PHYSICS 267: Higher Representations in Physics and Mathematics

Logistics:
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Office hours in Lyman 227 available during working hours, by appt.
In any given week, a few interested students may (and are encouraged to!) participate in videotaped Q&A discussions on any course-related topics at Bok Center for Learning (Science Center Building)

Prerequisites: In this course, all the methods used are elementary, finite-dimensional linear algebra, but used in very unusual ways. For instance, the course will not use any commutative diagrams.

Grading: Occasional homework assignments posted on Canvas (100%) but can be improved by class participation. No exams.

Outline: The course will consist of several big parts, some of which will be developed concurrently.

Part I: Topological quantum field theory (TQFT). Dissection of space will translate into mathematical structures and mathematical theorems. The topics covered will be:

1. TQFT in 1D: projections and orthogonalization
2. TQFT in 2D: algebras with traces, an elementary construction of Verlinde-type invariants for surfaces
3. TQFT in 2D: the computational algorithms. A fast way to block-diagonalize algebras, later used to find irreducible representations of a finite group or algebra.
4. TQFT in 3D: topological automorphisms of tori viewed as the action of the modular group connected to number theory
5. TQFT in 3D: Wormholes interpreted as the Drinfeld quantum double and its braiding

Part II: Elementary representation theory. We will discover the group representation theory from scratch, on a few simple examples, the cyclic and dihedral groups. We shall concentrate in these explicit cases on interactions between representations, analogous to Clebsch-Gordan coefficients and Racah-Wigner 3j and 6j symbols, directly computable in these simple examples. We shall describe the representations in the language of wire diagrams in the plane. ***No previous knowledge of representation theory is needed!*** We shall extend these techniques explicitly to the spinor group SU(2) and to the interaction between its representations.

Part III: Quantum symmetries of graphs. We show that finite graphs have, besides the classical graph homomorphisms and automorphisms, quantum automorphisms which apply, in the spirit of quantum mechanics, to linear combinations of edges using statistical mechanics type computations in plaquette models as Boltzmann weights. The graphs which have a finite number of quantum automorphisms are the same as the graphs describing the crystallography of simple Lie groups (***no previous knowledge assumed***). We show that a finite graph endowed with such plaquettes has a parallel transport like the one of Levi-Civita in general relativity. We compute the flat part of such finite graphs. We connect this structure to invariants found by theoretical physicists to representation of Hurwitz in number theory. We use these graph automorphisms to give a new, direct, and elementary construction from scratch the crystallographic structure of simple Lie groups.

Part IV: Higher representation theory. The previous theory can be interpreted as a construction of quantum subgroups of SU(2). We now construct, extending these methods, quantum subgroups of SU(3), SU(4), …. These would provide the higher analogs of the Coxeter-Dynkin diagrams, describing classical crystallography and group representations. These graphs answer a bottle-of-champagne classification question put by theoretical physicists DiFrancesco and Zuber. Starting from these, we build from scratch a higher (category) version of representation theory: new root lattices, weight lattices, diagonal elements, as well as higher matrices. The usual matrix algebra and representation theory is implicitly built, we show, on SL(2) as an underlying, or subjacent, group. The higher representation theory uses SL(3), SL(4), …, or any semisimple Lie group underlying its structure. We describe a new point of view on the Gelfand-Tsetlin explicit construction of representations of SL(N). The new higher theory describes relations between representations like the Racah-Wigner coefficients for the first time, combinatorically and crystallographically.

Conclusions: We finally show that underlying the whole classical and higher representation theory is a discrete form of Gaussian and Riemannian curvature, which allows a very simple geometrical description of it. Just like the 6j symbols of classical representations are describing interactions between punctures in 2D quantum field theory, the higher representation theory is aimed at providing, as internal symmetry of matter, the algebraic structure underlying quantum field theory in physical (3+1) dimensions. As the matter in this model would be described by Riemannian curvature, this would provide a concrete mechanism through which matter curves space.