W*-rigidity paradigms for embeddings of II$_1$ factors

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II₁ factors

Fundamental work of Murray – von Neumann, 1943

- Weakly closed ∗-subalgebras $M \subset B(H)$. Nowadays called von Neumann algebras.
- Bicommutant theorem: $M' = \{ T \in B(H) \mid ST = TS \text{ for all } S \in M \}$ and $M = M''$.
- Factor: $Z(M) = M \cap M' = \mathbb{C}1$.
- Factors come in different types with $B(H)$ being of type I.
- $\text{II}_1$ factors: factors with a tracial state $\tau : M \to \mathbb{C}1$.

Continuous dimension $\tau(p) \in [0, 1]$.

One of the most fascinating mathematical structures, sometimes extremely rich in symmetries, sometimes extremely rigid.
Group factors and group measure space factors

Geometric / group theoretic / ergodic / Lie theoretic initial data \( \leadsto \) \( \text{II}_1 \) factor.

**Group II\(_1\) factors**

Let \( G \) be a countable group with infinite conjugacy classes (icc).

- Left regular representation: \( \lambda : G \to U(\ell^2(G)) : \lambda_g \delta_h = \delta_{gh} \).
- \( \mathbb{C}[G] = \text{span}\{\lambda_g \mid g \in G\} \) and \( L(G) = \text{weak closure } \mathbb{C}[G] \).

**Group measure space II\(_1\) factors**

Let \( G \) be an infinite group and \( G \curvearrowright (X, \mu) \) a free, ergodic, pmp action.

- \( A = L^\infty(X) \) and \( M = A \rtimes G \).
- Generated by \( A \subset M \) and \( L(G) \subset M \) with \( u_g^* F(\cdot) u_g = F(g \cdot) \).
Isomorphism and embedding problems for II$_1$ factors

**Many-to-one paradigm**

Large classes of very distinct initial data may give isomorphic II$_1$ factors.

**One-to one / W*-rigidity paradigm**

Large classes of initial data such that $S \ni S \mapsto M(S) \in$ II$_1$ is one-to-one.

- Up to isomorphism of II$_1$ factors.
- Up to stable isomorphism $N \cong M^t$.

Recall: for $0 < t < 1$, take $p \in M$ with $\tau(p) = t$ and put $M^t = pMp$.

In general, $M^t = p(M_n(\mathbb{C}) \otimes M)p$.

- Up to virtual isomorphism, up to embeddability, ..., see later.
Many-to-one results: hyperfiniteness

Murray-von Neumann: a II$_1$ factor $M$ is hyperfinite if there exists an increasing sequence $A_n \subset M$ of finite dimensional $*$-subalgebras with $\bigcup_n A_n$ weakly dense in $M$.

Canonical construction: $R = M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes \cdots$.

**Theorem (Murray-von Neumann, 1943)**

- All hyperfinite II$_1$ factors are isomorphic!
- Every II$_1$ factor $M$ contains copies of $R \hookrightarrow M$.

“The possibility exists that any factor in the case II$_1$ is isomorphic to a sub-ring of any other such factor.”

**Notation:** $N \hookrightarrow M$ if there exists an embedding of $N$ into $M$.

Also, $N \hookrightarrow_s M$ if there exists a $t > 0$ with $N \hookrightarrow M^t$. 
Many-to-one results: amenability

- A group $G$ is called **amenable** if there exists a translation invariant mean on $G$, i.e. $m(gU) = m(U)$.

- A von Neumann algebra $M \subset B(H)$ is called **amenable** if there exists a (not necessarily normal) conditional expectation $P : B(H) \to M$.

- A group II$_1$ factor $L(G)$ is amenable if and only if $G$ is amenable.

- If $M$ is an amenable II$_1$ factor and $N \hookrightarrow_s M$, then $N$ is amenable.

If $\Lambda$ is nonamenable and $\Gamma$ is amenable, then $L(\Lambda) \not\hookrightarrow_s L(\Gamma)$.

**Theorem (Connes, 1976)**

All amenable II$_1$ factors are isomorphic!

Thus, for all amenable icc groups $G$, we have $L(G) \cong R$. Thus, if $N \hookrightarrow_s R$, then $N \cong R$!
## First non-embeddability results

### Connes-Jones, 1983

If $\Lambda$ has Kazhdan’s property (T) and if $\Gamma$ has the Haagerup property, then $L(\Lambda) \not\hookrightarrow_s L(\Gamma)$. For example, $L(SL(3, \mathbb{Z})) \not\hookrightarrow_s L(\mathbb{F}_\infty)$.  

The intrinsic definition of property (T) for II$_1$ factors.

### Cowling-Haagerup, 1988

If $\Gamma_n$ is a lattice in $Sp(1, n)$, then $L(\Gamma_n) \not\hookrightarrow_s L(\Gamma_m)$ for $n > m$.  

The Cowling-Haagerup constant is decreasing under (stable) embeddings.  

More rigid objects do not embed in less rigid objects.
First non-embeddability results

Countable-to-one results

- (Popa, 1986) On icc property (T) groups, $\Gamma \mapsto L(\Gamma)$ is countable-to-one.

- (Ozawa, 2002) There is no separable II$_1$ factor $N$ such that $M \hookrightarrow N$ for all separable II$_1$ factors.

  On separable II$_1$ factors, the relation $\hookrightarrow$ has no upper bound.

Open problems.

- **Conjecture.** If $n > m$, then $L(\text{PSL}(n, \mathbb{Z})) \not\hookrightarrow_s L(\text{PSL}(m, \mathbb{Z}))$.

- Does $L(\mathbb{F}_2) \hookrightarrow M$ for any nonamenable II$_1$ factor $M$? (von Neumann – Day problem for II$_1$ factors)

- Which II$_1$ factors $M$ embed into $L(\mathbb{F}_2)$?
Embeddability of Bernoulli crossed products

**Theorem (Popa-V, 2021)**

Let $\Gamma = F_n$ be a free group and $(A_0, \tau)$ amenable (e.g. abelian).

We build: $M(A_0, \tau) = (A_0, \tau) \otimes_{\Gamma} (\Gamma \times \Gamma)$.

Then, $M(B_0, \tau) \hookrightarrow_s M(A_0, \tau)$ if and only if the initial data embed: $(B_0, \tau) \hookrightarrow (A_0, \tau)$.

- With $A_0 = \mathbb{C}^2$ and $\tau(x, y) = ax + (1 - a)y$ : mutually non embeddable.

- With $A_0 = L^\infty([0, a] \cup \{1\})$ and $\tau = \text{Lebesgue on } [0, a]$ and atom $1 - a$ at 1, we get $M_a \hookrightarrow_s M_b$ iff $a \leq b$.

- With $A_0 = \mathbb{R} \oplus \mathbb{R}$ and varying $\tau$ : all mutually embeddable, but not stably isomorphic.

We now exploit/generalize this much further.
Fundamental group of a II$_1$ factor

**Murray-von Neumann:** the fundamental group $\mathcal{F}(M)$ is the subgroup of $\mathbb{R}^*_+$ given by $\mathcal{F}(M) = \{t > 0 \mid M^t \cong M\}$.

- (Murray-von Neumann, 1943) $\mathcal{F}(\mathbb{R}) = \mathbb{R}^*_+$.  
- (Connes, 1980) If $\Gamma$ is icc with property (T), then $\mathcal{F}(L(\Gamma))$ is countable.
- (Popa, 2001) With $M = L^\infty(\mathbb{T}^2) \rtimes \text{SL}(2, \mathbb{Z})$, we have $\mathcal{F}(M) = \{1\}$.
- (Popa, 2003) Any countable subgroup of $\mathbb{R}^*_+$ as fundamental group.
- (Popa-V, 2008) Many uncountable (Borel) subgroups of $\mathbb{R}^*_+$ as fundamental group.
- (Popa-V, 2011) For any $M = L^\infty(X) \rtimes \mathbb{F}_2$, we have $\mathcal{F}(M) = \{1\}$.

Wide open: intrinsic description of possible fundamental groups!
One-sided fundamental group

Notation: $\mathcal{F}_s(M) = \{ t > 0 \mid M \hookrightarrow M^t \}$. Always: $\mathbb{N} \subset \mathcal{F}_s(M)$. Put $\mathcal{F} = \mathcal{F}_s(M)$.

- Compose embeddings: $\mathcal{F} \cdot \mathcal{F} \subset \mathcal{F}$.
- Direct sum of embeddings: $\mathcal{F} + \mathcal{F} \subset \mathcal{F}$.
- Infinite direct sums of embeddings: if $\mathcal{F} \cap (0, 1) \neq \emptyset$, then $\mathcal{F} = \mathbb{R}_+^*$.

Only known computations give $\mathcal{F}_s(M) = \mathbb{N}$ or $\mathcal{F}_s(M) = \mathbb{R}_+^*$.

Theorem (Popa-V, 2021)

Let $\mathbb{N} \subset \mathcal{F} \subset [1, +\infty)$ with $\mathcal{F}$ countable, $\mathcal{F} + \mathcal{F} \subset \mathcal{F}$ and $\mathcal{F} \cdot \mathcal{F} \subset \mathcal{F}$.

Let $M$ be one of our Bernoulli crossed products. Define $P = \left( *_{s \in \mathcal{F}} M^{1/s} \right) * L(\mathbb{F}_\infty)$.

Then, $\mathcal{F}_s(M) = \mathcal{F}$. Example: $\mathbb{N} + \sqrt{p} \mathbb{N}$. 

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Wilder embeddability results

We have seen: “base space” \((A_0, \tau) \sim \text{Bernoulli crossed product } M(A_0, \tau)\).

Now: augmentation functor assigning to any infinite group \(\Gamma\) an icc group \(H_\Gamma\).

**Theorem (Popa-V, 2021)**

Let \(\Gamma\) be an infinite group.

- Put \(G_\Gamma = \mathbb{F}_{1+|\Gamma|}\) freely generated by \(a_0\) and \((a_g)_{g \in \Gamma}\).
- Define \(\pi : G_\Gamma \to \mathbb{Z} \times \Gamma\) by \(\pi(a_0) = 1 \in \mathbb{Z}\) and \(\pi(a_g) = g \in \Gamma\).
- Put \(N_\Gamma = \pi^{-1}(\Gamma)\).
- Define \(H_\Gamma = (\mathbb{Z}/2\mathbb{Z})^J \rtimes (G_\Gamma \times G_\Gamma)\) with \(J = (G_\Gamma \times G_\Gamma)/\Delta(N_\Gamma)\).

We have \(L(H_\Lambda) \hookrightarrow_s L(H_\Gamma)\) if and only if \(\Lambda \hookrightarrow \Gamma\).
Partially ordered sets of $\text{II}_1$ factors

We already constructed $(M_t)_{t \in \mathbb{R}}$ with $M_t \rightarrow_{s} M_r$ iff $t \leq r$. Now: $(M_i)_{i \in I}$ for $(I, \leq)$.

- A subset $I_0 \subset I$ of a partially ordered set is called sup-dense if every element of $I$ is the supremum of a subset of $I_0$.

  Example: $\mathbb{Q} \subset \mathbb{R}$ is sup-dense.

- We say that $(I, \leq)$ is separable if $(I, \leq)$ admits a countable sup-dense subset.

**Theorem (Popa-V, 2021)**

For any separable partial order $(I, \leq)$, we construct a concrete family of separable $\text{II}_1$ factors $(M_i)_{i \in I}$ such that $M_i \rightarrow_{s} M_j$ if and only if $i \leq j$ if and only if $M_i \rightarrow_{s} M_j$.

With $\omega_1$ the first uncountable ordinal, also $(\omega_1, \leq)$ can be realized as a chain of separable $\text{II}_1$ factors w.r.t. $\rightarrow_{s}$ and $\rightarrow$. 
II$_1$ factors without nontrivial self-embeddings

- Stable self-embeddings: $M \hookrightarrow M^d$ with $d > 0$.

- We say that $\theta : M \hookrightarrow M^d$ is **trivial** if $d \in \mathbb{N}$ and $\theta$ is unitarily conjugate to $M \to M_d(\mathbb{C}) \otimes M : a \mapsto 1 \otimes a$.

- We consider now: II$_1$ factors $M$ for which all stable self-embeddings are trivial.

  They have trivial fundamental group, trivial outer automorphism group, no nontrivial finite index subfactors, etc.

**Theorem (Popa-V, 2021)**

Let $G = A_\infty$, the group of finite, even permutations of $\mathbb{N}$. Put $\Gamma = G \ast G$.

For an appropriate (generalized) Bernoulli action $\Gamma \times \Gamma \curvearrowright (X, \mu)$, the II$_1$ factor $M = L^\infty(X) \rtimes (\Gamma \times \Gamma)$ has no nontrivial stable self-embeddings.
The semiring of stable self-embeddings

Given a II$_1$ factor $M$, consider all embeddings $\theta : M \hookrightarrow M^d$, $d > 0$.

- The same as Hilbert $M$-bimodules $M\mathcal{H}_M$ with $\dim_{-M}(\mathcal{H}) = d < +\infty$.
- Identify embeddings that are unitarily conjugate (or bimodules that are isomorphic).
- Composition of embeddings, direct sum of embeddings.
- We get the semiring $\text{Emb}_s(M)$.
- No nontrivial self-embeddings means: $\text{Emb}_s(M) = \mathbb{N}$.

Theorem (Popa-V, 2021)

For many semigroups $S$, we explicitly construct II$_1$ factors $M$ with $\text{Emb}_s(M) \cong \mathbb{N}[S]$. 
Outer automorphism groups

The embeddings semiring $\text{Emb}_s(M)$ encodes in particular $\text{Out}(M) = \text{Aut}(M)/\text{Inn}(M)$, the group of outer automorphisms of $M$.

Theorem (Popa-V, 2021)

The following Polish groups arise as $\text{Out}(M)$ for a full II$_1$ factor $M$.

- All closed subgroups of the Polish group $\text{Sym}(\mathbb{N})$ of all permutations of $\mathbb{N}$.
- Unitary groups $U(\mathcal{N})$ of von Neumann algebras.
- Locally compact, totally disconnected groups.
- Compact groups.

Wide open: intrinsic characterization of Polish groups that arise as $\text{Out}(M)$.
Complete intervals of II$_1$ factors

A **complete interval** in $\langle \text{II}_1, \hookrightarrow_s \rangle$ means:

- a family of II$_1$ factors $(M_i)_{i \in I}$ indexed by a partially ordered set $(I, \leq)$;
- $M_i \hookrightarrow_s M_j$ if and only if $i \leq j$;
- if $N$ is any II$_1$ factor and $M_i \hookrightarrow_s N \hookrightarrow_s M_j$, there is a unique $k \in I$ and $t > 0$ with $N \cong M_k^t$ and $i \leq k \leq j$.

A **lattice** is a partially ordered set $(I, \leq)$ in which every $a, b \in I$ have a supremum $a \vee b$ and infimum $a \wedge b$.

**Theorem (Popa-V, 2021)**

We concretely realize any finite lattice as a complete interval in $\langle \text{II}_1, \hookrightarrow_s \rangle$.

More generally: any separable, algebraic lattice.