Dynamics of NLS with Bounded Initial Data

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The Nonlinear Schrödinger Equation on $\mathbb{R}$ or $\mathbb{Z}$

$$i\frac{\partial}{\partial t}u(t,x) = i\dot{u}(t,x) = -\Delta u(t,x) + u(t,x) + |u|^2 u(t,x),$$

with initial data $u(0,x) = u_0(x),\ x \in \mathbb{R}\, or\, \mathbb{Z}$

On $\mathbb{R}$, assume that $u_0(x)$ is uniformly smooth and bounded.

On $\mathbb{Z}$, Δ is the finite difference Laplacian.

Key Questions:

Does there exist a solution?

Describe properties of the solution.

What is its growth as a function of time?

Examples of initial data

Examples on R

1) Periodic function: $u_0(x) = \sum_n a_n e^{inx}$.

2) Quasi-periodic $u_0(x) = a\cos(x) + b\cos(\alpha\, x)$, α *irrational*

3) Smooth Random $u_0(x) = \sum_{j\in\mathbb{Z}} a_j e^{-(x-j)^2}$, $x \in \mathbb{R}$, $|a_j| \leq 1$

Examples on Z

1) Bounded: $|u_0(x)| \leq C$, $x \in \mathbb{Z}$.

2) Quasi-periodic: $u_0(x) = a\cos(\alpha\pi x)$, $x \in \mathbb{Z}$, α *irrational*

3) Random: $u_0(x)$ independent identically distributed (iid), $x \in \mathbb{Z}$.

4) $u_0(x)$ in equilibrium.

Conservation laws for NLS

Mass: M(t)= $\int_R |u(t,x)|^2 dx$

Energy: E(t)= $\frac{1}{2}\int_R \{|\nabla u(t,x)|^2 + |u(t,x)|^2 + \frac{1}{2}|u(t,x)|^4\} dx$

On the lattice $\mathbb{Z}$ the integrals are replaced by sums.

Conservation law : M(t) and E(t) are **independent of t**.
They are essential for proving global existence of solutions when they are finite.

However, Mass and Energy are **infinite** for the cases above.

The *periodic case* can be formulated on a circle or torus $\mathbb{T}^d$ **finite energy**. Solutions have been extensively analysed by Bourgain + others.

Polynomial upper bounds on the growth of $H^s(\mathbb{T}^d)$ for smooth initial conditions. H^1 is bounded by conservation of energy but when $s > 1$ the H^s norm may grow.

Motivation and Comments

There is an enormous mathematical literature on NLS. Almost all papers assume that the initial data has finite energy so that conservations laws can be used. In physical situations such as light transmission in very long fibers, or dynamics of ocean waves the finite energy assumption does not seem so natural. Perhaps one should assume that the energy is only locally finite. This assumption is probably insufficient to take an infinite volume limit for NLS. One will presumably have to assume higher regularity and / or some randomness in the initial data. The theory of wave turbulence (Zakharov) is now being developed by mathematicians.

Buckmaster, Collot, Deng, Germain, Hani, Lukkarinen, Masmoudi, Shatah, and Spohn have made progress in showing that NLS with random data can be described for certain time scales by **effective kinetic equations**. Feynman Graph estimates.

Outline of Talk

A) Results for quasi-periodic

B) Nonlinear Wave equation - Finite propagation speed

C) Linear Schrödinger Dynamics

D) Global existence and bounds for the lattice NLS

E) Global existence for Regularized NLS on $\mathbb{R}$

F) Local existence for NLS with analytic data - Newton Iteration

G) Conclusions and Conjectures (Driven anharmonic oscillator)

Results for Quasi-Periodic Data on $\mathbb{R}$

$$u_0(x) = a\cos(x) + b\cos(\sqrt{2}\,x)$$

Tadahiro Oh: Local existence proved for quasi-periodic data and global existence for a class of *limit periodic* data.

Damanik, Goldstein, Binder, Luckic:
Global existence of **KdV** for *small amplitude* data with good Diophantine frequency. Solution bounded and almost periodic in time. Relies of **integrability** of 1D KdV. Deift conjecture.

Remark: Global existence for large data qp data **not known** for KdV or NLS.

Wei-Min Wang: There exist small amplitude solutions to 1D NLS-type equations which are quasi-periodic in space and time. KAM method. Applies to non-integrable equations.

Nonlinear Wave equation (NLW) on $\mathbb{R}$

$$u_{tt}(t,x) - u_{xx}(t,x) + u^3(t,x) = 0, \;\; u(0,x),\, u_t(0,x) \in C^2(R)$$

Finite Propagation Speed:

$u(t, x_0)$ only depends on $u_0(x)$ for $|x - x_0| \leq t$.

Thus the energy is effectively finite at any time.

$$E(u, u_t) = \frac{1}{2}\int (\partial_x u(t,x))^2 dx + \frac{1}{2}\int (u_t(t,x))^2 dx + \frac{1}{4}\int u(t,x)^4 dx.$$

Lemma: If the data uniformly in $C^1(R)$ then NLW has a unique solution and $|u(t,x)| \leq Ct^{1/3}$.

Problem: NLS and KdV **do not** have finite propagation speed.
Rough data can propagate very rapidly.
Or solution may become rough via nonlinearity. Speed: k^2/t.

Dynamics of Linear Schrödinger $\mathbb{R}$

$$i\,\dot{u}(t,x) = -\Delta u(t,x), \quad u(0,x) = u_0(x), \text{ bounded}$$

Lemma: If $u_0(x)$ in $C^2(R)$ then

$$|u(t,x)| \leq Ct^{3/2} ||\nabla_x^2 u_0(x)||_\infty \, .$$

Random case:

$$u_0(x) = \sum_j a_j e^{-(x-j)^2}, \;\; a_j \in \mathbb{C}, |a_j| \; \leq 1$$

then $|u(t,x)| \leq Ct^{1/2}$. The a_j can be chosen to cancel the phases so that you cannot improve $t^{1/2}$. Similar results hold on the lattice.

If the a_j are independent random variables, of mean 0 then $\mathbb{E}[|u(t,x)|^2] \leq C$.

Sketch of Proof of Lemma

Let $\chi_j(x), j \in \mathbb{Z}$ be a partition of unity, $\chi_j(x)$ supported near j.

Let $f_j = f\chi_j$ with f uniformly in $C^2(R)$.
Then $\sum_j [e^{it\Delta} f\chi_j]$ converges pointwise since by integration by parts

$$|e^{it\Delta} f_j|(x) \leq C[|x-j|+1]^{-3/2}\ t^{3/2}|\nabla^2 f_j|.$$

Alternatively use functional analysis of commutators

$$|x\, e^{it\Delta} f_0| = \int_0^t |e^{i(t-s)\Delta}[x,\Delta]e^{is\Delta} f_0|\, ds \leq C\ t|f_0'|$$

Here we used $[x,\Delta] = -2\frac{d}{dx}$ and $x\, f_0$ is bounded.

Dynamics of NLS on the Lattice $\mathbb{R}$

Let $-\Delta = \partial^* \partial$ be the finite difference Laplacian on $\mathbb{Z}$. Here

$$\partial f(x) = f(x+1) - f(x), \textit{ and } \partial^* f(x) = f(x-1) - f(x)$$

The lattice NLS is given by

$$i \frac{\partial}{\partial t} \psi(t,x) = i\dot{\psi}(t,x) = -\Delta \psi(t,x) + |\psi|^2 \psi(t,x), \; x \in \mathbb{Z}$$

Proposition: If $|u(0,x)| \leq A$, then $|u(t,x)| \leq C\,A\,t^{1/4}$. Moreover, the following space average is bounded:

$$\frac{1}{t} \sum_{|x-x_0| \leq t} |u(t,x)|^4 \leq Const. \quad all \; x_0 \in \mathbb{Z}$$

Remark: For the focusing Lattice NLS: $|u(t,x)| \leq CA\,t^{1/2}$

Sketch of Proof of Proposition

Define a **Local Mass**:

$$M(t) = \sum_x |u(t,x)|^2 e^{-F(t,x)}$$

where

$$F(t,x) = \frac{\sqrt{(x-x_0)^2+1}}{R(2t_0-t)}, \quad t_0 \geq t$$

Then using integration by parts and using the fact that Δ **is a bounded operator** on the lattice

$$\frac{dM(t)}{dt} \leq \frac{3M(t)}{R(2t_0-t)}, \quad \textit{thus} \;\; M(t_0) \leq \frac{3\ln 2}{R} M(0)\,.$$

The proposition now follows easily from this bound.

Global existence for Regularized NLS

Consider the regularized Hamiltonian, formally given by

$$\int_{\mathbb{R}} \{\frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{4}|u_\phi(t,x)|^4\} dx$$

Where $u_\phi = u \star \phi(x)$ with ϕ smooth symmetric function eg. e^{-x^2}.

Theorem If $u_0(x)$ lies in C^4 then there is a global solution with $|u(t,x)| \leq Ct^{8/3}$.

The proof uses a local energy norm:

$$E(t,x_0) = \int_{\mathbb{R}} \{\chi(\frac{x-x_0}{R})[\frac{1}{2}|u_x^2| + \frac{1}{4}|u_\phi|^4]\}\, dx$$

where $\chi(x) \geq 0$ is smooth and compactly supported.

Remark: The proof is more complicated because the derivative is unbounded.

Local existence for Real Analytic Data

Suppose that $u_0(x)$ is bounded and analytic in a strip of width 3. For example: the smooth random or quasi-periodic data.

Apply Newton iteration to solve NLS. Start $u_1(t,x) = e^{it\Delta} u_0$. The linear correction to u_1 is $\xi_2(t,x)$

$$i\frac{\partial \xi_2}{\partial t} = -\Delta \xi_2 + 2|u_1|^2 \xi_2 + u_1^2 \bar{\xi}_2 + R_1, \qquad \xi_2(0,x) = 0$$

$$R_1 = -i\frac{\partial u_1}{\partial t} - \Delta u_1 + |u_1|^2 u_1 = |u_1|^2 u_1$$

and set $u_2(t,x) = u_1(t,x) + \xi_2(t,x)$. The next correction ξ_3

$$i\frac{\partial \xi_3}{\partial t} = -\Delta \xi_3 + 2|u_2|^2 \xi_3 + u_2^2\, \bar{\xi}_3 + R_2, \qquad \xi_3(0,x) = 0,$$

$$R_2(t,x) = 2|\xi_2^2|u_1 + \xi_2^2\bar{u}_2 + |\xi_2^2|\xi_2\,.$$

The corrections ξ_m satisfy linear time dependent equations. The remainders R_m are quadratic in ξ_m. This makes the iteration converge super-exponentially.

To solve the equations for the $\xi, \bar{\xi}$ it is convenient to introduce matrix notation:

$$A_n = \begin{pmatrix} \xi_n \\ \bar{\xi}_n \end{pmatrix},\ B_n = \begin{pmatrix} R_n \\ -\bar{R}_n \end{pmatrix}$$

$$M_0 = \begin{pmatrix} -\Delta & 0 \\ 0 & \Delta \end{pmatrix},\ \ V_n(t,x) = \begin{pmatrix} 2|u_n|^2 & u_n^2 \\ -\bar{u}_n^2 & -2|u_n|^2 \end{pmatrix}.$$

Then the equations for ξ have the form

$$i\frac{\partial A_{n+1}}{\partial t} = M_n(t,x)\,A_{n+1} + B_n, \qquad M_n = M_0 + V_n.$$

Let $\Phi(t, t_1)$ be the fundamental solution operator:

$$i\frac{\partial \Phi(t,t_1)}{\partial t} = M_n \Phi_n(t,t_1), \qquad \Phi_n(t_1,t_1) = Id.$$

We solve

$$A_{n+1} = -i\Phi_n(t,0)\int_0^t \Phi_n(s,0)^{-1} B_n(s)ds.$$

To get smoothness use

$$[\frac{d}{dx}, \Phi_n(t_2,t_1)] = -i\int_{t_1}^{t_2} \Phi_n(t_2,s)\frac{d}{dx}V_n(s,x)\Phi_n(s,t_1)ds$$

We also need

$$[x, \Phi_n(t_2, t_1)] = 2i \int_{t_1}^{t_2} \Phi_n(t_2, s) \frac{d}{dx} \Phi_n(s, t_1) ds.$$

Norm used to control iteration:

$$\|f\|(r, p) = \sup_x \sum_{n \geq 0, 0 \leq q \leq p} |\frac{f^{(n+q)}(x)}{n!}| r^n$$

Let $r_{n+1} < r_n$ and set $\delta_n = r_n - r_{n+1}$. Then,

General Lemma: $\|f\|(r_{n+1}, p) \lesssim_p \|f\|(r_n, 0)\delta_n^{-p}$.

Key Lemma: For $|t| \leq 1$ and any $r_{n+1} > 0$,

$$\|\Phi_n(t,0)B_n\|(r_{n+1},0) \lesssim e^{t|V_n(t)|}\|B_n\|(r_{n+1},3)[1+\|V_n\|(r_{n+1},3)].$$

To prove bounds on $A_m = (\xi_m, \bar{\xi}_m)$ recall $B_m = (R_m, \bar{R}_m) \approx \xi_m^2$. Let

$$\epsilon_1 = ||u_1||(r_1, 0) \text{ and } \epsilon_k = ||A_k||(r_k, 0) .$$

If $\epsilon_1 > 0$ is sufficiently small, $p = 3$ and $\delta_n = n^{-2}$,

$$\epsilon_{n+1} \leq C\epsilon_n^2 n^4 \leq C^n (n!)^4 \epsilon_1^{2^n} \to 0,$$

as $n \to \infty$. Thus, $u_n = u_1 + \xi_2 + ... + \xi_n$ converges as $n \to \infty$

ODE Question:

$$\ddot{x}(t) + p(t)x + 2x^3 = 0, \quad p(t) \geq 1 \, and \, |\dot{p}(t)| \leq C.$$

Let

$$E(t) = \dot{x}^2 + p(t)x^2 + x^4$$

Then

$$\dot{E} = \dot{p}x^2 \leq CE^{1/2} \Rightarrow E(t) \leq Ct^2$$

This is a crude estimate like the ones we have proved for NLS.

If p(t) were very smooth and periodic or quasi-periodic then E(t) might grow much more slowly.

Can one get better bounds on growth rates for lattice NLS, NLWE with quasi-periodic or nice random data?

Random initial Data on Lattice

For random iid data on a periodic box of side L,
the **average local energy** is bounded in time uniformly in L.

What about bounds on higher moments of local energy?

If the data are in equilibrium (defocusing) then all moments of the local energy are bounded in t.

In 3D if data is iid, how to describe the formation of a condensate? (Data must be sufficiently large). We would like to describe loss of spatial independence as $t \to \infty$.

This is a hard question. Are there useful classical toy models one can analyze? For example large N, where N is the number of components of $u = (u_1, u_2, ...u_N)$.

THE END

More Properties of Φ

$$\|\Phi_n f\|_{L^2} \leq e^{|t||V_n(t)|}\|f\|_{L^2}, \quad \textit{where } |V_n(t)| = \sup_{s\leq t, x} |V_n(s,x)|.$$

Also we have the identity;

$$e^{a\frac{d}{dx}}\Phi_{n,0} f = \Phi_{n,a} e^{a\frac{d}{dx}} f,$$

where $\Phi_{n,a}$ is the fundamental solution for $M_0 + V_n(t, x+a)$.

By making a complex we can track analyticity.