Topological order, tensor networks and subfactors

Dedicated to the memory of Vaughan Jones

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Topological order and operator algebras

Physicists study fusion categories in condensed matter physics in connection to topological order and gapped Hamiltonians. We clarify relations between their approaches and subfactor theory of Jones and identify the role of higher relative commutants.

Outline of the talk:

1. Representation theory, particles and subfactors
2. Gapped Hamiltonian, MPS and PEPS
3. Work of Bultinck et al.
4. Subfactors and bi-unitary connections
5. MPO, a tube algebra and a modular tensor category
6. PMPO, PEPS and higher relative commutants

Group representations and particles

A particle behaves like a group representation. For two irreducible representations π, σ of a group G, we have a tensor product representation $\pi \otimes \sigma$ of G and its irreducible decomposition. This is a fusion of particles. The trivial representation corresponds to the vacuum. The contragredient representation corresponds to an antiparticle. We have pair creation and annihilation of a particle and an antiparticle.

If the group G is finite, we have only finitely many irreducible representations of G, and they are closed in taking tensor products and contragredient representations. The trivial/contragredient representation is like the identity/inverse element.

Group representations and bimodules

Let M be a factor, which is a simple von Neumann algebra on a Hilbert space. A representation theory of M is not exciting. The correct setting is a theory of bimodules over M. An M-M bimodule ${}_M H_M$ is a Hilbert space H with left and right actions of M. For ${}_M H_M$ and ${}_M K_M$, we have a natural notion of a relative tensor product ${}_M H \otimes_M K_M$. We also have irreducible decomposition, the trivial representation ${}_M M_M$ and the contragredient representation ${}_M \bar{H}_M$.

For a finite group G, its representation theory is completely imitated by that of bimodules over a certain factor M. However, the latter theory is much more general and diverse.

Subfactors and bimodules

Study of a subfactor $N \subset M$ (with finite Jones index) was initiated by Jones and many rich structures in connection to low-dimensional topology and quantum physics have been discovered in the last 40 years. It gives a "quantum" version of Galois theory through representation theory. The correct representation theory of a subfactor $N \subset M$ is given by theory of bimodules. A subfactor $N \subset M$ gives a bimodule ${}_N M_N$. (We should actually take a Hilbert space completion of M.)

We take a relative tensor power of ${}_N M_N$ and look at irreducible N-N bimodules arising in this way. If we have only finitely many such bimodules, we say the subfactor is of finite depth.

Fusion categories and subfactors

Irreducible N-N bimodules arising from a subfactor $N \subset M$ with finite Jones index and finite depth give a fusion category which is similar to that of representations of a finite group.

That is, we have relative tensor products, the trivial representation and the contragredient representation. However, we do not have commutativity of relative tensor products. That is, ${}_N H \otimes_N K_N$ and ${}_N H \otimes_N K_N$ are not isomorphic in general, while $\pi \otimes \sigma$ and $\sigma \otimes \pi$ are always unitarily equivalent. We have an abstract axiomatization of a (unitary) fusion category and any such a category is realized with bimodules over a certain factor.

Gapped Hamiltonian, MPS and PEPS

From a viewpoint of operator algebras, gapped Hamiltonians are studied in the context of an infinite tensor product of matrix algebras $M_d(\mathbb{C})$. A gapped Hamiltonian is a sequence of self-adjoint matrices in finite tensor products and they have a fixed gap between the lowest eigenvalues and the next eigenvalues. Various functional analytic studies have been made.

A matrix product state (MPS) is a certain expression of a state using a trace of a matrix product. It is useful to express a ground state of a gapped Hamiltonian explicitly for a one-dimensional system. A projected entangled pair state (PEPS) is its higher dimensional analogue. We are now interested in the 2-dimensional case here.

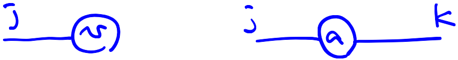
Topological order and anyons

A topological phase is a homotopy equivalence class of gapped Hamiltonians. One topological phase has topological order and we have finitely many quasi-particles called anyons there.

Such a system of anyons is described with a fusion category with a non-degenerate braiding (a modular tensor category). That is, the tensor product operation is commutative in a subtle way. Such a system is expected to be useful for constructing a topological quantum computer. One anyon corresponds to one irreducible object of a modular tensor category. Such a modular tensor category also arises from an operator algebraic conformal field theory (K-Longo-Müger).

Tensor networks

A vector (v_j) has one index j and a matrix (a_{jk}) has two indices j, k. They are graphically represented as follows.



Similarly, we can consider circles with 3 legs, 4 legs, and so on. Such an object is called a tensor.

The (j, l) entry of the matrix product of (a_{jk}) and (b_{kl}) is given by $\sum_k a_{jk} b_{kl}$. This is pictorially represented as follows and called a contraction.

We are now interested in tensors with 3 and 4 legs.

Work of Bultinck et al.

Bultinck-Mariëna-Williamson-Şahinoğlu-Haegemana-Verstraete (2017) considered an algebra of matrix product operators (MPOs) arising from a system of physically nice tensor networks. Such an algebra is called a matrix product operator algebra (MPOA). They considered a fusion category of matrix product operators and presented a graphical method to construct an interesting system of anyons. They get a modular tensor category and discuss its physical significance.

They are aware of its similarity to an old work of Ocneanu in subfactor theory. We clarify the role of subfactor theory in their work and its extensions.

Commuting squares and subfactors

Fix a subfactor $N \subset M$ with finite Jones index and finite depth. The diagram

$$\begin{array}{ccc} M' \cap M_k & \subset & M' \cap M_{k+1} \\ \cap & & \cap \\ N' \cap M_k & \subset & N' \cap M_{k+1} \end{array}$$

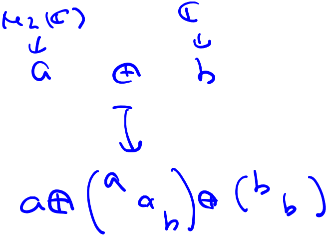
gives a square of finite dimensional C^*-algebras. Such a square satisfies a special property called a commuting square. If N or M has a nice approximation property with finite dimensional subalgebras, then the above commuting square for a single, sufficiently large k has complete information to recover the original $N \subset M$ (Popa). So classification of subfactors is reduced to the classification of certain commuting squares of finite dimensional C^*-algebras.

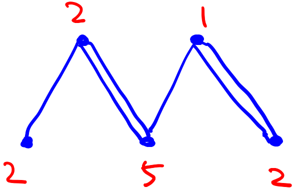
From a commuting square to a subfactor

Consider the following commuting square of finite dimensional C^*-algebras.
$$\begin{array}{ccc} A & \subset & B \\ \cap & & \cap \\ C & \subset & D. \end{array}$$
We can apply the Jones basic construction horizontally and vertically to get a double sequence $\{A_{k,l}\}_{k,l=0,1,2,\dots}$ of finite dimensional C^*-algebras with $A_{k,l} \subset A_{k+1,l}$ and $A_{k,l} \subset A_{k,l+1}$. Then we get two subfactors $A_{0,\infty} \subset A_{1,\infty}$ and $A_{\infty,0} \subset A_{\infty,1}$ as certain limit algebras. If the original commuting square arises from $N \subset M$ as in the previous slide, then the former is anti-isomorphic to $N \subset M$ and the latter is isomorphic to $N \subset M$.

Bi-unitary connections

For an inclusion $A \subset B$ of finite dimensional C^*-algebras, we draw the Bratteli diagram as follows, where a dot represents a direct summand.



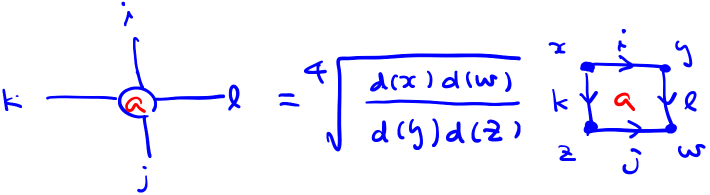


Look at the Bratteli diagrams of a commuting square. Choose one edge from each diagram corresponding to each of the four inclusions. We then get a complex number from the 4 edges. This is a notion of a bi-unitary connection considered by Ocneanu and Haagerup.

Bi-unitary connections and 4-tensors

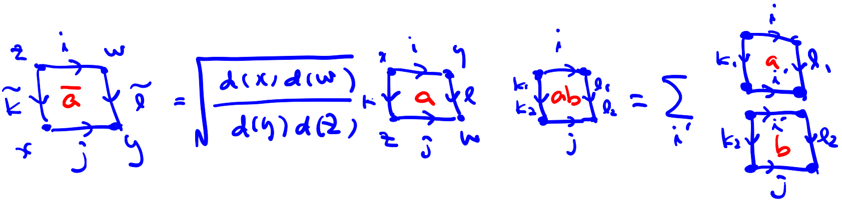
If one of the two subfactors $A_{0,\infty} \subset A_{1,\infty}$ and $A_{\infty,0} \subset A_{\infty,1}$ arising from a bi-unitary connection is of finite depth, so is the other (Sato). From now on, we assume this holds.

We would like to define a 4-tensor so that the labels for a wire are given by the edges of a Bratteli diagram and the value of a 4-tensor with four wires labeled is given by that of the square for a bi-unitary connection.



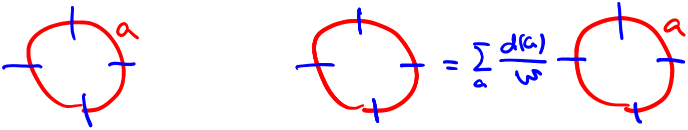
Bi-unitary connections, a fusion category and a 4-tensor

We have natural notions of a tensor product and irreducible decomposition for such bi-unitary connections corresponding to those for bimodules. In this way, we have a fusion category of bi-unitary connections equivalent to that of bimodules (Asaeda-Haagerup).



MPO and PMPO

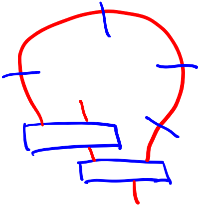
Using such 4-tensors arising from a bi-unitary connection, we define a matrix product operator (MPO) and a projector matrix product operator (PMPO) as follows, where the length of a circle happens to be 4.

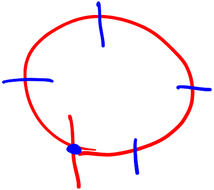


The product structure of these MPOs is isomorphic to that of relative tensor products of bimodules. We have a so-called zipper condition for these MPOs arising from this product structure.

An anyon algebra

Bultinck et al. introduced a new algebra called an anyon algebra for a family of 4-tensors and obtained a modular tensor category describing a topological order.





They have physical arguments to show braiding and the Verlinde formula in this setting and work out examples arising from a finite group and a 3-cycle. They are aware of its similarity to Ocneanu's tube algebra in subfactor theory.

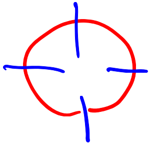
Identification of the two machineries

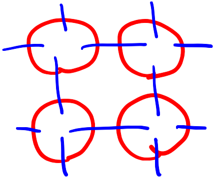
Suppose we start with a subfactor $N \subset M$ with finite Jones index and finite depth. We then have a family of bi-unitary connections as above. We can next show that this family produces 4-tensors satisfying all the settings of Bultinck et al. simply by changing the normalization constants arising from the Perron-Frobenius eigenvector entries.

Theorem *Ocneanu's tube algebra for the fusion category arising from a subfactor and the anyon algebra of Bultinck et al. arising from its bi-unitary connections are isomorphic. In particular, the two fusion rules are identical and the Verlinde formula also holds for the setting of anyon algebra of Bultinck et al.*

PEPS arising from a bi-unitary connection

For a PMPO of length 4, we have 4 exterior wires and 4 internal wires. By combining the internal 4 wires into one, we get a 5-tensor from this diagram.

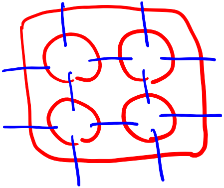


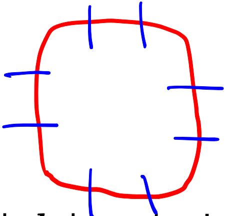


Bultinck et al. studied a PEPS on a square lattice as in the above picture, where the square lattice is of size 2×2 now. This gives a ground state of a local Hamiltonian.

PEPS and PMPO

For the PEPS in the right figure in the previous slide, a PMPO of length 8 acts from the exterior wires and PMPOs of length 4 acts from the internal wires. So this PEPS lives in the intersections of the ranges of these PMPOs.

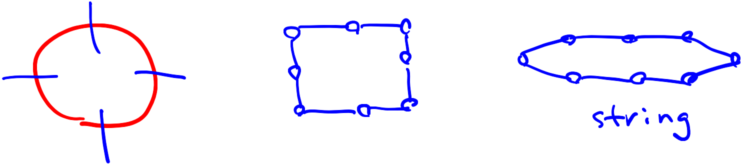




The range of a PMPO of length k has physical significance and we would like to identify this range with something important and well-studied in subfactor theory: higher relative commutants of a subfactor.

PMPO and the higher relative commutants

Theorem *The range of a PMPO of length k is naturally identified with the higher relative commutants $A'_{\infty,0} \cap A_{\infty,k}$ of the subfactor $A_{\infty,0} \subset A_{\infty,1}$ since an element in this range is identified with a flat field of strings in subfactor theory.*



Note that we have two subfactors $A_{0,\infty} \subset A_{1,\infty}$
$A_{\infty,0} \subset A_{\infty,1}$ in this study. The former appears in the anyon algebra and the latter appears here. They are known to be complex conjugate Morita equivalent.