Decay estimates and complete Bakry-Émry theory

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Plan

- Quantum dynamical systems
- Relative Entropy
- (modfied) Log-Sobolev inequalities
- Classical Laplacians
- Complete Bakry Émery Theorem
- Examples from graphs
- Open problems

QMS–Quantum Markov Semigroup

- Let $\mathcal{M} \subset \mathbb{B}(H)$ be a W^*-algebra (closed in WOT and *) and $\tau : \mathcal{M} \to \mathbb{C}$ a faithful normal, tracial state, i.e. $\tau(ab) = \tau(ba)$. Then $(\mathcal{M}, \tau)$ is called a noncommutative probability space (algebra of observables).
- A quantum Markov semigroup is given by a family $(T_t)_{t \geq 0}$ such that

$$T_t T_s \; = \; T_{t+s}$$

and each T_t is completely positive, i.e.

$$x \; \geq \; 0 \Rightarrow id \otimes T_t(x) \; \geq \; 0$$

for $x \in \mathbb{M}_n(\mathcal{M})$.
This means T_t remains positivity preserving if coupled with an auxiliary system. Moreover, $f_x(t) = T_t(x)$ is continuous with respect to the strong operator topology.

Additional assumption

- We will assume in addition that

$$\tau(T_t(a)b) \;=\; \tau(aT_t(b))\,.$$

  and denote by L the selfadjoint generator, i.e. $T_t = e^{-tL}$.
- In the probabilistic literature a semigroup $T_t = e^{-tL}$ with $L = L^*$ is called reversible. In quantum information theory reversible often refers to $T_t(a) = e^{itH}ae^{-itH}$, which is fully reversible in time.
- Warning: Assuming that the generator is selfadjoint maybe physically unrealistic, but interesting cases (time states) can be deduced from the analysis of these sa QMS (joint with Rouzé).

Gradient form and fixpoint algebra

- Let us define the gradient form

$$2\Gamma_L(a, b) \;=\; L(a^*)b + a^*L(b) - L(a^*b)\;.$$

- Selfadjoint QMS admit a subalgebra $M_{fix} \subset M$ of fixpoints of the evolution. The conditional expectation (orthogonal projection) is denoted by E_{fix}.

Classical examples

- (Heat semigroup) $-\Delta = \sum_j \frac{d^2}{dx_j}$ on $L_\infty(\mathbb{R}^n, dx)$ with gradient $\Gamma(f_1, f_2) = \sum_j \frac{d}{dx_j}(\overline{f}_1)\frac{d}{dx_j}(f_2)$.
- $\Delta = -\frac{d^2}{dt^2}$ on $\mathbb{T}$ the one dimensional torus.
- Let (M, g) a Riemanian manifold of dimension d and $(X_j)_{j=1}^d$ a vector field representing an orthonormal basis on the tangent space at every point. Then $-\Delta_{LB} = \sum_{j=1}^d X_j^2$ is the Laplace Beltrami operator with gradient form

$$\Gamma(f_1, f_2) = \sum_j X_j(f_1)X_j(f_2) \ .$$

- L is selfadjoint with respect to the volume vol_g. Let $\mu = e^{-\varphi}\mathrm{vol}_g$. Then

$$(\Delta_\varphi(f_1), f_2) \ = \ \int \sum_{j=1}^d X_j(f_1)^* X_j(f_2) d\mu$$

defines a selfadjoint generator for μ.

Discrete Examples

- Let $V = \{1, ..., n\}$ and $E \subset \{1, ..., n\}^2$ be a set of edges. Then the non-normalized graph Laplacian is given by

$$A_E(f)(x) \;=\; \sum_{y,(x,y)\in E} f(y) - f(x)\,.$$

The generator is selfadjoint if E is undirected (symmetric).

- A_E can be extended (in several ways) to the matrix algebra $\mathbb{M}_n = \mathbb{B}(\mathbb{C}^n)$. For example let

$$Y_e \;=\; i(|r\rangle\langle s| + |s\rangle\langle r|)$$

the Pauli (type) matrix corresponding to an edge $e = (r, s)$.

$$L_E(a) \;=\; \sum_{e\in E} Y_e^2 a + aY_e^2 - 2Y_e a Y_e \;=\; \sum_{e\in E}[Y_e[Y_e^*, a]]$$

defines a 'second order differential operator' on matrices equipped with the normalized trace $n\tau(x) = tr(x)$.

Church Thesis

- The usual model for quantum computation is via unitary evolution.

- Semigroups can be successfully modelled by unitaries (e.g. Eisert+co).
- Childs used continuous times evolution in connection with complexity and search algorithm.
- The aim of this talk is to prove decay estimates with entanglement.

Relative entropy

- Let (M, τ) be an algebra with a trace and $\sigma, \rho \geq 0$. Then the relative entropy

$$D(\rho|\sigma) = \tau(\rho \log \rho) - \tau(\rho \log \sigma)$$

is positive provided $\tau(\rho) = \tau(\sigma)$.
- Despite the delicate definition $\mathrm{supp}(\sigma) \subset \mathrm{supp}(\rho)$, relative entropy is a versatile tool, mimicking a distance between states:

$$D(\rho|\sigma) = 0 \quad \Leftrightarrow \quad \rho = \sigma \ .$$

- black wholes?
- Let (Ω, μ) be a probability spaces and $\int f d\mu = 1$. Then

$$D(f|1) = \int f \log f d\mu$$

can be used as 'energy functional' on the state space.
- For $(\mathbb{M}_n, \tau_n)$ we see that $D_\tau(n\rho|1) = \log n - H(\rho)$ holds for the von Neumann entropy $H(\rho) = tr(\rho \log \rho)$.

Log-Sobolev inequality

- Let $T_t = e^{-tL}$ be a QMS. Then the log-Sobolev constant is the largest constant λ such that

$$\lambda D(\rho|E_{fix}(\rho)) \;\le\; 2\mathcal{E}(\sqrt{\rho}) \;=\; 2\tau(L(\sqrt{\rho})\sqrt{\rho})\,.$$

- The modified log-Sobolev constant $\mathrm{MLSI}(L)$ is the largest λ such that

$$\lambda D(\rho|E_{fix}(\rho)) \;\le\; \mathcal{E}(\rho, \log\rho) \;=\; \tau(L(\rho)\log\rho)\,.$$

- **Remark:** For sa generator on a Riemanian manifold $2\,\mathrm{LSI}(L) = \mathrm{MLSI}(L)$.
- Combining results from Hörmander, Stein and Rothschield it is knwown that every Sub-Laplacian given by a bracket generating vector field admits a lower bound on the LSI constant-with is almost intractable.
- Gross, Jaffe,... overview of overview.

Complete Log-Sobolev constant

- The complete Log-Sobelev constant is

$$\text{CLSI}(L) \ = \ \inf_M \text{MLSI}(L \otimes id_M) \ ,$$

  where the infimum is taken over all noncommutative probability spaces.

- The MLSI-constant measures decay to equilibrium for relative entropy

$$D(T_t(\rho)|E_{fix}(\rho)) \ \leq \ e^{-t\,\text{CLSI}(L)} D(\rho|E_{fix}(\rho)) \ .$$

  Therefore the complete Log-Sobolev constant ensures in the presence of entanglement.

- The complete Log-Sobolev constant is tensor stable

$$\text{CLSI}(L_1 \otimes 1 + 1 \otimes L_2) \ \geq \ \min\{\text{CLSI}(L_1), \min \text{CLSI}(L_2)\}$$

Return to equilibrium

- There are different ways to measure time to equilibrium, for example the mixing time $t(\varepsilon) = \inf t$ such that

$$\| T_t(\rho) - E_{fix}(\rho)\|_1 \ \leq \ \varepsilon$$

  for all input densities ρ is called mixing time.

- The 1-norm

$$\|a - b\|_1 \ = \ 2 \sup_{0 \leq X \leq 1} [\tau(aX) - \tau(bX)]$$

  characterize distinguishability with respect to von Neumann measurements.

- Similarly, we consider the complete mixing time $t_{11}^{cb}(\varepsilon)$ such that

$$\|(T_t \otimes id)\rho^{AM} - (E_{fix} \otimes id)(\rho^{AM})\|_1 \ \leq \ \varepsilon$$

  holds for all states in a coupled system ρ^{AM}.

From CLSI to mixing time

- Pinsker's inequality

$$2\|\rho - \sigma\|_1^2 \;\leq\; D(\rho|\sigma)\,.$$

- Let $T_t : \mathbb{M}_n \to \mathbb{M}_n$ be a sa QMS. Then

$$t_{11}^{cb}(\varepsilon) \;\leq\; \frac{1 + \log\log n - \log\varepsilon}{\mathrm{CLSI}(L)}\,.$$

provides an estimate for the mixing time in presence of entanglement.

- **Remark:** The mixing time discrete semigroup $(\Phi^n)_{n\,\geq\,0}$ is used to estimate running times for Metropolis quantum search algorithm (Temme+co (Nature)).

Hypercontractivity and graphs

- A semigroup is called λ-hypercontractive if
$$\|T_t : L_2 \to L_{p(t)}\| \;\le\; 1$$
holds for $t \le \lambda \log \sqrt{p-1}$.
- For ergodic systems, we have (Gross '75)
$$\mathrm{LSI}(L) \;\ge\; \lambda \quad \Leftrightarrow \quad \mathrm{HC}(L) \;\ge\; \lambda\,.$$
- For the normalized graph-Laplacian Saloff-Coste and Diaconis '96 showed on a graph with n vertices showed that
$$\frac{1}{2\,\mathrm{LSI}(L)} \;\le\; t_{12}(1/2) \;\le\; \frac{4+\ln\ln n}{\mathrm{LSI}(L)}$$
holds for the $(1,2)$ return time
$$t_{12} \;=\; \inf\{t : \|T_t - E : L_1 \to L_2\| \le \frac{1}{2}\}\,.$$
- Hypercontractivity and LSI is closely related to concentration of meausure and Talagrand's inequality.
- The classical route!

Complete Bakry-Émry theory

Remark

The hypercontractivity approach fails miserably for matrix-valued functions (Bardet-GJLR). Indeed, the famous Rothaus Lemma fails because certain noncommutative vector-valued $L_p(L_2)$ have bad geometric properties (uniform convexity).

Theorem

(Barkry-Émry) Let (M, g) be a manifold and $\mu = e^{-\varphi}\mathrm{vol}_g$ such that

$$\mathrm{Ricci}(M) + \mathrm{Hess}(\varphi) \;\geq\; \kappa$$

Then $(L_\infty(M, \mu)$ satisfies κ-LSI.

Bochner-Lichenowitz-Formula

- The key ingredient is given by the Bochner-Lichnerowitz formula
$$\delta(L(f)) \;=\; \hat{L}(\delta(f)) + \mathrm{Rc}_\varphi(\delta(f))$$
and $(\mathrm{Rc}_\varphi(\omega), \omega) \;\geq\; \kappa(\omega, \omega)$ holds for all differential forms.
- (complete BE-Thm) Let L be a generator and $\delta : N \to \hat{N}$ be a derivation in a finite von Neummann algebra containing N such that
  i) $E_N(\delta(x)^*\delta(y)) = \Gamma_L(x, y)$ (then we call $(N, \delta, \hat{\tau})$ is a derivation triple);
  ii) $e^{-t\hat{L}}(x) = e^{-tL}(x)$,
  iii) $\delta(Lx) = \hat{L}(\delta(x)) + \mathrm{R}(\delta(x))$ holds for an N-N bimodule map and
  $$(\frac{R + R^*}{2}\omega, \omega) \;\geq\; \kappa(\omega, \omega)$$
  holds for all differential forms $\omega = \sum_j \delta(a_j)b_j$.

  Then L satisfies $\mathrm{CLSI}(L) \;\geq\; 2\kappa$.

Geometric Ricci curvature

- We say that L admits a lower Ricci bound κ, if there exists a derivation triple, a generator of a semigroup $\hat{L}$, and a bimodule map as above such that $\frac{R+R^*}{2} \geq \kappa id$.
- For $\mathbb{T}$ and $\Delta = -\frac{d^2}{dt^2}$ the pest possible bound is $\kappa = 0$.
- For $\sqrt{\Delta}$ the best possible lower bound is $\kappa = 1$.
- Using free probability we can show that $L = (id - E)$ admits lower bound $1/2$ which is optimal for $\mathbb{T}$.
- The generator $(I - E)$ on $\ell_\infty(\mathbb{Z}_2)$ admits $\kappa = 1$.

Examples beyond positive curvature

- Thm (GJMcBrannon-in progress) Assume that $\mathrm{R} + \mathrm{R}^* \geq \lambda id$ and $\lambda > -\infty$, and $\| T_t - E : L_1 \to L_2 \|_{cb} \leq ct^{-d/2}$. Then L satisfies CLSI(L) > 0.
- Cor All compact Riemanian manifolds satisfy CLSI(Δ_{LB}) > 0.
- Cor All compact Riemanian foliations saitsfying the Hörmander bracket generating condition satisfy CLSI(Δ_H) > 0.
- Conjecture All selfadjoint Lindbladian on a matrix algebra satisfy CLSI(L) > 0.

Representation Theory

❄ A sa Lindbladian on a matrix algebra is given by

$$L(\rho) \;=\; \sum_{j=1}^{d}[X_j,[X_j,\rho]] \;=\; \sum_{j=1}^{d} X_j^2\rho + \rho X_j^2 - 2X_j\rho X_j \;.$$

❄ We may define differential operators on $SU(n)$ via $X_j(f) = \frac{d}{dt}f(e^{itX_j}g)|_{t=0}$ and sub-Laplacian

$$\Delta_{\mathcal{X}} \;=\; \sum_{j=1}^{d} -X_j^2 \;.$$

❄ The collective Lindbladian on $\mathbb{M}_{n^m}$ is given by

$$L(m) \;=\; \sum_{j=1}^{d}\sum_{k=1}^{m} 1\otimes\cdots\otimes \underbrace{X_j}_{k\text{-th position}} \otimes\cdots \;.$$

Transference

- "Thm" assume that the $X_j = iZ_j$, Z_j are real and $Z_j* = -Z_j$. Then
$$\mathrm{CLSI}(\Delta_{\mathcal{X}}) \stackrel{?}{=} \inf_m \mathrm{CLSI}(L(m)) .$$

- We can not really prove this for CLSI but for $\mathrm{CLSI}^+ = \inf_{p>1} \mathrm{C}_p\mathrm{SI}$, a certain p-variation of complete Log-Sobolev constants.

Transferred Laplacian

- The easy part of the theorem starts with a vector field $\mathcal{X} = \{X_1, ..., X_d\}$ in the Lie algebra of a compact Lie group, and $\Delta_{\mathcal{X}} = -\sum_j X_j^2$.
- Let $u : G \to \mathbb{M}_N$ be a representation, $X_j^u = \frac{d}{idt} u(e^{itX_j})|_{t=0}$ and $L_{\mathcal{X}}^u = \sum_j [X_j^u, [X_j^u, \;]]$. Then we have a commuting diagram

$$\begin{array}{ccc} L_\infty(SU(n), \mathbb{M}_N) & \stackrel{e^{-t\Delta_{\mathcal{X}}} \otimes id}{\to} & L_\infty(SU(n), \mathbb{M}_N) \\ \uparrow_\pi & & \uparrow_\pi \\ \mathbb{M}_N & \stackrel{e^{-tL_{\mathcal{X}}^u}}{\to} & \mathbb{M}_N \end{array}$$

- In other words the Lindbladian defines a dynamical substem, provided we can add matrix-valued coefficients. Ergodicity is naturally violated.

Examples

Let Δ_{LB} the Laplace Beltrami operator for $SU(n)$. Then

$$L^u(\rho) \;=\; 2n(\rho - \frac{tr(\rho)}{n}1)\,.$$

By Bardet/Spohn, we know that for $u = id$

$$\mathrm{CLSI}(L^u) \;\geq\; 2n$$

which is better then the estimate from the curvature bound $2\frac{2n}{4} = n$.

The estimate $\mathrm{CLSI}(L^u) \;\geq\; n$ holds for every representation.

Small Hörmander system

The cyclic graph induced by $\mathbb{Z}_n$

is given by the n selfadjoint elements

$$X_j = i(|j\rangle\langle j+1| - |j+1\rangle\langle j|)$$

The value for $\mathrm{CLSI}(\Delta_{X_1,\dots,X_n})$ is unknown.
However, for the representation on $\mathbb{M}_n$ we know that

$$\mathrm{CLSI}(L_{\mathbb{Z}_n}) \sim cn^2 \ .$$

using result for $\mathbb{T}$, lifting if to the graph-Laplacian, and then lifting it again to the matrix algebra.

Graph Laplacian

- Let $V = \{1, ..., n\}$ and $E \subset V \times V$ a symmetric set of edges. Recall $Y_e = i(|r\rangle\langle s| + |s\rangle\langle r|)$ for $e = (r, s)$. Then

$$\mathrm{CLSI}(\sum_{e \in E} [Y_e, [Y_e, \ ]]) \ \geq \ \frac{1}{cdL(T)^2}$$

  where $L(T)$ is the length of a spanning tree for the graph and d its degree.
- Using tensor stability we deduce estimate for higher dimensional lattices, for example for peridoc lattice in $\mathbb{Z}_{n^d}$.
- The remark above applies to Talagrand's concentration inequalities on $\{-1, 1\}^n$ and recovers his concentration inequality $\mu(A)\mu(B) \ \leq \ e^{-c\mathrm{dist}(A,B)^2}$.

What is missing?

- The connection to transport inequality and the geometry of the state space.
- Carlen-Maas, Datta, Rouzé, Henson, worked on application in quantum information, and moreover quantum HWI-inequalities.
- Before our work the only known examples were $I - E$, and gaussian (fermionic and bosonic) estimates. Good examples for positive Ricci curvature on graphs still very are rare.
- Prove $\mathrm{CLSI}(L) > 0$.
- Connection to unitary evolution given by data from graph.
- Connection to complexity theory?

Thank you for listening!

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