Recent progress on random field Ising model

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Based on joint works with
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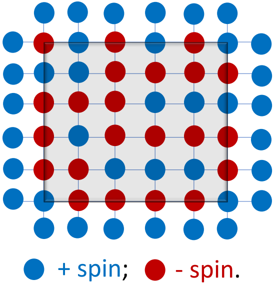
Mathematical Picture Language Seminar

(Lenz–)Ising model: a model for magnetization

• $\mathbb{Z}^d$: d-dimensional lattice (nearest neighbor graph);
• Λ_N: box with side length $2N$ centered at origin o;
• configuration $\sigma \in \Omega = \{-1, 1\}^{\Lambda_N}$.
• $E(A, B)$ for $A, B \subset \mathbb{Z}^d$: edges between A and B;



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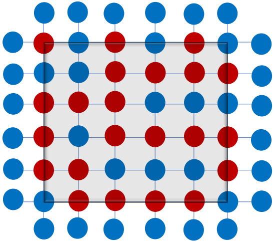

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Hamiltonian with plus (resp. minus) boundary condition:

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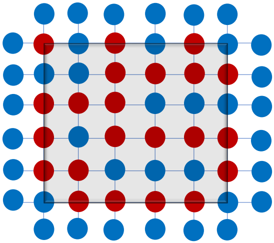

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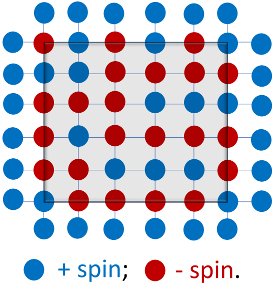

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Observation: Ising measure favors configurations with more agreeing neighboring pairs.

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• Much more on Ising model has been understood, but not our focus today.

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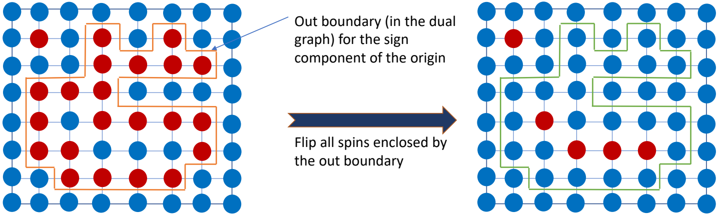

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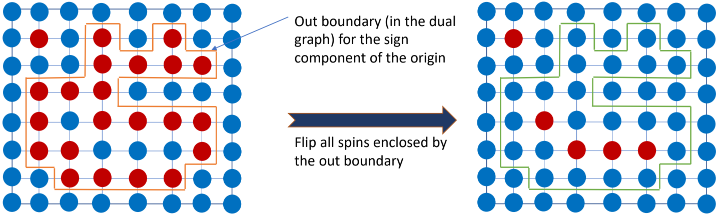

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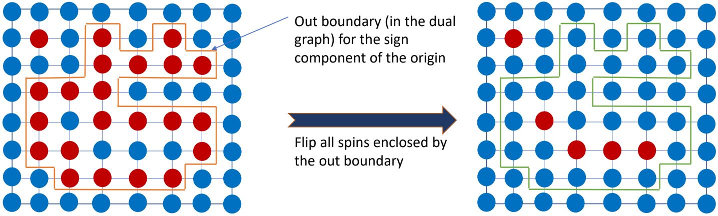

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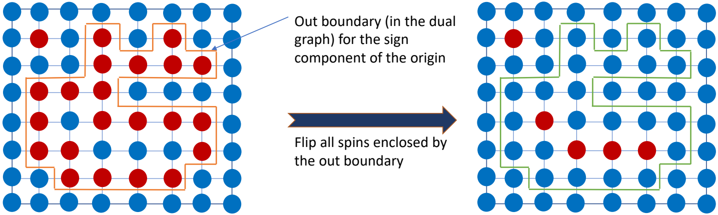

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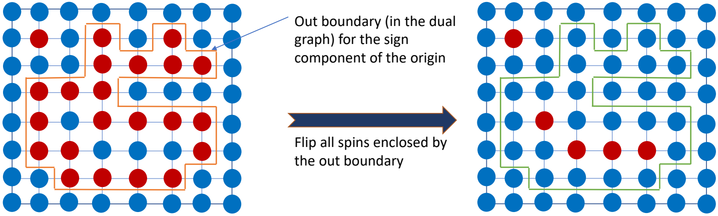

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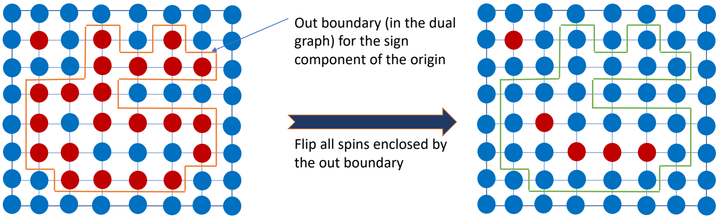

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• **Conclusion** (summing over ℓ): at low temperature, the origin agrees with the boundary condition with good probability.

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The RFIM Hamiltonian on Λ_N with external field ϵh and plus (resp. minus) boundary condition:

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Main question today: how does the random field affects the long range order? I.e., what is the limiting behavior of $m_{T,\Lambda_N,\epsilon}$?

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For small ϵ, it is much more delicate and challenging (the focus for the rest of the talk).

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• For $d = 2$ with small ϵ, we need the collective influence from disorder on a large set to fight against the boundary effect. But why should they collaborate?

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• Deterministically $|\Delta F| \leq 2|\partial\Lambda_N| = 16N$ (via direct comparison of Hamiltonians with plus and minus boundary conditions for the same configuration).

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Debate among physicists on decay rate: polynomial decay in some regime (Berezinskii–Kosterlitz–Thouless transition) v.s. exponential decay for all ϵ?

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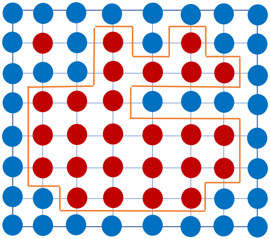

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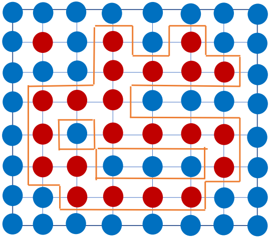

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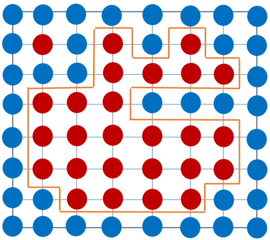

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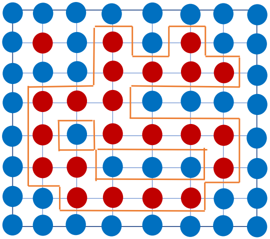

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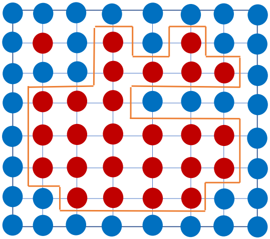

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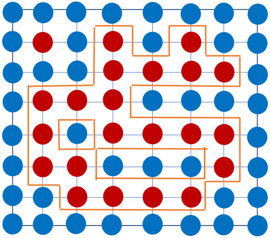

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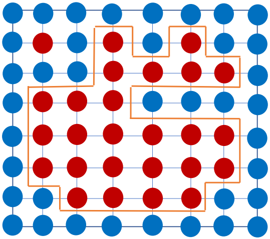

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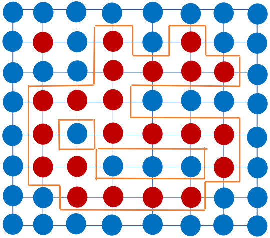

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D.–Zhuang 21: a simple proof without renormalization group theory (also gives a new result for random field Potts model).

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Goal: show $\mathbb{Q}^{+}(\sigma_o = -1) \ll 1$ for small ϵ, T and $d \geq 3$.

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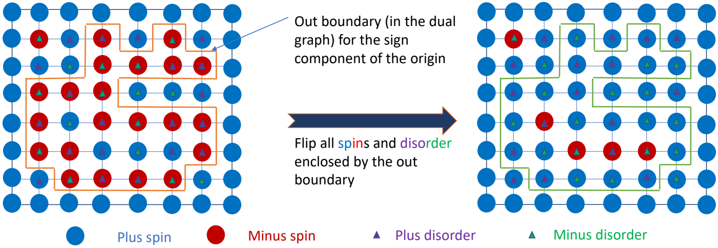

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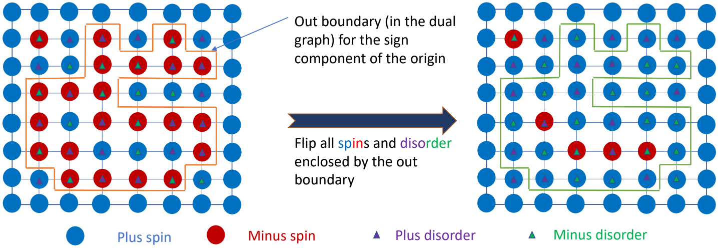

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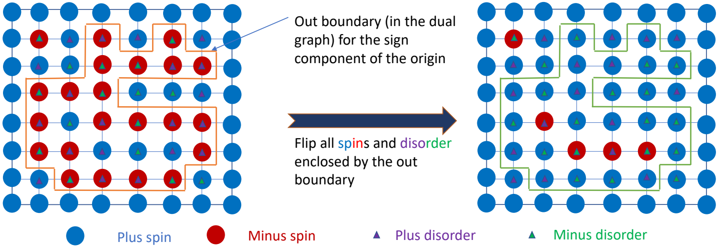

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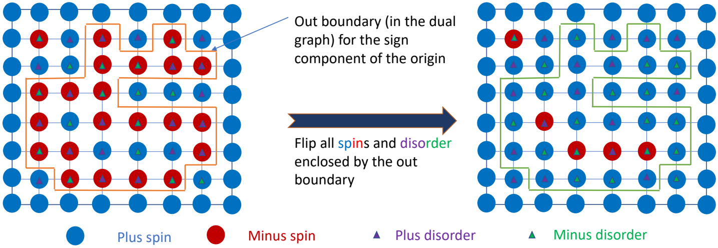

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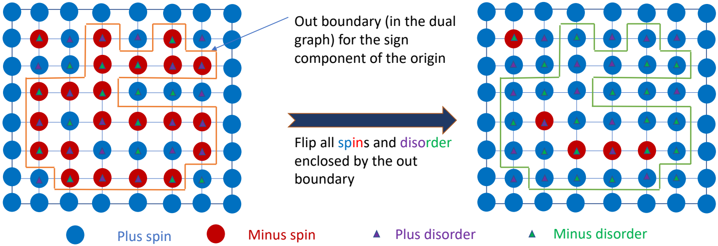

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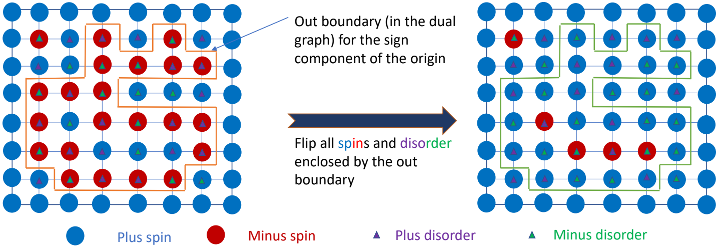

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A caveat: the partition function for the Ising model is changed since the external field is changed!

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$$\nu^+(h,\sigma) = \prod_v \frac{1}{\sqrt{2\pi}} e^{-\frac{h_v^2}{2\epsilon^2}} \prod_v \frac{e^{-\frac{1}{T} H^+_{\Lambda_N,h}(\sigma)}}{\mathcal{Z}^+_{T,\Lambda_N,h}}$$

(recall $\mathcal{Z}^+_{T,\Lambda_N,h} = \sum_\sigma e^{-\frac{1}{T} H^+_{\Lambda_N,h}(\sigma)}$ is the partition function).

- The density for disorder is unchanged after flipping.
- The Hamiltonian $H^+_{\Lambda_N,h}(\sigma)$ decreased by 2ℓ after flipping a component with boundary size ℓ; but what about the partition function $\mathcal{Z}^+_{T,\Lambda_N,h}$?

Solution: show that the change of the free energy $-\frac{1}{T} \log \mathcal{Z}^+_{T,\Lambda_N,h}$ is bounded by ℓ after flipping **any** component with boundary size ℓ.

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- Adapting the proof of Fisher–Fröhlich–Spencer 84 verbatim: with high probability, the change of free energy after flipping the sign of disorder in **any simply connected** component of boundary size ℓ is bounded by ℓ.

A closer look at three dimensions

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- This is hard since our understanding for critical 3D Ising without disorder remains limited.

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Remark. D.–Zhuang 21: for 3D random field Potts model with weak disorder, long range order exists at low temperatures.

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Question: what about scaling limits for interfaces of random field Ising model?