphism group, no nontrivial finite index subfactors, etc.

Theorem (Popa-V, 2021)

Let $G = A_\infty$, the group of finite, even permutations of $\mathbb{N}$. Put $\Gamma = G * G$.

For an appropriate (generalized) Bernoulli action $\Gamma \times \Gamma \curvearrowright (X, \mu)$, the II_1 factor $M = L^\infty(X) \rtimes (\Gamma \times \Gamma)$ has no nontrivial stable self-embeddings.

The semiring of stable self-embeddings

Given a II_1 factor M, consider all embeddings $\theta : M \hookrightarrow M^d$, $d > 0$.

- The same as Hilbert M-bimodules ${}_M\mathcal{H}_M$ with $\dim_{-M}(\mathcal{H}) = d < +\infty$.
- Identify embeddings that are unitarily conjugate (or bimodules that are isomorphic).
- Composition of embeddings, direct sum of embeddings.
- We get the semiring $\mathrm{Emb}_s(M)$.
- No nontrivial self-embeddings means: $\mathrm{Emb}_s(M) = \mathbb{N}$.

Theorem (Popa-V, 2021)

For many semigroups $\mathcal{S}$, we explicitly construct II_1 factors M with $\mathrm{Emb}_s(M) \cong \mathbb{N}[\mathcal{S}]$.

Outer automorphism groups

The embeddings semiring $\mathrm{Emb}_s(M)$ encodes in particular $\mathrm{Out}(M) = \mathrm{Aut}(M)/\mathrm{Inn}(M)$, the group of outer automorphisms of M.

Theorem (Popa-V, 2021)

The following Polish groups arise as $\mathrm{Out}(M)$ for a full II_1 factor M.

- All closed subgroups of the Polish group $\mathrm{Sym}(\mathbb{N})$ of all permutations of $\mathbb{N}$.
- Unitary groups $\mathcal{U}(N)$ of von Neumann algebras.
- Locally compact, totally disconnected groups.
- Compact groups.

$\rightsquigarrow$ Wide open: intrinsic characterization of Polish groups that arise as $\mathrm{Out}(M)$.

Complete intervals of II_1 factors

A **complete interval** in $(\mathbf{II_1}, \hookrightarrow_s)$ means:

- a family of II_1 factors $(M_i)_{i\in I}$ indexed by a partially ordered set $(I, \leq)$;
- $M_i \hookrightarrow_s M_j$ if and only if $i \leq j$;
- if N as **any** II_1 factor and $M_i \hookrightarrow_s N \hookrightarrow_s M_j$, there is a unique $k \in I$ and $t > 0$ with $N \cong M_k^t$ and $i \leq k \leq j$.

A **lattice** is a partially ordered set $(I, \leq)$ in which every $a, b \in I$ have a supremum $a \vee b$ and infimum $a \wedge b$.

Theorem (Popa-V, 2021)

We concretely realize any finite lattice as a complete interval in $(\mathbf{II_1}, \hookrightarrow_s)$.

More generally: any separable, algebraic lattice.