Is Relativity Compatible with Quantum Theory?

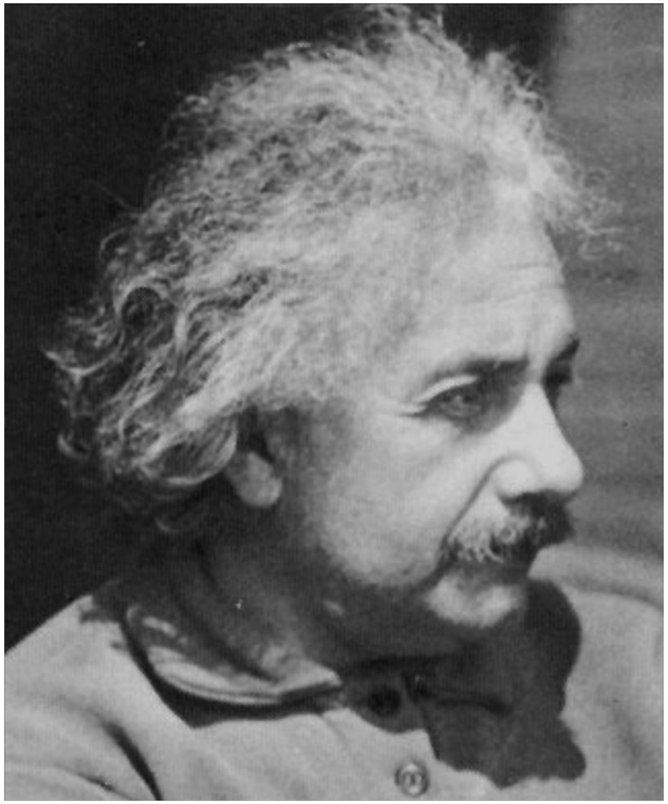
Arthur Jaffe
Harvard University

Literature in the Mathematical Sciences
Harvard CMSA – Tsinghua YMSC
December 2, 2020

Outline

1. Problem and Results
   a. Physics problem: special relativity + quantum theory
   b. Proposal
   c. Mathematical obstructions
   d. Potential methods
   e. Existence theorems
   f. Properties of solutions
2. Brief Remarks about Methods
   a. Classical-quantum duality
   b. Perron-Frobenius analysis in infinite dimensions
   c. Phase cell localization and Independence
   d. Reflection positivity and Multiple reflection bounds
3. What’s Next?

Major pedestals of 20th century physics



$$E = mc^2$$

Special Relativity 1905

$$i\hbar \frac{d\psi}{dt} = H\psi$$

Quantum Theory 1926

Isolate:

Make: Physics = Mathematics

240-year history: 1687-1926 Newton to Schrödinger

- Newton: Gravitational Physics Calculus
- Maxwell: Electromagnetism Symmetry
- Gibbs/Boltzmann: Statistical Physics Probability
- Einstein: Relativity Algebra, Geometry
- Schrödinger: Quantum Physics Analysis, PDE

- Wigner 1960: "On the Unreasonable Effectiveness.."

- SO WHAT ABOUT Relativity + Quantum Theory?

Hilbert's 6th Problem: Axiomatize Physics

- Formulated by Hilbert in 1900 at Paris ICM
- Einstein: "But the creative principle resides in mathematics…."
- Wightman: "A great physical theory is not mature until it has been put in precise mathematical form."

Dirac Equation P.A.M. Dirac, Proc. Roy. Soc. A (1928)

"Relativistic wave equation" for the electron wave function ψ, given Coulomb potential *A(x)*.

$$\left(\sum_\mu \gamma^\mu \left(i\hbar \frac{\partial}{\partial x^\mu} - eA_\mu(x) \right) - mc \right) \psi(x) = 0$$

This explained hydrogen-atom energy levels

–almost!

It omits interaction that would come if ψ and *A* satisfy coupled Dirac + Maxwell equations.

Interaction = QFT

Quantum Field Theory

Born, Heisenberg, Jordan, Dirac (1925-27)

Forces between particles (or between particles and fields) arise from equations for the fields.

A particle is a state of a field (associated with irreducible representation of the special relativity group).

Representations of Poincaré group

Wigner (following earlier work of Majorana) classified the positive-energy, irreducible representations of the semi-direct product $SL(2,C) \rtimes R^4$ of the covering group $SL(2,C)$ of the Lorentz group $SO(3,1)$ with translations on R^4.

Parameters of a particle emerge:
mass $m \geq 0$, half-integer spin s.

E. Majorana Nuovo Cimento (1932)
E. Wigner Annals of Mathematics (1939)
V. Bargmann and E. Wigner PNAS (1948)

Wigner 1964

An Ultimate Quantitative Test of QFT

Magnetic moment μ of an electron.

$$\mu = \kappa \frac{e\hbar}{mc} , \qquad \kappa_{\mathrm{Bohr}} = \frac{1}{2} , \qquad \kappa_{\mathrm{Dirac}} = 1 .$$

1926
non-relativistic

1928
relativistic

$$\kappa = 1.001 \quad \mathrm{Kusch}\ (1947)$$

70 years of measurements later:

$$\kappa = 1.00115965218073(\pm 28).$$

(2008)

part in 10^{12}, Gerald Gabrielse

A.I.P. “Physics Achievement of the Year: 2006” Improved 2008

Field Revolution: Ultimate Quantitative Test

Magnetic moment μ of an electron.

$$\kappa = 1.00115965218073(\pm 28).$$

Amazing: These QED (Quantum Electrodynamics) experiments agree completely with the Feynman rules for calculation in perturbation theory, also carried out over 70 years!!

Many physics Nobel laureates related to this question.

(Central to physics)

Bohr, Heisenberg, Schrödinger, Dirac, Pauli, Tomonaga, Feynman, Schwinger, Kusch, Lamb, Wigner, Ramsey, Dehmelt, Paul, Josephson, Gell Mann, Wilson, Glashow, Weinberg, Salam, ‘t Hooft, Veltman, Gross, Politzer, Wilczek, …..

The Big Mathematical Question

- As the "Feynman rules" for quantum electrodynamics agree with the most accurately measured phenomena in nature,
- And as the putative quantum field explanation of the measurements embodies special-relativity and quantum theory,
- Our belief in logical science mandates that we determine whether (or not) relativistic QFT makes mathematical sense.
- BIG Question: QED, QCD, YM, or Standard Model

What Do We Know?

d = dimension of space-time

- **Goal: Understand how to distinguish what we know, from what we believe we know.**
- **✓ d=2**
- **✓ d=3**
- **? d=4**

  d>4

I now describe some results.

Focus on Early Work; Tentacles to Today

Apologies in advance for important work I forget to mention

Formulate as Mathematics; 3 methods (+30 years)

I. Axioms for Fields A.S.Wightman (Phys.Rev.1956)

- Quantum theory (Hilbert space of vector states)
- Field $\varphi(x)$ is an operator-valued distribution
- Positive energy H and unique ground state vector Ω
- Covariance under unitary representation of $SL(2,C) \rtimes R^4$
- Locality $[\varphi(x), \varphi(y)]_\pm = 0\ ,\ \text{for } x - y \text{ spacelike}$
- Some technical regularity assumptions
- Output: Spin/Statistics Theorem and the PCT Theorem
  R.Streater-A.Wightman book (Benjamin 1964), R.Jost book (AMS 1965)

II. Axioms for Algebraic Quantum Theory:

R. Haag, D. Kastler, J.Math.Phys.(1964); R.Haag book (Springer 1992)

- Study bounded functions of the field and the von Neumann algebras that they generate.

Formulate as Mathematics (+30 years)

III. Axioms for Scattering Theory: (assume isolated single particles in spectrum) H.Lehmann, K.Symanzik, W.Zimmermann (Nuovo Cimento, 1955-1959)

R.Haag (Phys. Rev., 1958) and D.Ruelle (Helv. Phys. Acta, 1962)

H.Araki, K.Hepp, D.Ruelle (Helv. Phys. Acta, 1962)

Problem: NO INTERESTING MATHEMATICAL EXAMPLES

In 1950’s this goal appeared beyond mathematical reach.

Wightman Seminar 1965

4 Natural Methods: 2 Seem Promising

- ✗ Algebraic Solutions or Inverse Scattering
  - A few known examples (W.Thirring, J.Schwinger,…)
  - All have uninteresting scattering.
  - No indication physical theories are integrable
- ✗ Representations of $[q_k,p_j] = i\delta_{kj}$ (CCR) (A.Wightman, J.Lew) Too many inequivalent representations. (no Lagrangian connection)
- ✓ Hilbert Space & Specific Equations
  - Analysis of operators, and PDE A. Wightman, I. Segal
- ✓ Path Integrals K.Symanzik, J. Schwinger
  - Schrödinger → Diffusion Probability Theory, analysis of measures

Focus here on the Simplest Case, d=2

Free Field: (Klein Gordon equation): particles without interaction

$$\varphi_{tt}(x,t) - \varphi_{xx}(x,t) + \varphi(x,t) = 0$$

Canonical time-zero (CCR): $[q_k, p_j] = i\delta_{kj}$. Generalize to

$$[\varphi(x,0), \varphi_t(y,0)] = i\delta(x-y)$$

Well-Known Solution for linear equation by Fourier transformation. Particle interpretation arises from Fourier analysis of the field.

For interaction, need non-linearity. Introduce simplest non-linearity: Perturbation with parameter $0 \leqslant \lambda$ (to give positive energy).

$$\varphi_{tt}(x,t) - \varphi_{xx}(x,t) + \varphi(x,t) + \lambda\varphi^3(x,t) = 0$$

Clearly the non-linear term will be singular.

φ^4 Equation, $d = 2$

$$\varphi_{tt}(x,t) - \varphi_{xx}(x,t) + \varphi(x,t) + \lambda\varphi^3(x,t) = 0$$

Classical Hamiltonian: $H = \int_{t=0} \left(\frac{1}{2}\varphi_t^2 + \frac{1}{2}\varphi_x^2 + \frac{1}{2}\varphi^2 + \frac{\lambda}{4}\varphi^4 \right) dx$

$$= H_0 + \frac{\lambda}{4}\int_{t=0} \varphi^4 dx = H_0 + H_I$$

Quantum Hamiltonian: $H = H_0 + \frac{\lambda}{4}\int_{t=0} :\varphi^4: dx = H_0 + H_I$

"Normal ordering" $\quad :\varphi^4:(x,0) = \varphi^4(x,0) - 6\,c\,\varphi^2(x,0) + 3\,c^2$

$$= (\varphi^2(x,0) - 3\,c)^2 - 6\,c^2$$

Hermite polynomial. $\langle \Omega_0, :\varphi^4: \Omega_0 \rangle = 0$, $\quad \Omega_0$ ground state of H_0

Lower bound on quantum perturbation H_I: $-\frac{3\lambda}{2}|\text{Volume}|\, c^2$

Where does the constant c come from?

1. Coulomb singularity in Yukawa Green's function:

$$C(x-y) = (-\Delta + m^2)^{-1}(x,y) \sim \begin{cases} 1\,, & \text{if } d=1 \\ \ln|x-y|\,, & \text{if } d=2 \\ \frac{1}{|x-y|^{d-2}}\,, & \text{if } d \geqslant 3 \end{cases}\,, \text{ as } |x-y| \to 0$$

Lowest orders of perturbation theory in λ for the equation involves both

$$c = C(0) = \begin{cases} O(1)\,, & \text{if } d=1 \\ \infty\,, & \text{otherwise} \end{cases}\,, \quad \text{and} \quad \|C\|_{\mathbf{L}^4} = \begin{cases} O(1)\,, & \text{for } d=1,2 \\ \infty\,, & \text{otherwise} \end{cases}\,.$$

2. Redefining φ^4 as $:\varphi^4:$ is sufficient "renormalization" for the d=2 equation. Other renormalizations are needed with Dirac equation in d=2, or with d > 2.

Some Early Results: Hilbert Space Methods

- A.J. Theory on T^s with mollification Thesis (1965)
  - H_I self-adjoint operator Jour. Math. Phys. (1966)
    - H_I unbounded above and below; could H_0 stabilize?
- **E.Nelson Endicott House Meeting Proceedings (1966)

Theorem: For φ^4 Hamiltonian on $S^1 \times \mathbb{R}$, there exists a constant $M = M(|S^1|) < \infty$, such that

$$0 \leqslant H + M = H_0 + H_I + M$$

1965
Nelson
Epstein
Wightman

2013
Nelson
A.J.
Dyson

More Early Results: Hilbert Space Methods

- **J.Glimm and A.J.

Physical Review (1968)

Theorem: For Hamiltonian $H_L = H_0 + \lambda \int_{-L}^{L} : \varphi^4 : dx$, with $0 \leqslant L < \infty$,

$$H_L^* = H_L^{**}$$

Enables unitary evolution $e^{itH_L^{**}}$ and field in terms of initial values $\varphi(x,t) = e^{itH_L^{**}}\,\varphi(x,0)\,e^{-itH_L^{**}}$. This is the *local* Hamiltonian solution.

Annals of Mathematics (1970)

Theorem: For $f = \bar{f} \in C_0^\infty(\mathbb{R}^2)$ and $\varphi(f) = \int e^{itH_L}\varphi(x,0)e^{-itH_L} f(x,t)dxdt$,

$$\varphi(f)^* = \varphi(f)^{**}$$

For L sufficiently large, $\varphi(f)^{**}$ is independent of L. Also

$[\varphi(f)^{**}, \varphi(g)^{**}] = 0$, for f and g with space-like separated supports.

Theorem: The Hamiltonian H_L has a unique ground eigenstate Ω_L.

- J.Cannon and A.J.

Comm. Math. Phys. (1970)

Theorem: The Haag-Kastler axioms of algebraic QFT hold.

More Early Results: Hilbert Space Methods

- J.Glimm and A.J.

Acta Mathematica (1971)

Theorem: The vacuum states ω_L defined by the ground state eigenvectors Ω_L of H_L

$$\omega_L(\,\cdot\,) = \langle \Omega_L,\ \cdot\ \Omega_L \rangle\ ,$$

have a weakly convergent subsequence to a vacuum state ω as $L \to \infty$.

Theorem: Representation of time zero fields locally, unitarily equivalent to Fock representation.

Theorem: There exists $0 < M < \infty$ such that

$$-ML \leqslant H_L = H_0 + \lambda \int_{-L}^{L} {:}\varphi^4{:} dx$$

Jour. Math. Phys. (1972)

Cargèse Summer School 1979

Kurt Symanzik's 1968 Varenna lectures
His Evaluation of Methods

Euclidean Quantum Field Theory (*).

K. SYMANZIK (**)

New York University, Courant Institut of Mathematical Sciences - New York, N.Y.

Introduction.

Euclidean quantum field theory (EQFT) is ordinary relativistic local quantum field theory (QFT) *e.g.* quantum electrodynamics (QED), gA^4 theory as in Jaffe's lecture, etc., looked at in a particular way. Namely, the objects considered are not the ones, *e.g.* Wightman's functions (W-functions), the Hamiltonian, scattering amplitudes etc. one would like to learn about (an approach concentrating on these objects we call for brevity Minkowski quantum field theory (MQFT)), but the analytic W-functions restricted to Schwinger points, which are the ones with imaginary time, real space components. We shall find that these functions have, compared to W-functions at real points, relatively simple properties. These simplifications have been known since long and often been exploited, *e.g.* in renormalization theory and in discussions of solutions of Bethe-Salpeter equations. Specifically, when one treats a Lagrangian QFT in perturbation theory, the Feynman integrals one encounters are known to have to be defined by certain limiting processes (*e.g.* the $i\varepsilon$-technique), and only then can they be recognized as being convergent or divergent. This complication is lastly caused by the indefinite Minkowski space metric. In contrast, the integrals for the W-functions restricted to Schwinger points, called Schwinger functions (S-functions) or Euclidean Green's functions, are unambiguously either absolutely convergent or absolutely divergent. Also, as we shall see, the S-functions are singular on much smaller manifolds than W-functions (at real points).

I assume that you share by now the opinion of some lecturers that it would be desirable to have examples of QFT's. The only way known to construct such examples obeying the familiar axioms is to start from a local-invariant

(*) Supported in part by the National Science Foundation, NSF-GP 7434.
(**) Present address: DESY, Hamburg 52.

Lagrangian density for covariant fields. The feature exploited in EQFT is that the field equations and canonical commutation relations (CCR) imply differential equations of elliptic type for S-functions while the differential equations for W-functions or, more usual, Feynman amplitudes, are of hyperbolic type. For elliptic differential equations certain powerful analytic tools are available that have no (mathematically meaningful) counterpart for hyperbolic differential equations.

On the other hand, EQFT has the basic disadvantage of dealing with not directly physically interpretable quantities. However, the existence of S-functions with properties to be given later for a specific Lagrangian QFT is a necessary condition for the existence of an MQFT to that Lagrangian. Thus, EQFT, provided its tools are sufficiently sharpened, may be an effective test for the admissibility of certain Lagrangians. In particular, the famous renormalization constants, as far as they are not rigidly tied to the mass spectrum or other

GLIMM-JAFFE	EQFT
A: works directly with physically relevant quantities (Hamiltonian, field operators, state space).	D: existence of S-functions only necessary for existence of (MQF) theory. W-functions are only analytic continuations of S-functions.
D: analysis proceeds through noncovariant steps, such that relativistic invariance must be proved separately. The vacuum and objects related to it, as well as divergences occurring in the noncovariant approach through canonical variables, present a nontrivial problem.	A: covariance is preserved throughout. The vacuum presents no problem, and only the divergences familiar from covariant renormalization theory occur.
A: concepts employed (of linear analysis) are relatively familiar. Many established techniques are available, and many results have been obtained.	D: concepts (belonging to probability theory), are relatively unfamiliar. Few established techniques are available, and few results have (as yet) been obtained.
A: Higher-spin theories, *e.g.* QED of spin-$\frac{1}{2}$ particles, offer little new complication compared with scalar theories.	D: Most theories involving particles with spin, *e.g.* QED of spin-$\frac{1}{2}$ particles or four-fermion coupling, offer considerable complications compared with scalar theories or spin-0 QED.

(A: advantage; D: disadvantage).

Common features: Locality simplifies the analysis. Merely renormalizable theory (*e.g.* gA^4 theory in 4 dimensions) presents much greater difficulties than a superrenormalizable theory.

1966 lunch after Courant Institute seminar
W.Zimmermann, J.Lascoux, K.Symanzik
?, J.Perkus?, G.Brown, D.Zwanziker, behind

Euclidean Revolution

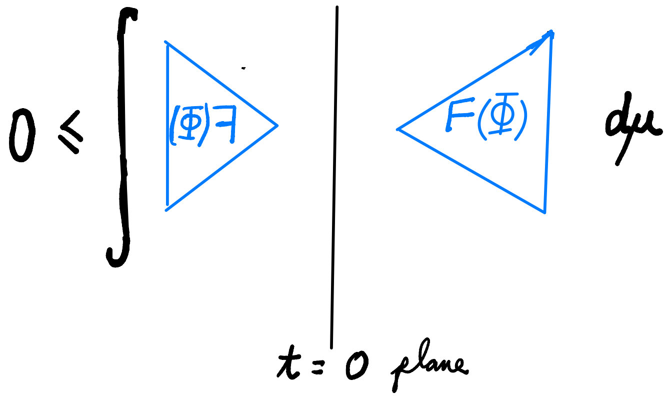
- In spite of Symanzik's negative assessment, E.Nelson persisted to work with his method. Four years later he showed that a random field, with a global Markov property, gives a QFT.
  Jour. Funct. Anal. (1973)
- F.Guerra showed that Nelson's method displayed an extraordinarily helpful symmetry.
  Phys. Rev. Lett. (1972)
- Problems: Need global Markov property. Only scalar fields; no fermions (e.g. Yukawa interaction.)
- Other Methods: F.Guerra, L.Rosen, and B.Simon used correlation methods to study Euclidean infinite volume limits.

Amazing Elementary Discovery: RP

- I studied Nelson's paper in 1972 with my postdoctoral fellows Konrad Osterwalder & Robert Schrader. I hoped for a more robust method. This led to their discovery of the "reflection-positivity (RP) property."
- This began the IVth (Euclidean) theme for QFT axioms.

K.Osterwalder and R.Schrader: Comm. Math. Phys. (1973), (1975)

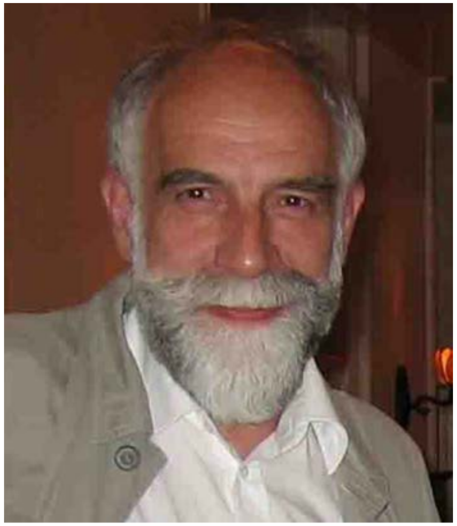
Theorem: Assume expectations are (i) Euclidean covariant, (ii) reflection positive, and (iii) obey some analytic regularity conditions. This is **equivalent** to the Wightman axioms, supplemented by additional analytic regularity conditions, but without uniqueness of the vacuum. If correlations decay, then the vacuum is unique.



$$d\mu = \frac{1}{Z_I} e^{-\mathfrak{A}_I(\Phi)} d\mu_0(\Phi)$$

K.Osterwalder and R. Schrader in CT, Sept. 11, 1992

Berlin, Feb. 2009



Jürg Fröhlich, Zurich, June 2006

Reflection Positivity

- RP solved many problems:
  - RP miracle: it gives the correct inner product for every quantum theory.
  - Markov properties unnecessary. Many different fields included.
  - RP automatically assures Hilbert space, positive *H*, and vacuum Ω.
  - RP gives reflection inequality in quantum theory, useful for many things.
  - Ultimately RP became important in gauge theory, in statistical physics, in representation theory, and in other subjects including planar algebras.

Party 1976 at the Newton, MA home of Konrad and Vreni Osterwalder on their departure from Harvard for ETH.
Left to right, front row first:
K.Osterwalder, E.Seiler, A.J., J.Fröhlich
R.Sénéor, V.Osterwalder, M.Sénéor
C.Sénéor, J.Feldman, M.Seiler, E.Fröhlich

Back to Seeking a Full Theory

Tom Spencer: PhD Courant 1972; Postdoctoral Fellow Harvard 1974-5

A Highpoint

Finally, one has the first relativistic QFT with non-trivial scattering.

Theorem: (Wightman Theory.) [**J.Glimm, A.J., and T.Spencer, Annals of Mathematics (1974)**] There exists $\lambda_{\max} > 0$, such that for all $\lambda \in [0, \lambda_{\max}]$, the $\lambda\varphi^4$ theory exists and satisfies the Wightman axioms (in strengthened form) and the Haag-Ruelle axioms for scattering theory. This includes:

A Highpoint

Finally, one has the first relativistic QFT with non-trivial scattering.

Theorem: (Wightman Theory.) [**J.Glimm, A.J., and T.Spencer, Annals of Mathematics (1974)**] There exists $\lambda_{\max} > 0$, such that for all $\lambda \in [0, \lambda_{\max}]$, the $\lambda\varphi^4$ theory exists and satisfies the Wightman axioms (in strengthened form) and the Haag-Ruelle axioms for scattering theory. This includes:

1. The vacuum is unique: The Hamiltonian $H = H^* \geqslant 0$ and momentum $P = P^*$ have a unique vector Ω (up to a multiple) for which $H\Omega = P\Omega = 0$.

A Highpoint

Finally, one has the first relativistic QFT with non-trivial scattering.

Theorem: (Wightman Theory.) [**J.Glimm, A.J., and T.Spencer, Annals of Mathematics (1974)**] There exists $\lambda_{\max} > 0$, such that for all $\lambda \in [0, \lambda_{\max}]$, the $\lambda\varphi^4$ theory exists and satisfies the Wightman axioms (in strengthened form) and the Haag-Ruelle axioms for scattering theory. This includes:

1. The vacuum is unique: The Hamiltonian $H = H^* \geqslant 0$ and momentum $P = P^*$ have a unique vector Ω (up to a multiple) for which $H\Omega = P\Omega = 0$.

2. The Hamiltonian has a lower and and upper mass gap: The mass operator $M = \sqrt{H^2 - P^2}$ has an isolated eigenvalue $m > 0$ corresponding to single particle states.

A Highpoint

Finally, one has the first relativistic QFT with non-trivial scattering.

Theorem: (Wightman Theory.) [**J.Glimm, A.J., and T.Spencer, Annals of Mathematics (1974)**] There exists $\lambda_{\max} > 0$, such that for all $\lambda \in [0, \lambda_{\max}]$, the $\lambda\varphi^4$ theory exists and satisfies the Wightman axioms (in strengthened form) and the Haag-Ruelle axioms for scattering theory. This includes:

1. The vacuum is unique: The Hamiltonian $H = H^* \geqslant 0$ and momentum $P = P^*$ have a unique vector Ω (up to a multiple) for which $H\Omega = P\Omega = 0$.
2. The Hamiltonian has a lower and and upper mass gap: The mass operator $M = \sqrt{H^2 - P^2}$ has an isolated eigenvalue $m > 0$ corresponding to single particle states.
3. The field is well-behaved: For real functions $f \in C_0^\infty(\mathbb{R}^2)$, the fields $\varphi(f)$ have a common, dense domain $\mathcal{D}$ on which their closures are self adjoint, $\varphi(f)^* = \varphi(f)^{**}$.

A Highpoint

Finally, one has the first relativistic QFT with non-trivial scattering.

Theorem: (Wightman Theory.) [**J.Glimm, A.J., and T.Spencer, Annals of Mathematics (1974)**] There exists $\lambda_{\max} > 0$, such that for all $\lambda \in [0, \lambda_{\max}]$, the $\lambda\varphi^4$ theory exists and satisfies the Wightman axioms (in strengthened form) and the Haag-Ruelle axioms for scattering theory. This includes:

1. The vacuum is unique: The Hamiltonian $H = H^* \geqslant 0$ and momentum $P = P^*$ have a unique vector Ω (up to a multiple) for which $H\Omega = P\Omega = 0$.

2. The Hamiltonian has a lower and and upper mass gap: The mass operator $M = \sqrt{H^2 - P^2}$ has an isolated eigenvalue $m > 0$ corresponding to single particle states.

3. The field is well-behaved: For real functions $f \in C_0^\infty(\mathbb{R}^2)$, the fields $\varphi(f)$ have a common, dense domain $\mathcal{D}$ on which their closures are self adjoint, $\varphi(f)^* = \varphi(f)^{**}$.

4. The field is covariant: There is a positive-energy representation $U(g)$ of the Poincaré group on the Hilbert space of the theory, and the field $\varphi(x)$ transforms covariantly, $U(g)\varphi(x)U(g^{-1}) = \varphi(gx)$.

A Highpoint

Finally, one has the first relativistic QFT with non-trivial scattering.

Theorem: (Wightman Theory.) [**J.Glimm, A.J., and T.Spencer, Annals of Mathematics (1974)**] There exists $\lambda_{\max} > 0$, such that for all $\lambda \in [0, \lambda_{\max}]$, the $\lambda\varphi^4$ theory exists and satisfies the Wightman axioms (in strengthened form) and the Haag-Ruelle axioms for scattering theory. This includes:

1. The vacuum is unique: The Hamiltonian $H = H^* \geqslant 0$ and momentum $P = P^*$ have a unique vector Ω (up to a multiple) for which $H\Omega = P\Omega = 0$.
2. The Hamiltonian has a lower and and upper mass gap: The mass operator $M = \sqrt{H^2 - P^2}$ has an isolated eigenvalue $m > 0$ corresponding to single particle states.
3. The field is well-behaved: For real functions $f \in C_0^\infty(\mathbb{R}^2)$, the fields $\varphi(f)$ have a common, dense domain $\mathcal{D}$ on which their closures are self adjoint, $\varphi(f)^* = \varphi(f)^{**}$.
4. The field is covariant: There is a positive-energy representation $U(g)$ of the Poincaré group on the Hilbert space of the theory, and the field $\varphi(x)$ transforms covariantly, $U(g)\varphi(x)U(g^{-1}) = \varphi(gx)$.
5. The field is local: If f and g have space-like separated supports, $[\varphi(f)^{**}, \varphi(g)^{**}] = 0$.

A Highpoint

Finally, one has the first relativistic QFT with non-trivial scattering.

Theorem: (Wightman Theory.) [**J.Glimm, A.J., and T.Spencer, Annals of Mathematics (1974)**] There exists $\lambda_{\max} > 0$, such that for all $\lambda \in [0, \lambda_{\max}]$, the $\lambda\varphi^4$ theory exists and satisfies the Wightman axioms (in strengthened form) and the Haag-Ruelle axioms for scattering theory. This includes:

1. The vacuum is unique: The Hamiltonian $H = H^* \geqslant 0$ and momentum $P = P^*$ have a unique vector Ω (up to a multiple) for which $H\Omega = P\Omega = 0$.

2. The Hamiltonian has a lower and and upper mass gap: The mass operator $M = \sqrt{H^2 - P^2}$ has an isolated eigenvalue $m > 0$ corresponding to single particle states.

3. The field is well-behaved: For real functions $f \in C_0^\infty(\mathbb{R}^2)$, the fields $\varphi(f)$ have a common, dense domain $\mathcal{D}$ on which their closures are self adjoint, $\varphi(f)^* = \varphi(f)^{**}$.

4. The field is covariant: There is a positive-energy representation $U(g)$ of the Poincaré group on the Hilbert space of the theory, and the field $\varphi(x)$ transforms covariantly, $U(g)\varphi(x)U(g^{-1}) = \varphi(gx)$.

5. The field is local: If f and g have space-like separated supports, $[\varphi(f)^{**}, \varphi(g)^{**}] = 0$.

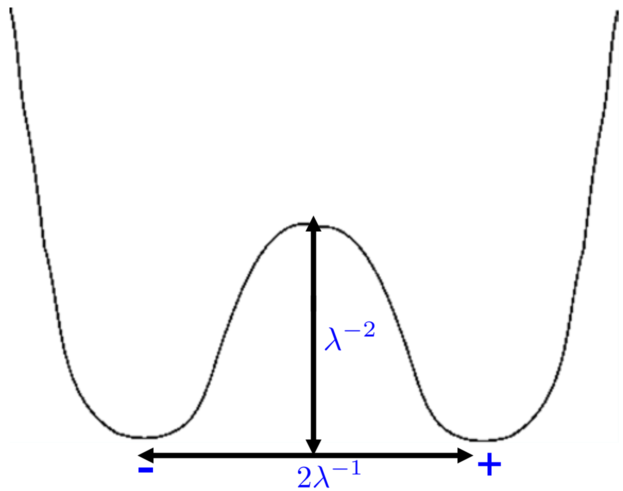
6. Perturbation theory is asymptotic: [J.P.Eckmann, H.Epstein, J.Fröhlich, Ann l'Inst. Henri Poincaré (1976) and K.Osterwalder, R.Sénéor, Helv. Phys. Acta (1976)] Multi-particle scattering theory for this theory agrees in perturbation theory with the expressions in standard physics texts.

Examples of Axioms

Rudolf Haag and Arthur Wightman
Boulder 1982

What's Next?

Double-Well Potential: Non-Uniqueness?



Classical Potential

$$V(\varphi) = \lambda^{-2}\left(\lambda^2\varphi^2 - 1\right)^2$$

$\lambda \to 0$

$V \to$ "Ising" distribution

Perron-Frobenius ensures "tunneling" and a unique ground state in usual quantum theory. (no spin)

G-J showed that Perron-Frobenius still holds for finite volume scalar QFT.

The Ising model suggests this may not be true in infinite volume, and physicists believed this.
If true, then symmetry $\varphi \to -\varphi$ of H would be broken in the ground state, and $\langle\varphi\rangle \neq 0$.

Quantum Potential: Problem from $c = \infty$.
Discussed during a 1972 meeting in Moscow with expert on phase transitions in lattice statistical physics, R. Dobrushin.

CONSTRUCTION OF A ONE-DIMENSIONAL QUANTUM FIELD BY MEANS OF A CONTINUOUS MARKOV FIELD

R. L. Dobrushin and R. A. Minlos

Moscow State University. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 7, No. 4, pp. 81-82, October-December, 1973. Original article submitted May 11, 1973.

Announcement, but no published proof.

Another Highpoint
Phase Transitions Exist in QFT

Theorem: ($\varphi \mapsto -\varphi$ Symmetry Breaking.) **[J.Glimm, A.J., and T.Spencer, Comm. Math. Phys. (1975), and two papers on clustering in Ann. Phys. (1976)]** There exists $\lambda_{\max} > 0$, such that for all $\lambda \in [0, \lambda_{\max}]$, the Euclidean $\lambda^{-2}\left(\lambda^2\varphi^2 - 1\right)^2$ measure exists as a weak limit of finite-volume measures with interaction on increasing space-time squares Λ, and with the field on the boundary of Λ equal to one of $-\lambda^{-1}$, 0, or λ^{-1}. This gives three different limiting measures $\omega_-, \omega_0, \omega_+$ respectively, for the same interaction.

$$\underbrace{\boxed{\Lambda}}_{\text{Fix field on boundary to equal } 0 \text{ or } \pm\lambda^{-1}}$$

Another Highpoint
Phase Transitions Exist in QFT

Theorem: ($\varphi \mapsto -\varphi$ Symmetry Breaking.) **[J.Glimm, A.J., and T.Spencer, Comm. Math. Phys. (1975), and two papers on clustering in Ann. Phys. (1976)]** There exists $\lambda_{\max} > 0$, such that for all $\lambda \in [0, \lambda_{\max}]$, the Euclidean $\lambda^{-2}\left(\lambda^2\varphi^2 - 1\right)^2$ measure exists as a weak limit of finite-volume measures with interaction on increasing space-time squares Λ, and with the field on the boundary of Λ equal to one of $-\lambda^{-1}$, 0, or λ^{-1}. This gives three different limiting measures $\omega_-, \omega_0, \omega_+$ respectively, for the same interaction.

1. The states $\omega_\pm$ give Wightman theories: The states $\omega_\pm$ are extremal and satisfy the OS axioms. The corresponding two theories obtained from the OS-construction satisfy the Wightman axioms with a unique vacuum state.

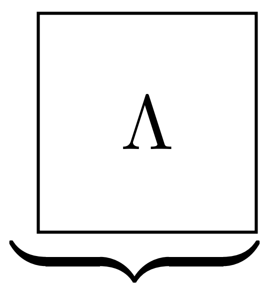
$$\underbrace{\boxed{\Lambda}}_{\text{Fix field on boundary to equal } 0 \text{ or } \pm\lambda^{-1}}$$

Another Highpoint
Phase Transitions Exist in QFT

Theorem: ($\varphi \mapsto -\varphi$ Symmetry Breaking.) **[J.Glimm, A.J., and T.Spencer, Comm. Math. Phys. (1975), and two papers on clustering in Ann. Phys. (1976)]** There exists $\lambda_{\max} > 0$, such that for all $\lambda \in [0, \lambda_{\max}]$, the Euclidean $\lambda^{-2}\left(\lambda^2\varphi^2 - 1\right)^2$ measure exists as a weak limit of finite-volume measures with interaction on increasing space-time squares Λ, and with the field on the boundary of Λ equal to one of $-\lambda^{-1}$, 0, or λ^{-1}. This gives three different limiting measures $\omega_-, \omega_0, \omega_+$ respectively, for the same interaction.

1. The states $\omega_\pm$ give Wightman theories: The states $\omega_\pm$ are extremal and satisfy the OS axioms. The corresponding two theories obtained from the OS-construction satisfy the Wightman axioms with a unique vacuum state.

2. Broken symmetry $\varphi \to -\varphi$: The field has vacuum expectation $\omega_+(\varphi(x)) = -\omega_-(\varphi(x))$ close to $\pm\lambda^{-1}$.



Fix field on boundary to equal 0 or $\pm\lambda^{-1}$

Another Highpoint
Phase Transitions Exist in QFT

Theorem: ($\varphi \mapsto -\varphi$ Symmetry Breaking.) **[J.Glimm, A.J., and T.Spencer, Comm. Math. Phys. (1975), and two papers on clustering in Ann. Phys. (1976)]** There exists $\lambda_{\max} > 0$, such that for all $\lambda \in [0, \lambda_{\max}]$, the Euclidean $\lambda^{-2}\left(\lambda^2\varphi^2 - 1\right)^2$ measure exists as a weak limit of finite-volume measures with interaction on increasing space-time squares Λ, and with the field on the boundary of Λ equal to one of $-\lambda^{-1}$, 0, or λ^{-1}. This gives three different limiting measures $\omega_-, \omega_0, \omega_+$ respectively, for the same interaction.

1. The states $\omega_\pm$ give Wightman theories: The states $\omega_\pm$ are extremal and satisfy the OS axioms. The corresponding two theories obtained from the OS-construction satisfy the Wightman axioms with a unique vacuum state.

2. Broken symmetry $\varphi \to -\varphi$: The field has vacuum expectation $\omega_+(\varphi(x)) = -\omega_-(\varphi(x))$ close to $\pm\lambda^{-1}$.

3. $\omega_\pm$ theories have a non-zero mass gap: Correlations for the Euclidean theories from $\omega_\pm$ decay exponentially with a rate $m > 0$, corresponding to a mass gap $(0, m)$ in the spectrum of H.

Another Highpoint
Phase Transitions Exist in QFT

Theorem: **($\varphi \mapsto -\varphi$ Symmetry Breaking.)** **[J.Glimm, A.J., and T.Spencer, Comm. Math. Phys. (1975), and two papers on clustering in Ann. Phys. (1976)]** There exists $\lambda_{\max} > 0$, such that for all $\lambda \in [0, \lambda_{\max}]$, the Euclidean $\lambda^{-2}\left(\lambda^2\varphi^2 - 1\right)^2$ measure exists as a weak limit of finite-volume measures with interaction on increasing space-time squares Λ, and with the field on the boundary of Λ equal to one of $-\lambda^{-1}$, 0, or λ^{-1}. This gives three different limiting measures $\omega_-, \omega_0, \omega_+$ respectively, for the same interaction.

1. The states $\omega_\pm$ give Wightman theories: The states $\omega_\pm$ are extremal and satisfy the OS axioms. The corresponding two theories obtained from the OS-construction satisfy the Wightman axioms with a unique vacuum state.

2. Broken symmetry $\varphi \to -\varphi$: The field has vacuum expectation $\omega_+(\varphi(x)) = -\omega_-(\varphi(x))$ close to $\pm\lambda^{-1}$.

3. $\omega_\pm$ theories have a non-zero mass gap: Correlations for the Euclidean theories from $\omega_\pm$ decay exponentially with a rate $m > 0$, corresponding to a mass gap $(0, m)$ in the spectrum of H.

4. The state ω_0: The vacuum state ω_0 gives $\omega_0(\varphi(x)) = 0$. Correlations in ω_0 do not decay. Presumably $\omega_0 = \frac{1}{2}(\omega_- + \omega_+)$ is an even mixture of the extremal states $\omega_\pm$.

Another Highpoint
Phase Transitions Exist in QFT

Theorem: ($\varphi \mapsto -\varphi$ **Symmetry Breaking.**) **[J.Glimm, A.J., and T.Spencer, Comm. Math. Phys. (1975), and two papers on clustering in Ann. Phys. (1976)]** There exists $\lambda_{\max} > 0$, such that for all $\lambda \in [0, \lambda_{\max}]$, the Euclidean $\lambda^{-2}\left(\lambda^2\varphi^2 - 1\right)^2$ measure exists as a weak limit of finite-volume measures with interaction on increasing space-time squares Λ, and with the field on the boundary of Λ equal to one of $-\lambda^{-1}$, 0, or λ^{-1}. This gives three different limiting measures $\omega_-, \omega_0, \omega_+$ respectively, for the same interaction.

1. The states $\omega_\pm$ give Wightman theories: The states $\omega_\pm$ are extremal and satisfy the OS axioms. The corresponding two theories obtained from the OS-construction satisfy the Wightman axioms with a unique vacuum state.

2. Broken symmetry $\varphi \to -\varphi$: The field has vacuum expectation $\omega_+(\varphi(x)) = -\omega_-(\varphi(x))$ close to $\pm\lambda^{-1}$.

3. $\omega_\pm$ theories have a non-zero mass gap: Correlations for the Euclidean theories from $\omega_\pm$ decay exponentially with a rate $m > 0$, corresponding to a mass gap $(0, m)$ in the spectrum of H.

4. The state ω_0: The vacuum state ω_0 gives $\omega_0(\varphi(x)) = 0$. Correlations in ω_0 do not decay. Presumably $\omega_0 = \frac{1}{2}(\omega_- + \omega_+)$ is an even mixture of the extremal states $\omega_\pm$.

Phase Transitions in φ^6 theory K.Gawędzki, Comm. Math. (established) Phys. (1978) S.Summers, Inst. H. Poincaré (1981). (cluster expansion)

Weak-Coupling Phase Diagrams

- For many lattice statistical mechanical models, S.Piragov and Ya.G.Sinai analyzed their phase diagrams. Could a similar analysis be relevant for QFTs?

Theorem: [**"Pirogov-Sinai"-type phase diagram**]
[John Imbrie, thesis and two papers in Comm. Math. Phys.(1981).]
Let $0 \leqslant W$ be a polynomial of even degree with r distinct zeros; let $V = W + P$, where P is a polynomial perturbation of degree $(r-1)$; and let $0 < \lambda$ be sufficiently small. Consider the $\lambda^{-2}V(\lambda\varphi)$ QFT in dimension $d = 2$.

Weak-Coupling Phase Diagrams

- For many lattice statistical mechanical models, S.Piragov and Ya.G.Sinai analyzed their phase diagrams. Could a similar analysis be relevant for QFTs?

Theorem: [**"Pirogov-Sinai"-type phase diagram**]
[John Imbrie, thesis and two papers in Comm. Math. Phys.(1981).]
Let $0 \leqslant W$ be a polynomial of even degree with r distinct zeros; let $V = W + P$, where P is a polynomial perturbation of degree $(r-1)$; and let $0 < \lambda$ be sufficiently small. Consider the $\lambda^{-2}V(\lambda\varphi)$ QFT in dimension $d = 2$. This theory has $k \in \{r, r-1, \ldots, 1\}$ distinct phases for the parameters of P lying in $\binom{r}{k}$ hypersurfaces of dimension $(r-k)$ in the space of perturbations.

Let's apply this to the prior $\lambda\varphi^4$ example. Then:

$$V = W + P \ , \quad W = (\varphi^2 - 1)^2 \ , \quad P = -\mu\varphi$$

Weak-Coupling Phase Diagrams

- For many lattice statistical mechanical models, S.Piragov and Ya.G.Sinai analyzed their phase diagrams. Could a similar analysis be relevant for QFTs?

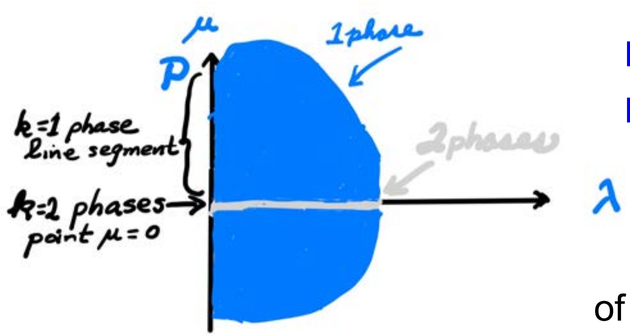
I took this photo of Kenneth Wilson, John Imbrie, and Gerard 't Hooft on an excursion we made during the 1981 Cargèse, Corsica summer school.

Theorem: [**"Pirogov-Sinai"-type phase diagram**]
[John Imbrie, thesis and two papers in Comm. Math. Phys.(1981).]
Let $0 \leqslant W$ be a polynomial of even degree with r distinct zeros; let $V = W + P$, where P is a polynomial perturbation of degree $(r-1)$; and let $0 < \lambda$ be sufficiently small. Consider the $\lambda^{-2}V(\lambda\varphi)$ QFT in dimension $d = 2$. This theory has $k \in \{r, r-1, \ldots, 1\}$ distinct phases for the parameters of P lying in $\binom{r}{k}$ hypersurfaces of dimension $(r-k)$ in the space of perturbations.

Let's apply this to the prior $\lambda\varphi^4$ example. Then:

$$V = W + P\ , \quad W = (\varphi^2 - 1)^2\ , \quad P = -\mu\varphi$$

$$\lambda^{-2}V(\lambda\varphi) = \lambda^{-2}(\lambda^2\varphi^2 - 1)^2 - \lambda^{-1}\mu\varphi$$



r=2, k=2; 1 point (dim 0)
r=2, k=1; 2 lines (dim 1)

Topologically:
Phase diagram is the corner of a hypercube in r-1 dimensions

d=3

- Space-time dimension 3 is qualitatively harder: normal ordering is insufficient for renormalization.
- Wave function renormalization entails a "change of representation" in finite volume.
- This example gives the only known family of Euclidean-invariant, RP measures on $\mathfrak{S}'(R^3)$.

Theorem: (Stability.) **[J.Glimm and A.J., *Fort. d. Physik* (1973), republished in Collected Papers, Vol.2, Birkhäuser Boston (1985)]** In $d = 3$ dimensional space-time, consider the renormalized $\lambda\varphi^4$ Hamiltonian H_L, with $0 < \lambda$ and with non-zero interaction localized in a fixed spatial square $[-L, L]^2$. Then H_L is bounded from below.

Theorem: (Full Field Theory.) **[J.Feldman and K.Osterwalder, *Ann. Phys.* (1976)]** The renormalized $\lambda\varphi^4$ field theory, with $0 < \lambda$ sufficiently small, exists in $d = 3$ dimensional space-time. This theory satisfies the Wightman axioms with a positive mass gap. It comes from a Euclidean-invariant, RP-positive measure on $\mathcal{S}'(\mathbb{R}^3)$.

Celebration Exhibit in the Cabot Science Library at Harvard

Glimm-Jaffe "Quantum Physics"
Springer Monograph
Publication 1981

Some Other Work: Much with RP

- Phase transitions in stat. physics also Continuous symmetry breaking J.Fröhlich, B.Simon, T.Spencer, R.Israel, E.Lieb, A.Sokal, D.Brydges, J.Bricmont, J.-R.Fontaine, J.Lebowitz, R.Kotecky, M.Biscup, ….
- RP and representation theory E.Seiler, J.Fröhlich, A.Klein, L.Landau, P.Jorgensen, G.Olafsson, K.-H.Neeb,….
- Complex Fields A.J., C.Jäkel, R.Martinez
- Algebras, Parafermions, and Ground states in SM, E.Lieb, N.Macris, B.Nachtergale, A.Kitaev, F.Pedrocchi, D.Loss, S.Chesi, A.J.
- Quantum Information A.J., Z.Liu, A.Wozniakowsi Topological proof A.J. & Z.Liu, Comm. Math Phys. (2017)
- Relations to subfactor theory and Quantum Fourier Analysis V.Jones, Z.Liu, J.Wu, C.Jiang, Y.Ren, S.Palcoux, A.J., (1999-2020)

Zhengwei Liu, A.J.. Zurich, July 2, 2019

Some Other Work: Models

- F.Guerra, L.Rosen, B.Simon
- J.Fröhlich
- Many results

Some Other Work: Phase Transitions

- Contin. symmetry unbroken d=2 Dobrushin, Shlossman
- Kosterlitz-Thouless transition J.Fröhlich, T.Spencer
- Critical dimension random field Ising J.Imbrie

Some Other Work: Hypercontractivity

- E.Nelson discovered this in 1965
- Hypercontractivity for fermions L.Gross
- Quantum Information settings C.King

Some Other Work: Phase Cell Loc.

- Many partial results J.Magnen, V.Rivasseau, R.Sénéor

Some Other Work: SUSY

- Many partial results on d=2 SUSY, but no Wightman theory. O.McBryan, A.Lesniewski, J.Weitsman, A.J., S.Janowsky, J.Imbrie

Non-Commutative Geometry

- The super-symmetric, interacting field theories give examples of Connes' entire cyclic cohomology through the JLO cocycle. A.J., A.Lesniewski, K.Osterwalder, Comm. Math. Phys. (1988); K.Ernst, P.Feng, A.J., A.Lesniewski, Jour. Funct. Anal. (1990); A.J. Advances in Mathematics (1998) and (2000); A.J. Comm. Math. Phys. (2000); A.J., PNAS (2000)

Algebraic QFT

Many results: S.Doplicher, R.Haag, J.Roberts, R.Longo, D.Buchholz, K.Fredenhagen,….

Some Other Work: Ren. Group

- Various models D.Brydges, J.Dimock, T.Hurd, K.Gawędzki, A.Kupiainen, G.Gallavotti, G.Benfatto, M.Cassandro, F.Nicolò, E.Olivieri, E.Presutti, P.Faria daVeiga,C.deCalan, D.Moser, A.Abdesselam

Other Approaches & Partial Results

- Scaling Limit: Exact Solutions B.McCoy, T.T.Wu (1972), also Russian work.
- Stochastic PDE and Quantization (Regular Structures) M.Hairer and collaborators (2015-2020)

Some Brief Remarks on Methods

Key Word: Estimates

1. Classical-Quantum Duality from OS

$$\underbrace{\int_{\mathcal{S}'} \overline{\psi(\Phi(0))}\psi(\Phi(t))d\mu}_{\text{Classical Probability Theory with RP Property}} = \underbrace{\langle \psi(\varphi), e^{-tH}\psi(\varphi)\rangle_{\mathcal{H}}}_{\text{Quantum Mechanical Hilbert Space}}$$

Evolution of Feynman-Kac-Symanzik-Nelson to Osterwalder-Schrader

Allow classical methods to estimate quantum operators.

2. Generalize Perron-Frobenius to Infinite Dimensions
3. Combine operator algebra methods with Hilbert space methods

More Brief Remarks on Methods

4. Localization of Estimates (refined from d=2 to d=3)

   Localization in phase space (need small position in 3d)

   Limit by uncertainty: borderline in d=4

   Phase-cell localization more robust—but less elegant than—renormalization group methods.

   Method Inductive, rather than iterative

5. RP & Multiple Reflection Bounds

for example: $\pm\,\varphi(f,0) \leqslant \|f\|\,(H+I)$, for $f=\bar{f}\in C_0^\infty$

for example: Estimate QFT deviation from Peierls' argument for φ^4 phase transition.

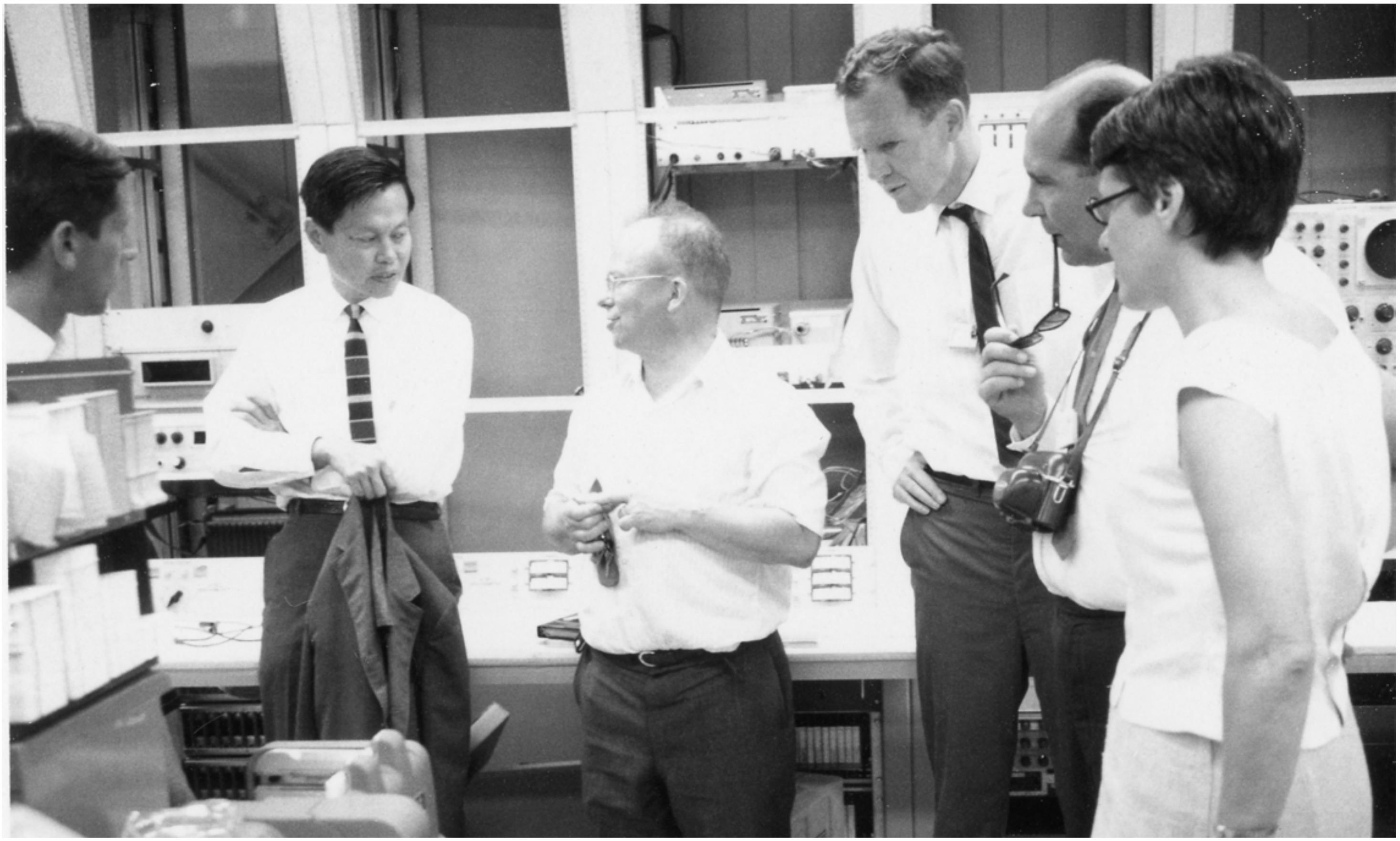
What's Next?

Today: Most physicists believe that the φ^4 and QED are mathematically inconsistent in d=4!

Physicists discovered a potential refinement of Sobolev inequalities for perturbation, in the case that the two sides of the inequalities have the same scaling properties—i.e. in the critical dimension. (2004 Nobel prize)

That is called "asymptotic freedom." The belief is that asymptotically-free theories may exist (such as non-abelian gauge theory with a simple gauge group), but not φ^4 or QED.

Yang-Mills Theory for d=4?

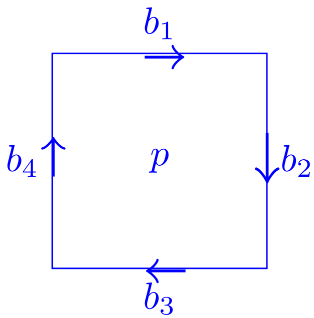


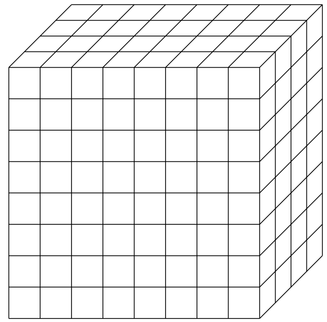
Summer 1966 tour of SLAC main accelerator, just before its opening, during a conference at Stanford University.
H.Lehmann, C.N.Yang, W.Panofsky (director), A.Wightman, H.Joos, N.Byers

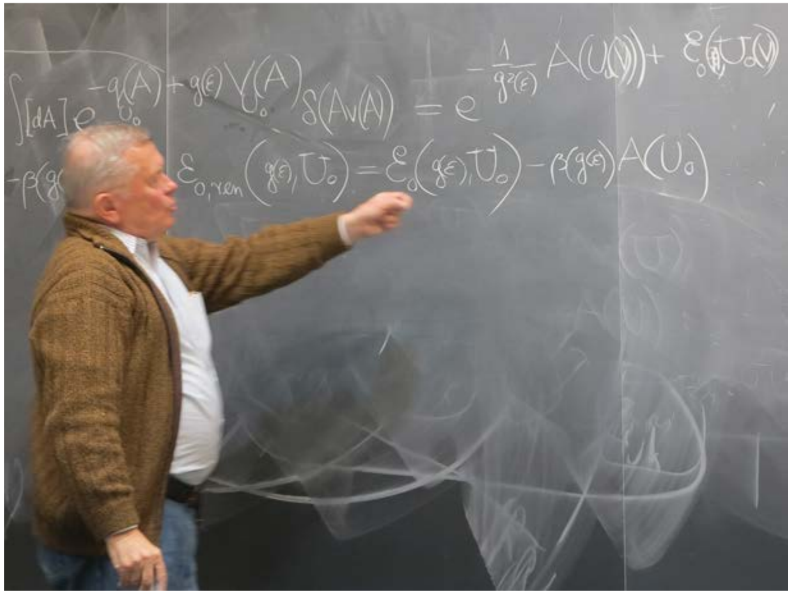
K. Wilson Action

$g \in \mathcal{G}$ Gauge Group

Assign a unitary representation of $g \in \mathcal{G}$ to oriented bonds bonds, with $U_{g^{-1}}(b) = U_g(b^{-1})$.







Balaban seminar: Harvard, November 27, 2012

$$\mathfrak{A} = \frac{1}{g^2} \sum_{\text{plaquettes}} \mathfrak{A}(p)$$

$$\mathfrak{A}(p) = \mathrm{Tr} \prod_{b \in \partial p} U_g(b)$$

$$\mu = \frac{1}{Z} e^{-\mathfrak{A}} dg \,, \qquad dg \text{ is product Haar measure}$$

Reflection Positive: K.Osterwalder, E.Seiler (1976)

Study Effective Action d=2,3,4 T.Balaban (1982-1987) many papers several with co-authors; renormalization group (RG) methods gauge choice dependent on length scale; analyze effective actions BDH developed RG methods

Many partial d=4 results in finite volume

After Harvard seminar November 2012

Jon Dimock, December 2018

Higgs model d=2 T.Balaban, D.Brydges, J.Imbrie, A.J. (1984-1987)
QED in d=3 using Balaban's method J.Dimock arXiv:2009.01156

And Tomorrow

I believe:

1. The finite-volume SU(2) theory is within human reach.
2. The infinite volume limit will require new physics and new mathematics. See A.J and E.Witten (2001)

A. Weil on Riemann Hypothesis

1. When he was young he wanted to prove the RH.
2. As he got older, he wanted to see the proof.
3. Now he would only like to know that it has been proved.

I feel that way about YM_4!!

HOPE for TOMORROW

Some viewers of this meeting will be inspired to construct a mathematical YM-QFT in dimension 4, also with a mass gap! This will be interesting for both mathematics and physics! This probably requires some genius mathematicians, who also know physics!

Good Luck  谢谢