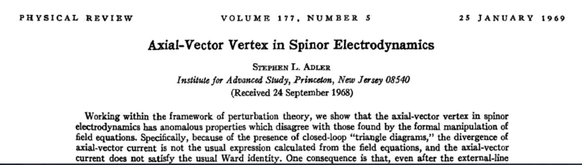
What is an anomaly?

Dan Freed
University of Texas at Austin

February 7, 2023

Adler, Bell-Jackiw

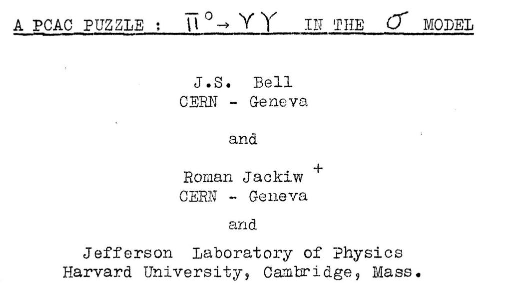


PHYSICAL REVIEW VOLUME 177, NUMBER 5 25 JANUARY 1969

Axial-Vector Vertex in Spinor Electrodynamics

Stephen L. Adler
Institute for Advanced Study, Princeton, New Jersey 08540
(Received 24 September 1968)

Working within the framework of perturbation theory, we show that the axial-vector vertex in spinor electrodynamics has anomalous properties which disagree with those found by the formal manipulation of field equations. Specifically, because of the presence of closed-loop "triangle diagrams," the divergence of axial-vector current is not the usual expression calculated from the field equations, and the axial-vector current does not satisfy the usual Ward identity. One consequence is that, even after the external-line



A PCAC PUZZLE : $\pi^0 \to \gamma\gamma$ IN THE σ MODEL

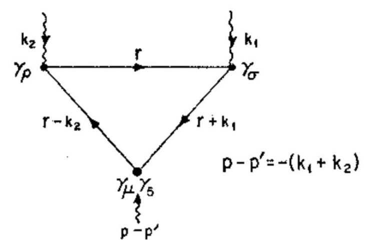
J.S. Bell
CERN - Geneva

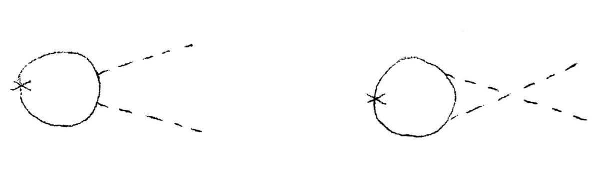
and

Roman Jackiw +
CERN - Geneva

and

Jefferson Laboratory of Physics
Harvard University, Cambridge, Mass.





Anomalies and the Atiyah-Singer index theorem

Nuclear Physics B
Volume 127, Issue 3, 12 September 1977, Pages 493-508

Axial anomaly and Atiyah-Singer theorem

N.K. Nielsen, Bert Schroer

PHYSICAL REVIEW D VOLUME 21, NUMBER 10 15 MAY 1980

Path integral for gauge theories with fermions

Kazuo Fujikawa
Institute for Nuclear Study, University of Tokyo, Tanashi, Tokyo 188, Japan
(Received 28 January 1980)

The Atiyah-Singer index theorem indicates that a naive unitary transformation of basis vectors for fermions interacting with gauge fields is not allowed in general. On the basis of this observation, it was previously shown that the path-integral measure of a gauge-invariant fermion theory is transformed nontrivially under the chiral transformation, and thus leads to a simple derivation of "anomalous" chiral Ward-Takahashi identities. We here clarify some of the technical aspects associated with the discussion. It is shown that the Jacobian factor in the path-integral measure, which corresponds to the Adler-Bell-Jackiw anomaly, is independent of any smooth regularization procedure of large eigenvalues of $\not{D}$ in Euclidean theory; this property holds in any even-dimensional space-time and also for the gravitational anomaly. The

Nuclear Physics B234 (1983) 269–330

GRAVITATIONAL ANOMALIES

Luis ALVAREZ-GAUMÉ[1]
Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138, USA
Edward WITTEN[2]
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544, USA

Received 7 October 1983

It is shown that in certain parity-violating theories in $4k+2$ dimensions, general covariance is spoiled by anomalies at the one-loop level. This occurs when Weyl fermions of spin-$\frac{1}{2}$ or -$\frac{3}{2}$ or self-dual antisymmetric tensor fields are coupled to gravity. (For Dirac fermions there is no trouble.) The conditions for anomaly cancellation between fields of different spin is investigated. In six dimensions this occurs in certain theories with a fairly elaborate field content. In ten dimensions there is a unique theory with anomaly cancellation between fields of different spin. It is the chiral $n=2$ supergravity theory, which is the low-energy limit of one of the superstring theories. Beyond ten dimensions there is no way to cancel anomalies between fields of different spin.

VOLUME 53, NUMBER 16 PHYSICAL REVIEW LETTERS 15 OCTOBER 1984

Anomalies in Nonlinear Sigma Models

Gregory Moore and Philip Nelson
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138
(Received 15 June 1984)

Certain nonlinear sigma models with fermions suffer from an anomaly similar to the one in non-Abelian gauge theory. We exhibit this anomaly using both perturbative and global methods. The affected theories are ill defined and hence unsuitable for describing low-energy dynamics. They include certain supersymmetric models in four-space dimensions.

ALGEBRAIC AND HAMILTONIAN METHODS IN THE THEORY OF NON-ABELIAN ANOMALIES

L.D. Faddeev and S.L. Shatashvili

The non-Abelian anomalies and the Wess–Zumino action are given a new interpretation in terms of infinitesimal and global cocycles of the representation of the gauge group acting on functionals of Yang–Mills fields. On the basis of this interpretation, two simple methods of nonperturbative calculation of the anomalies and the Wess–Zumino action are proposed.

Proc. Natl. Acad. Sci. USA
Vol. 81, pp. 2597–2600, April 1984
Mathematics

Dirac operators coupled to vector potentials

(elliptic operators/index theory/characteristic classes/anomalies/gauge fields)

M. F. ATIYAH† AND I. M. SINGER‡
†Mathematical Institute, University of Oxford, Oxford, England; and ‡Department of Mathematics, University of California, Berkeley, CA 94720

Contributed by I. M. Singer, January 6, 1984

THEOREM 4. *A gauge covariant $\mathcal{S}_r$(A) smooth in A exists if and only if the determinant line bundle of Ind $\not{\partial}$ is trivial—i.e.,* $d_2 = 0$ *in* $H^2(\mathfrak{A}/\mathcal{G}, Z)$ *or* $t_1 = 0$ *in* $H^1(\mathcal{G}, Z)$.

The characteristic forms $d_{2j} \epsilon H^{2j}(\mathfrak{A}/\mathcal{G}, Z)$ are obstructions to the existence of a covariant propagator for $\not{\partial}_{\mathfrak{A}/\mathcal{G}}$. We ask the question: Do the higher obstructions have physical significance?

Hamiltonian Interpretation of Anomalies

Philip Nelson[1]* and Luis Alvarez-Gaumé[2]
1 Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106, USA
2 Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138, USA

Abstract. A family of quantum systems parametrized by the points of a compact space can realize its classical symmetries via a new kind of nontrivial ray representation. We show that this phenomenon in fact occurs for the quantum mechanics of fermions in the presence of background gauge fields, and is responsible for both the nonabelian anomaly and Witten's SU(2) anomaly. This provides a hamiltonian interpretation of anomalies: in the affected theories Gauss' law cannot be implemented. The analysis clearly shows why there are no further obstructions corresponding to higher spheres in configuration space, in agreement with a recent result of Atiyah and Singer.

Faddeev's anomaly in Gauss's law

Graeme Segal

§1. General remarks

Faddeev [3] has pointed out that when a gauge theory is quantized the gauge operators act with anomalous commutation relations - so called "Schwinger terms" - on the Hilbert space $\mathcal{S}$ of states. In mathematical language this means that the Lie algebra $\mathcal{L}$ of the gauge group does not act on $\mathcal{S}$, but an extension of $\mathcal{L}$ by the vector space $\mathcal{F}$ of scalar-valued functions on the space of gauge fields does act. (Here $\mathcal{F}$ is regarded as an abelian Lie algebra.) The extension is described by a cocycle

$$c : \mathcal{L} \times \mathcal{L} \to \mathcal{F} .$$

In this note I shall explain how the cocycle c arises from simple topological considerations of a general kind. I am very grateful

Commun. Math. Phys. 100, 197—229 (1985)

Communications in Mathematical Physics

Global Gravitational Anomalies

Edward Witten*

Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544, USA

Abstract. A general formula for global gauge and gravitational anomalies is derived. It is used to show that the anomaly free supergravity and superstring theories in ten dimensions are all free of global anomalies that might have ruined their consistency. However, it is shown that global anomalies lead to some restrictions on allowed compactifications of these theories. For example,

not obvious. Usually, the only simple way to study a diffeomorphism π is to investigate the associated manifold $(M\times S^1)_\pi$ discussed in Sect. II. The simplest properties of $(M\times S^1)_\pi$ are invariants of a manifold B which has it for boundary. The only evident connection between $(M\times S^1)_\pi$ and B in which spinors play a role is the Atiyah-Patodi-Singer theorem concerning the η-invariant [29]. The η invariant can be defined as

$$\eta=\lim_{\varepsilon\to 0}\sum_{E_A\neq 0}(\operatorname{sign}E_A)\exp-\varepsilon|E_A|, \tag{22}$$

where E_A are the eigenvalues of the Dirac operator on $(M\times S^1)_\pi$. The Atiyah-Patodi-Singer theorem asserts (for the spin 1/2 case) that

$$\frac{\eta}{2}=\operatorname{index}_B(i\not{D})-\int_B\hat{A}(R), \tag{23}$$

hep-th/9907041, IASSNS-HEP-99-56

WORLD-SHEET CORRECTIONS VIA D-INSTANTONS

Edward Witten

School of Natural Sciences, Institute for Advanced Study

Olden Lane, Princeton, NJ 08540, USA

1.) Such a relation means that there is a three-manifold $U\subset Y$ whose boundary is the union of the C_i (or more generally a three-manifold U with a map $\phi:U\to Y$ such that the boundary of U is mapped diffeomorphically to the union of the C_i). In this situation, we can give a relation, which depends only on the gauge-invariant H-field and not on the mysterious B-field, for the product $\prod_{i=1}^s F(C_i)$.

First of all, though the factors $\exp\left(i\int_{C_i}B\right)$ are mysterious individually, for their product we can write an obvious classical formula that depends only on H and U:

$$\prod_{i=1}^s\exp\left(i\int_{C_i}B\right)=\exp\left(i\int_U H\right). \tag{2.25}$$

This expression depends on U, though this is not shown in the notation on the left hand side.

More subtle is the product of the Pfaffians. We recall that each fermion path integral $\mathrm{Pfaff}(\mathcal{D}_F(C_i))$ takes values in a complex line $\mathcal{L}_{C_i}$. However, according to a theorem of Dai and Freed [11], for every choice of a three-manifold U whose boundary is the union of the C_i (together with an extension of all of the bundles over U), there is a canonical trivialization of the product $\otimes_i\mathcal{L}_{C_i}$. This trivialization is obtained by suitably interpreting the quantity $\exp(i\pi\eta(U)/2)$, where $\eta(U)$ is an eta-invariant of a Dirac operator on U defined using global (Atiyah-Patodi-Singer) boundary conditions on the C_i. We write the trivialization

Two myths

Just in case...

Myth 1: Anomalies are only caused by fermionic fields

Myth 2: Anomalies are only associated to symmetries

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Just in case...

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Mythbuster 1: The flavor symmetry of QCD is anomalous—indeed, that anomaly involves fermions—but the anomaly persists in the effective theory of pions, which is a bosonic theory

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Myth 2: Anomalies are only associated to symmetries

Mythbuster 2: The theory of a free spinor field has an anomaly

Main thesis

Quantum theory is projective. Quantization is linear.

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QUANTUM THEORY IS PROJECTIVE. QUANTIZATION IS LINEAR.

The anomaly of a quantum theory expresses its projectivity

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The anomaly is a feature, not a bug ('t Hooft)

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The anomaly is a feature, not a bug ('t Hooft)

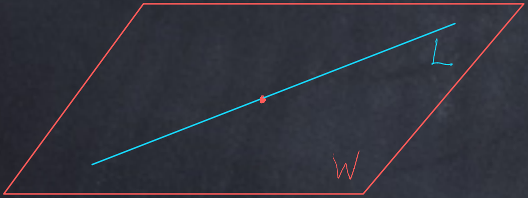
The anomaly is an obstruction only when quantizing

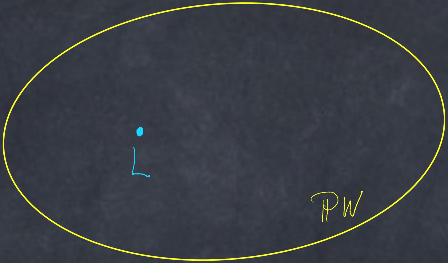
Outline

- Projective spaces, linearization, and symmetry
- Quantum mechanics as a projective system
- Quantum field theory as a projective system
- Invertible field theories
- Anomalies as an obstruction to quantization
- Anomaly of a spinor field

Projectivization of a linear space

W	(complex) vector space
$\mathbb{P}(W)$	projective space of lines $L \subset W$
$\mathrm{End}(W)$	algebra of linear maps $T \colon W \longrightarrow W$





Projectivization of a linear space

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If K is any line (1-dimensional vector space), then there are *canonical* isomorphisms

$$\begin{aligned} \mathbb{P}(W) &\longrightarrow \mathbb{P}(W \otimes K) \\ L &\longmapsto L \otimes K \end{aligned} \qquad\qquad \begin{aligned} \mathrm{End}(W) &\longrightarrow \mathrm{End}(W \otimes K) \\ T &\longmapsto T \otimes \mathrm{id}_K \end{aligned}$$

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A projective symmetry of $\mathbb{P}(W)$ has a $\mathbb{C}^\times$-torsor of lifts to a linear symmetry of W

Projective symmetries

$$\mathbb{C}^\times \longrightarrow \mathrm{GL} \longrightarrow \mathrm{PGL}$$

Short exact sequence of Lie groups

Projective symmetries

$$\begin{array}{ccccc} \mathbb{C}^\times & \longrightarrow & \mathrm{GL} & \longrightarrow & \mathrm{PGL} \\ & & & & \uparrow \\ & & & & G \end{array}$$

Short exact sequence of Lie groups

Lie group G of projective symmetries

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Pullback group extension; linear action of $\widetilde{G}$

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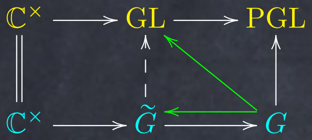

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Lift to linear symmetries $\longleftrightarrow$ splitting of group extension

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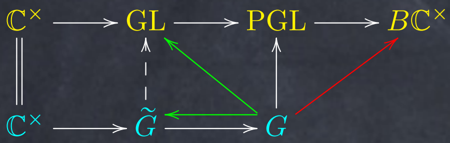

Short exact sequence of Lie groups

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Obstruction to lifting

Projective symmetries

$$\mathbb{C}^\times \longrightarrow \widetilde{G} \longrightarrow G \xrightarrow{\ \alpha\ } B\mathbb{C}^\times$$

$G \longrightarrow B\mathbb{C}^\times \quad \longleftrightarrow \quad$ group extension

Projective symmetries

$$
\begin{array}{ccccccc}
\mathbb{C}^\times & \longrightarrow & \mathrm{GL} & \longrightarrow & \mathrm{PGL} & \longrightarrow & B\mathbb{C}^\times \\
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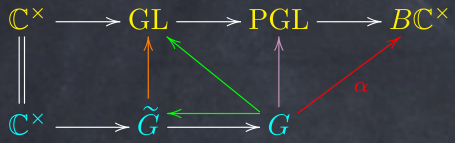
In QM one has analogs of the projective action

In QFT one has analogs of the anomaly α and the linear action

Projective symmetries



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In QFT one has analogs of the anomaly α and the linear action

The analog of the splitting is a linearization or trivialization of the anomaly α

Cohomological interpretation; splittings

$$G \xrightarrow{\alpha} B\mathbb{C}^\times$$

The projectivity has an equivalence class in $H^2(G;\mathbb{C}^\times)$ for some cohomology theory

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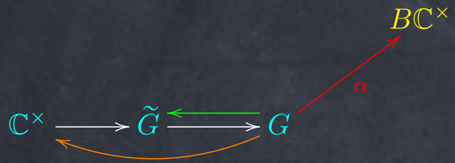
The extension is a "cocycle" for this cohomology class

Cohomological interpretation; splittings



The projectivity has an equivalence class in $H^2(G; \mathbb{C}^\times)$ for some cohomology theory

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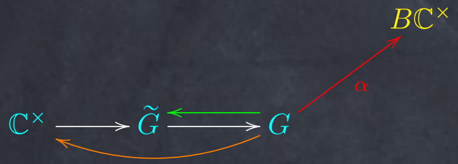
Splittings of the extension—trivializations of α—form a torsor over characters of G

Cohomological interpretation; splittings



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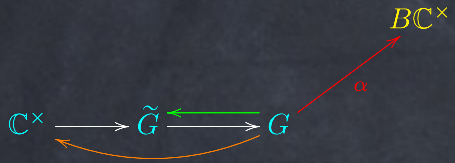

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Summary: Projectivity is a "suspended" *invertible* linear representation

What is a projective space?

Goal: Define a projective space $\mathbb{P}$ without committing to a linearization $\mathbb{P} \xrightarrow{\cong} \mathbb{P}(W)$

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An instance of that geometry is associated to a right H-torsor T by mixing: $X_T := T \times_H X$

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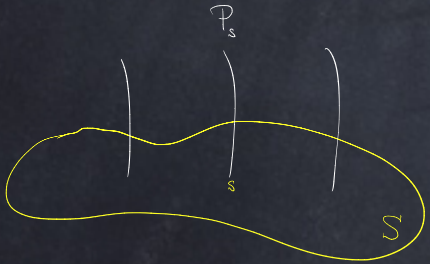

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Model geometries for complex projective space:

$\mathrm{PGL}_{n+1}\mathbb{C} \circlearrowright \mathbb{CP}^n$ (complex manifold)

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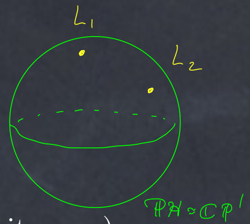
There are infinite dimensional analogs

Outline

- Projective spaces, linearization, and symmetry
- Quantum mechanics as a projective system
- Quantum field theory as a projective system
- Invertible field theories
- Anomalies as an obstruction to quantization
- Anomaly of a spinor field

Quantum mechanics as a linear system

$\dim \mathcal{H} = 2$



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$\mathbb{P}\mathcal{H}$	space of pure states
$H \in \mathrm{End}(\mathcal{H})$	Hamiltonian

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transition probability function ($\psi_i \in L_i$ unit norm)

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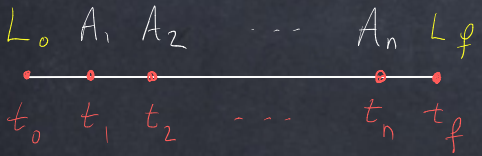
Probability: $p\left(L_f\ ,\ e^{-i(t_f - t_n)H/\hbar} A_n \cdots e^{-i(t_2 - t_1)H/\hbar} A_1 e^{-i(t_1 - t_0)H/\hbar} L_0\right) \in [0,1]$

$t_0 < t_1 < \cdots < t_n < t_f$ real numbers, $\quad A_1, \dots, A_n \in \mathrm{End}\,\mathcal{H}, \quad L_0,\ L_f \in \mathbb{P}\mathcal{H}$



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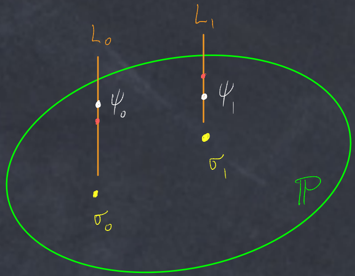
Probability: $p\left(L_f\,,\, e^{-i(t_f - t_n)H/\hbar} A_n \cdots e^{-i(t_2-t_1)H/\hbar} A_1 e^{-i(t_1-t_0)H/\hbar}\, L_0\right) \in [0,1]$

$t_0 < t_1 < \cdots < t_n < t_f$ real numbers, $A_1, \dots, A_n \in \mathrm{End}\,\mathcal{H}$, $L_0,\ L_f \in \mathbb{P}\mathcal{H}$

Amplitude: $\left\langle \psi_f\,,\, e^{-i(t_f - t_n)H/\hbar} A_n \cdots e^{-i(t_2-t_1)H/\hbar} A_1 e^{-i(t_1-t_0)H/\hbar}\, \psi_0 \right\rangle_{\mathcal{H}} \in \mathbb{C}$ if we choose vectors $\psi_0 \in L_0$, $\psi_f \in L_f$; as a function of L_0, L_f the amplitude lies in the hermitian line $(L_0 \otimes \overline{L_f})^*$; the probability is the norm square: $|\mathrm{Amplitude}|^2 = \mathrm{Probability}$

Quantum mechanics as a projective system

We only need a projective space, not a linear space:



$\mathbb{P}$ projective space

$\mathscr{A}_{\mathbb{P}}$ complex algebra

$H \in \mathrm{End}(\mathscr{A}_{\mathbb{P}})$ Hamiltonian

$$p \colon \mathbb{P} \times \mathbb{P} \longrightarrow [0,1]$$
$$\sigma_0\,,\ \sigma_1 \longmapsto |\langle \psi_0, \psi_1 \rangle|^2_{\mathcal{H}}$$

for any linearization $\mathbb{P} \xrightarrow{\cong} \mathbb{P}\mathcal{H}$

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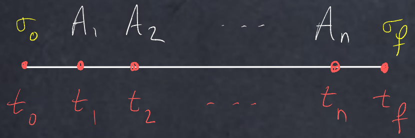

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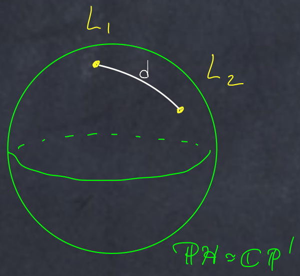
The symmetry/structure group of quantum mechanics

$\mathbb{P}$ — projective space

$p\colon \mathbb{P} \times \mathbb{P} \longrightarrow [0,1]$ — transition probability function

Fix a linearization $\mathbb{P} \xrightarrow{\cong} \mathbb{P}\mathcal{H}$; then the group $\mathrm{Aut}(\mathbb{P}, p)$ of maps $\mathbb{P} \longrightarrow \mathbb{P}$ preserving p is the isometry group of the Fubini-Study metric $d\colon \mathbb{P}\mathcal{H} \times \mathbb{P}\mathcal{H} \longrightarrow \mathbb{R}^{\geqslant 0}$ $\qquad \cos(d) = 2p - 1$



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Example: $\dim\mathcal{H}=2$, $\mathbb{P}=\mathbb{CP}^1\approx S^2$ (round metric), $\mathrm{Aut}(\mathbb{P},p)=\mathrm{O}_3$

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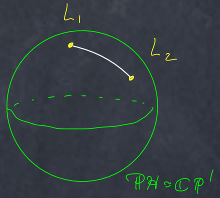

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Therefore, $\mathrm{P}\mathcal{Q}_n \circlearrowright \mathbb{CP}^n$ or $\mathrm{P}\mathcal{Q}_\infty \circlearrowright \mathbb{CP}^\infty$ is the model geometry for QM

Linearization and anomalies

$$\mathbb{T} \longrightarrow \mathrm{Q} \longrightarrow \mathrm{PQ} \xrightarrow{\alpha} \widetilde{B\mathbb{T}}$$

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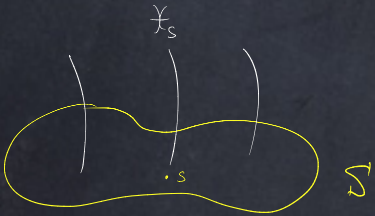

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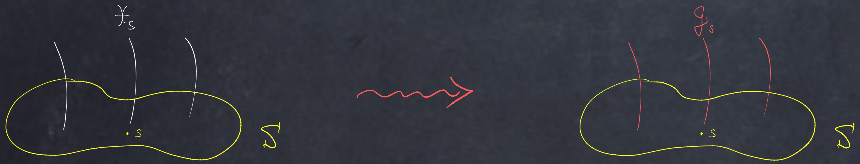

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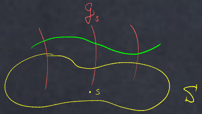

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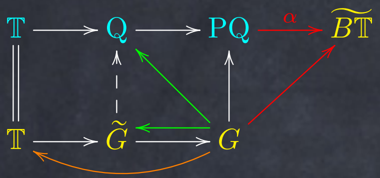

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For $S = *//G$ (single QM system with G-symmetry), reduce to group extension discussion

Outline

- Projective spaces, linearization, and symmetry
- Quantum mechanics as a projective system
- Quantum field theory as a projective system
- Invertible field theories
- Anomalies as an obstruction to quantization
- Anomaly of a spinor field

Wick-rotated QFT as a linear representation

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Man_n	category of smooth n-manifolds and local diffeomorphisms
sSet	category of simplicial sets

Definition: A *Wick-rotated field* is a sheaf

$$\mathcal{F}\colon \mathrm{Man}_n^{\mathrm{op}} \longrightarrow \mathrm{sSet}$$

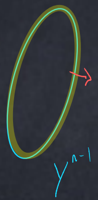
Examples: Riemannian metrics, G-connections, $\mathbb{R}$-valued functions, M-valued functions, orientations, spin structures, gerbes, ...

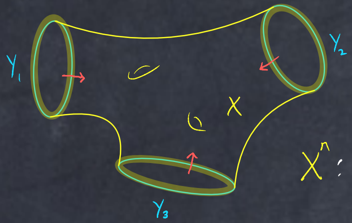
$\mathcal{F}$ can be a *collection* of fields; $\mathcal{F}(M)$ is the simplicial set of fields on an n-manifold M

Axiom System: $\mathrm{Bord}_n(\mathcal{F})$ bordism category

n dimension of spacetime

$\mathcal{F}$ background fields (orientation, Riemannian metric, ...)





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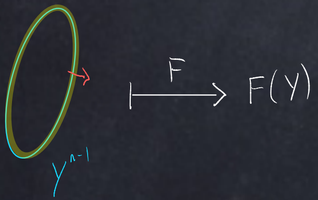

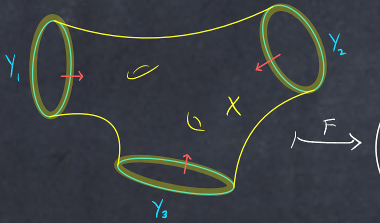

$X^n : Y_1 \sqcup Y_2 \sqcup Y_3 \to \emptyset^{n-1}$

$\stackrel{F}{\longmapsto} \left(F(X) : F(Y_1) \otimes F(Y_2) \otimes F(Y_3) \longrightarrow \mathbb{C} \right)$

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Kontsevich-Segal: recent paper with these axioms for *nontopological* theories
geometric form of Wick rotation via admissible complex metrics
theorem constructing theory on globally hyperbolic Lorentz manifolds

Wick-rotated QFT as a projective representation; the anomaly

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Projective theory $\overline{F}$

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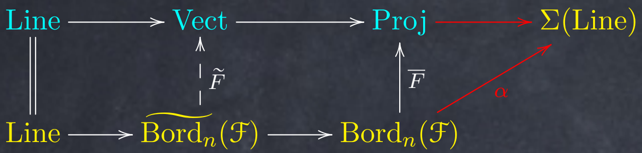

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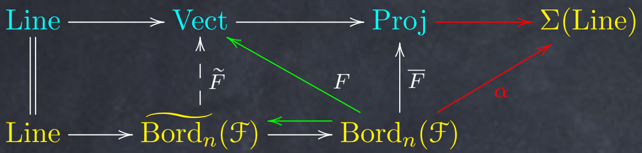

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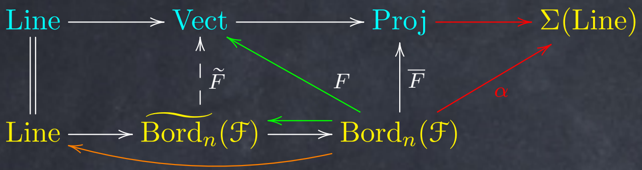

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Ratio of trivializations: an invertible n-dimensional theory

Segal: 1980s paper on 2d conformal field theory

$$\begin{array}{ccccc} \text{Line} & \longrightarrow & \text{Vect} & \longrightarrow & \text{Proj} \\ \Vert & & \uparrow \tilde{F} & & \uparrow \overline{F} \\ \text{Line} & \longrightarrow & \widetilde{\text{Bord}}_n(\mathcal{F}) & \longrightarrow & \text{Bord}_n(\mathcal{F}) \end{array}$$

For any modular functor E we have a map $E(X) \otimes E(Y) \to E(X \circ Y)$ when X and Y are composable morphisms in $\mathcal{C}$ with their boundaries compatibly labelled. So E defines an extension $\mathcal{C}^E$ of the category $\mathcal{C}$. An object of $\mathcal{C}^E$ is a collection of circles each with a label from Φ, and a morphism is a pair (X, ϵ), where X is an morphism in $\mathcal{C}$ and $\epsilon \in E(X)$.

Definition (5.2). A weakly conformal field theory is a representation of $\mathcal{C}^E$ for some modular functor E, satisfying conditions as in (4.4).

Anomaly as an invertible field theory

$\Sigma(\mathrm{Line})$

$\Sigma(\mathrm{Line})$ is a groupoid of gerbes, a categorification of Line

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$$\mathrm{Bord}_n(\mathcal{F}) \xrightarrow{\ \alpha\ } \Sigma(\mathrm{Line})$$

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The anomaly theory α is a *once-categorified* n-dimensional invertible field theory

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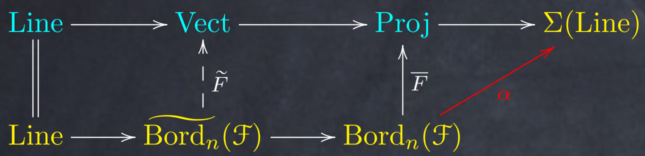

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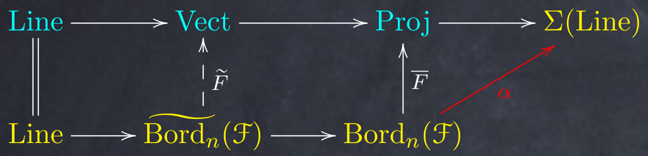
An n-dimensional theory $\overline{F}$ *relative* to α assigns $\overline{F}(X^n)\colon \mathbb{C} \longrightarrow \alpha(X^n)$ for X^n closed

(Note: *Relative* field theories are called *twisted* theories by Stolz-Teichner)

Anomaly as an invertible field theory



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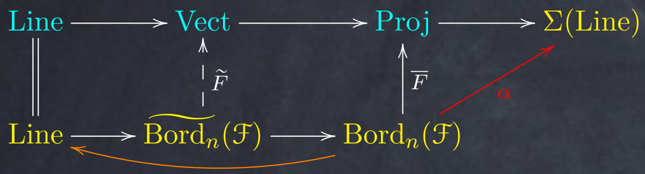

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Ratios of trivializations of α: a standard type of n-dimensional invertible theory

Extension of anomaly theory; relative theory $\longrightarrow$ boundary theory

$$
\begin{array}{ccc}
 & & \Sigma(\mathrm{Line}) \\
 & \overset{\alpha}{\nearrow} & \uparrow{\scriptstyle \tilde{\alpha}} \\
\mathrm{Bord}_n(\mathcal{F}) & \hookrightarrow & \mathrm{Bord}_{n+1}(\widetilde{\mathcal{F}})
\end{array}
$$

In many cases the once-categorified n-dimensional anomaly theory α has an extension to an $(n+1)$-dimensional theory $\tilde{\alpha}$

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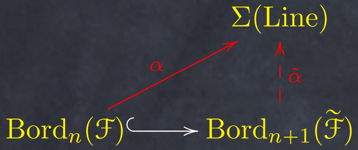

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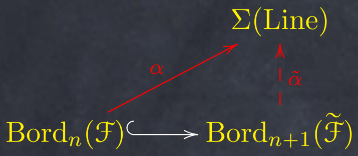
Extension of anomaly theory; relative theory $\longrightarrow$ boundary theory



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Anomaly theories $\alpha, \tilde{\alpha}$ are not in general topological; if so, topological tools are available

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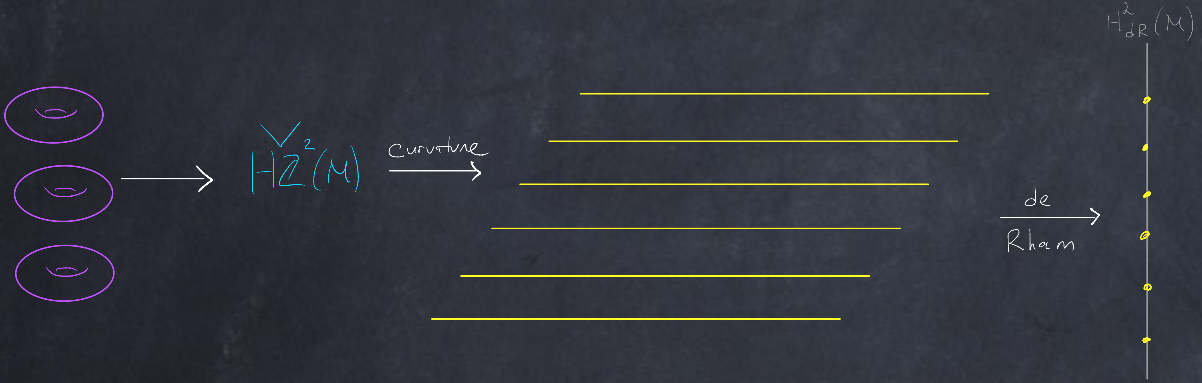

Invertible field theories

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The deformation class is accessible via homotopical methods

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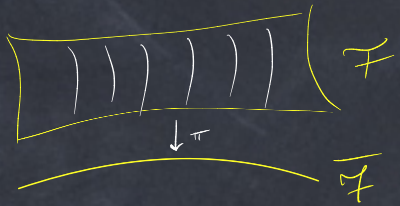

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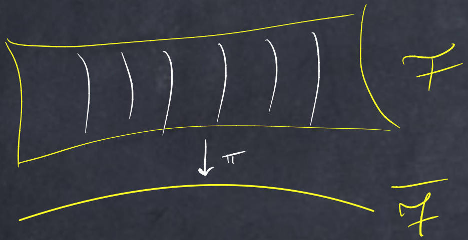

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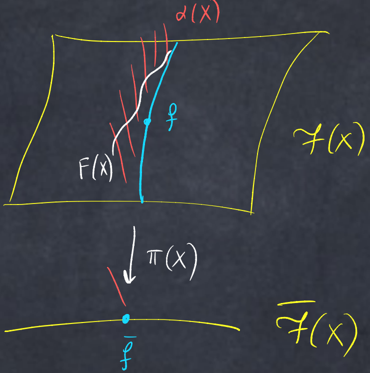

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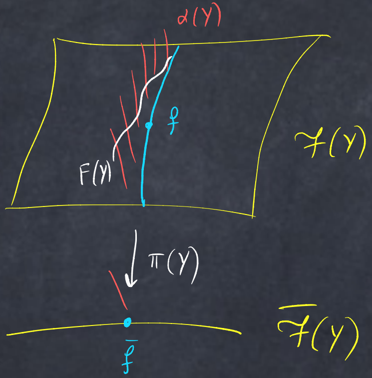

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To carry out quantization we must *descend* the projectivity/anomaly α:

$$\begin{array}{ccc} \mathrm{Bord}_n(\mathcal{F}) & \xrightarrow{\ \alpha\ } & \\ \big\downarrow & & \Sigma^{n+1} I\mathbb{C}^\times \\ \mathrm{Bord}_n(\overline{\mathcal{F}}) & \xrightarrow{\ \bar{\alpha}\ } & \end{array}$$

anomaly is obstruction to existence

descents form a torsor over n-dimensional theories

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- The anomaly of a QFT is itself a field theory, so obeys locality and, typically, unitarity

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$\mathbb{M}^n$	Minkowski spacetime (affine space, Lorentz metric)
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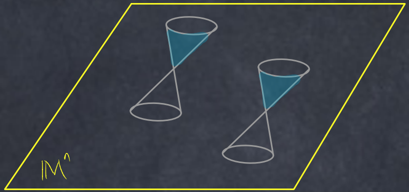

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- Let $M(\mathbb{S})$ denote the vector space of mass pairings. (It may be the zero vector space.) We can take $\mathcal{F} = \mathbf{Riem} \times \mathbf{Spin} \times M(\mathbb{S})$ and deduce the anomaly; see `arXiv:1905.09315` with Córdova-Lam-Seiberg

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