An elementary derivation of the periodic table of topological matter

Gian Michele Graf, ETH Zurich

Harvard Mathematical Picture Language Seminar
June 5, 2026

An elementary derivation of the periodic table of topological matter

Gian Michele Graf, ETH Zurich

Harvard Mathematical Picture Language Seminar
June 5, 2026

Based on joint work with Filippo Santi

Outline

The periodic table of topological matter

Plan of the proof

Derivation

The table, whatever it means

The table, whatever it means

	Symmetry			d							
CAZ Class	Θ	Σ	Π	0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	1	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

The table, whatever it means

CAZ Class	Symmetry Θ	Σ	Π	d = 0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	1	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Work by: Altland/Zirnbauer, Ryu/Schnyder/Furusaki/Ludwig, Kitaev, Schulz-Baldes, Thiang, Freed/Moore, Kellendonk

What is the table about? The broad picture

What is the table about? The broad picture

- Objects: Band insulators and superconductors

What is the table about? The broad picture

- Objects: Band insulators and superconductors within the approximation of independent particles, or BCS theory.

What is the table about? The broad picture

- Objects: Band insulators and superconductors within the approximation of independent particles, or BCS theory. They occur in spatial dimensions $d = 0, 1, 2 \ldots$

What is the table about? The broad picture

- Objects: Band insulators and superconductors within the approximation of independent particles, or BCS theory. They occur in spatial dimensions $d = 0, 1, 2 \ldots$
- Symmetries S: Objects enjoy symmetries S (involutions), among:

What is the table about? The broad picture

- Objects: Band insulators and superconductors within the approximation of independent particles, or BCS theory. They occur in spatial dimensions $d = 0, 1, 2 \ldots$
- Symmetries S: Objects enjoy symmetries S (involutions), among:
  - antiunitary symmetries:
    - $S = \Theta$: time-reversal
    - $S = \Sigma$: particle-hole conjugation

    come with even/odd parity: $S^2 = \sigma_S = \pm 1$

What is the table about? The broad picture

- Objects: Band insulators and superconductors within the approximation of independent particles, or BCS theory. They occur in spatial dimensions $d = 0, 1, 2 \ldots$
- Symmetries S: Objects enjoy symmetries S (involutions), among:
  - antiunitary symmetries:
    - $S = \Theta$: time-reversal
    - $S = \Sigma$: particle-hole conjugation

    come with even/odd parity: $S^2 = \sigma_S = \pm 1$
  - unitary symmetry:
    - $S = \Pi$: chiral symmetry (no parity)

What is the table about? The broad picture

- Objects: Band insulators and superconductors within the approximation of independent particles, or BCS theory. They occur in spatial dimensions $d = 0, 1, 2 \ldots$
- Symmetries S: Objects enjoy symmetries S (involutions), among:
  - antiunitary symmetries:
    - $S = \Theta$: time-reversal
    - $S = \Sigma$: particle-hole conjugation

    come with even/odd parity: $S^2 = \sigma_S = \pm 1$
  - unitary symmetry:
    - $S = \Pi$: chiral symmetry (no parity)
- Combinations X:
  - Symmetries occur in combinations
  - Rule “2 symmetries make 3”: $\Pi = \Theta\Sigma = \Sigma\Theta$

What is the table about? The broad picture

- Objects: Band insulators and superconductors within the approximation of independent particles, or BCS theory. They occur in spatial dimensions $d = 0, 1, 2 \ldots$
- Symmetries S: Objects enjoy symmetries S (involutions), among:
  - antiunitary symmetries:
    - $S = \Theta$: time-reversal
    - $S = \Sigma$: particle-hole conjugation

    come with even/odd parity: $S^2 = \sigma_S = \pm 1$
  - unitary symmetry:
    - $S = \Pi$: chiral symmetry (no parity)
- Combinations X:
  - Symmetries occur in combinations
  - Rule "2 symmetries make 3": $\Pi = \Theta\Sigma = \Sigma\Theta$
  - Number of possible combinations X:

What is the table about? The broad picture

- Objects: Band insulators and superconductors within the approximation of independent particles, or BCS theory. They occur in spatial dimensions $d = 0, 1, 2 \ldots$
- Symmetries S: Objects enjoy symmetries S (involutions), among:
  - antiunitary symmetries:
    - $S = \Theta$: time-reversal
    - $S = \Sigma$: particle-hole conjugation

    come with even/odd parity: $S^2 = \sigma_S = \pm 1$
  - unitary symmetry:
    - $S = \Pi$: chiral symmetry (no parity)
- Combinations X:
  - Symmetries occur in combinations
  - Rule "2 symmetries make 3": $\Pi = \Theta\Sigma = \Sigma\Theta$
  - Number of possible combinations X: $(1 + 2 \times 2) \times 2 = 10$.

    1: no symmetry
    2: $\sigma_S = \pm 1$
    2: $S = \Theta, \Sigma$
    2: Π present/absent

What is the table about? The broad picture

- Objects: Band insulators and superconductors within the approximation of independent particles, or BCS theory. They occur in spatial dimensions $d = 0, 1, 2 \ldots$
- Symmetries S: Objects enjoy symmetries S (involutions), among:
  - antiunitary symmetries:
    - $S = \Theta$: time-reversal
    - $S = \Sigma$: particle-hole conjugation

    come with even/odd parity: $S^2 = \sigma_S = \pm 1$
  - unitary symmetry:
    - $S = \Pi$: chiral symmetry (no parity)
- Combinations X: The tenfold way
  - Symmetries occur in combinations
  - Rule "2 symmetries make 3": $\Pi = \Theta\Sigma = \Sigma\Theta$
  - Number of possible combinations X: $(1 + 2 \times 2) \times 2 = 10$.

1: no symmetry

2: $\sigma_S = \pm 1$

2: $S = \Theta, \Sigma$

2: Π present/absent

What is the table about? The broad picture

- Objects: Band insulators and superconductors within the approximation of independent particles, or BCS theory. They occur in spatial dimensions $d = 0, 1, 2 \ldots$
- Symmetries S: Objects enjoy symmetries S (involutions), among:
  - antiunitary symmetries:
    - $S = \Theta$: time-reversal
    - $S = \Sigma$: particle-hole conjugation

    come with even/odd parity: $S^2 = \sigma_S = \pm 1$
  - unitary symmetry:
    - $S = \Pi$: chiral symmetry (no parity)
- Combinations X: The tenfold way
  - Symmetries occur in combinations
  - Rule “2 symmetries make 3”: $\Pi = \Theta\Sigma = \Sigma\Theta$
  - Number of possible combinations X: $(1 + 2 \times 2) \times 2 = 10$.

    1: no symmetry
    2: $S = \Theta, \Sigma$
    2: $\sigma_S = \pm 1$
    2: Π present/absent
- Altogether: Cartan-Altland-Zirnbauer (CAZ) classes (X, d)

What is the table about? The answer

- The goal:
  - To seek out the properties of objects that are invariant under deformations preserving their nature ($\to$ connected components)

What is the table about? The answer

- The goal:
  - To seek out the properties of objects that are invariant under deformations preserving their nature ($\rightarrow$ connected components)
  - Actually, to understand them in an elementary way

What is the table about? The answer

- The goal:
  - To seek out the properties of objects that are invariant under deformations preserving their nature ($\to$ connected components)
  - Actually, to understand them in an elementary way
- Index groups: There is a map from CAZ classes to (discrete, Abelian) groups

$$(X, d) \mapsto G$$

What is the table about? The answer

- The goal:
  - To seek out the properties of objects that are invariant under deformations preserving their nature ($\to$ connected components)
  - Actually, to understand them in an elementary way
- Index groups: There is a map from CAZ classes to (discrete, Abelian) groups

$$(X, d) \mapsto G$$

- Meaning: Within each CAZ class (X, d), its connected components are labelled by elements $g \in G$

What is the table about? The answer

- The goal:
  - To seek out the properties of objects that are invariant under deformations preserving their nature ($\to$ connected components)
  - Actually, to understand them in an elementary way
- Index groups: There is a map from CAZ classes to (discrete, Abelian) groups

$$(X, d) \mapsto G$$

- Meaning: Within each CAZ class (X, d), its connected components are labelled by elements $g \in G$
- In each case, $G = 0, \mathbb{Z}$, or $\mathbb{Z}_2$.

What is the table about? The answer

- The goal:
  - To seek out the properties of objects that are invariant under deformations preserving their nature ($\to$ connected components)
  - Actually, to understand them in an elementary way
- Index groups: There is a map from CAZ classes to (discrete, Abelian) groups

$$(X, d) \mapsto G$$

- Meaning: Within each CAZ class (X, d), its connected components are labelled by elements $g \in G$
- In each case, $G = 0, \mathbb{Z}$, or $\mathbb{Z}_2$. Objects corresponding to
  - $g = 0$ are topologically trivial;
  - $g \neq 0$ are interesting.

The periodic table of topological matter

CAZ Class	Symmetry Θ	Σ	Π	d = 0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	1	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Notation for symmetries $S = \Theta, \Sigma, \Pi$:

- 0 absent, ± 1 present
- (including parity, if applicable)

What to notice about the table

It's actually two tables (complex vs. real classes):

CAZ Class	Symmetry Θ	Σ	Π	d = 0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$

CAZ Class	Θ	Σ	Π	0	1	2	3	4	5	6	7
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	1	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

What to notice about the table

It's actually two tables (complex vs. real classes):

CAZ Class	Symmetry Θ	Σ	Π	d = 0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$

CAZ Class	Θ	Σ	Π	0	1	2	3	4	5	6	7
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	1	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Each table

- repeats periodically in the columns (periods 2, 8)
- has constant entries along diagonals

What to notice about the table

First row of each table repeated at bottom

CAZ Class	Symmetry Θ	Σ	Π	d = 0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

CAZ Class	Θ	Σ	Π	0	1	2	3	4	5	6	7
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	1	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$

What to notice about the table

First row of each table repeated at bottom

CAZ Class	Symmetry Θ	Σ	Π	d = 0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

CAZ Class	Θ	Σ	Π	0	1	2	3	4	5	6	7
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	1	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Tables are periodic in the rows, too

Classes that are physically realized (a selection)

	Symmetry			d							
CAZ Class	Θ	Σ	Π	0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	1	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Classes that are physically realized (a selection)

CAZ Class	Symmetry Θ	Σ	Π	d = 0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	1	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Index $n \in \mathbb{Z}$ is Chern number; Integer Quantum Hall effect and Thouless pumps ($d = 1 + 1$)

Classes that are physically realized

CAZ Class	Symmetry Θ	Σ	Π	d = 0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	1	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Classes that are physically realized

CAZ Class	Symmetry Θ	Σ	Π	d 0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	1	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Index $n \in \mathbb{Z}$ is 2nd Chern number; generalized Thouless pump ($d = 3 + 1$; proposal)

Classes that are physically realized

CAZ Class	Symmetry Θ	Σ	Π	d = 0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	1	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Classes that are physically realized

	Symmetry			d							
CAZ Class	Θ	Σ	Π	0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	1	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Index $n \in \mathbb{Z}$ is chiral; Su-Schrieffer-Heeger model; cavity polaritons

Classes that are physically realized

	Symmetry			*d*							
CAZ Class	Θ	Σ	Π	0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	1	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Classes that are physically realized

	Symmetry			*d*							
CAZ Class	Θ	Σ	Π	0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	1	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Index $s \in \mathbb{Z}_2 = \{0, 1\}$; Kitaev chain; topological phase $s = 1$ has Majorana edge modes

Classes that are physically realized

CAZ Class	Symmetry Θ	Σ	Π	d = 0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	1	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Classes that are physically realized

CAZ Class	Symmetry Θ	Σ	Π	d = 0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	1	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Index $s \in \mathbb{Z}_2 = \{0, 1\}$ is Fu-Kane-Mele index
$d = 2$: Kane-Mele model; HgTe quantum wells
$d = 3$: BiSb alloys

What are the objects being classified?

What are the objects being classified?

Objects are single-particle Hamiltonians H.

What are the objects being classified?

Objects are single-particle Hamiltonians H.

- Physically

What are the objects being classified?

Objects are single-particle Hamiltonians H.

- Physically, H determines a Fermi sea: All single-particle states of energy E below a certain spectral gap at Fermi energy μ are occupied:

What are the objects being classified?

Objects are single-particle Hamiltonians H.

- Physically, H determines a Fermi sea: All single-particle states of energy E below a certain spectral gap at Fermi energy μ are occupied:

- Independent fermions (electrons) move in a periodic lattice in dimension d with N orbitals per site.

What are the objects being classified?

Objects are single-particle Hamiltonians H.

- Physically, H determines a Fermi sea: All single-particle states of energy E below a certain spectral gap at Fermi energy μ are occupied:

- Independent fermions (electrons) move in a periodic lattice in dimension d with N orbitals per site.
- All band below (above) the gap are lumped together.

What are the objects being classified?

Objects are single-particle Hamiltonians H.

- Physically, H determines a Fermi sea: All single-particle states of energy E below a certain spectral gap at Fermi energy μ are occupied:

- Independent fermions (electrons) move in a periodic lattice in dimension d with N orbitals per site.
- All band below (above) the gap are lumped together. W.l.o.g. $\mu = 0$, $E = -1$, ($E = +1$)

What are the objects being classified?

Objects are single-particle Hamiltonians H.

- Physically, H determines a Fermi sea: All single-particle states of energy E below a certain spectral gap at Fermi energy μ are occupied:

- Independent fermions (electrons) move in a periodic lattice in dimension d with N orbitals per site.
- All band below (above) the gap are lumped together. W.l.o.g. $\mu = 0$, $E = -1$, ($E = +1$): bands are flattened.

What are the objects being classified?

Objects are single-particle Hamiltonians H.

- Physically, H determines a Fermi sea: All single-particle states of energy E below a certain spectral gap at Fermi energy μ are occupied:

- Independent fermions (electrons) move in a periodic lattice in dimension d with N orbitals per site.
- All band below (above) the gap are lumped together. W.l.o.g. $\mu = 0$, $E = -1$, ($E = +1$): bands are flattened.

Next: What is the mathematical formulation?

What are those objects being classified?

Objects are single-particle Hamiltonians H.

- Independent fermions move in a periodic lattice (dimension d; N orbitals).
- Bands are flattened to $E = \pm 1$.

What are those objects being classified?

Objects are single-particle Hamiltonians H.

- Independent fermions move in a periodic lattice (dimension d; N orbitals).
- Bands are flattened to $E = \pm 1$.

Corresponding structures:

- Brillouin zone $\mathbb{T}^d \ni k$: Torus of quasi-momenta
- Orbital space $\mathfrak{h} = \mathbb{C}^N$
- Bloch bundle $B = \mathbb{T}^d \times \mathfrak{h}$ (ambient bundle, trivial)
- Hilbert space

$$\mathcal{H} = L^2(\mathbb{T}^d; \mathfrak{h}) = \{\psi \mid \text{sections of } B\}$$

- Hamiltonian is fibered

$$H = \int_{\mathbb{T}^d}^{\oplus} H(k) dk \,, \quad H(k) : \mathfrak{h} \to \mathfrak{h} \,, \quad H(k)^2 = 1$$

- Spectral and bundle decompositions

$$H = P^+ - P^- \,, \quad E_k^{\pm} = \operatorname{ran} P^{\pm}(k) \,, \quad B = E^+ \oplus E^-$$

Symmetry protected Hamiltonians

- Symmetries act on quasi-momenta

$$\mathbb{T}^d \to \mathbb{T}^d\,, \quad k \mapsto \tau k := \begin{cases} -k, & (S = \Theta, \Sigma)\,, \\ k, & (S = \Pi) \end{cases}$$

Symmetry protected Hamiltonians

- Symmetries act on quasi-momenta

$$\mathbb{T}^d \to \mathbb{T}^d\,, \quad k \mapsto \tau k := \begin{cases} -k, & (S = \Theta, \Sigma)\,, \\ k, & (S = \Pi) \end{cases}$$

- Given a combination X of symmetries, picked among Θ, Σ, Π,

Symmetry protected Hamiltonians

- Symmetries act on quasi-momenta

$$\mathbb{T}^d \to \mathbb{T}^d\,, \quad k \mapsto \tau k := \begin{cases} -k, & (S = \Theta, \Sigma)\,, \\ k, & (S = \Pi) \end{cases}$$

- Given a combination X of symmetries, picked among Θ, Σ, Π, the Hamiltonian H is symmetric if

Symmetry protected Hamiltonians

- Symmetries act on quasi-momenta

$$\mathbb{T}^d \to \mathbb{T}^d\,, \quad k \mapsto \tau k := \begin{cases} -k, & (S = \Theta, \Sigma)\,, \\ k, & (S = \Pi) \end{cases}$$

- Given a combination X of symmetries, picked among Θ, Σ, Π, the Hamiltonian H is symmetric if

$$[H, \Theta] = 0\,, \quad \{H, \Sigma\} = 0\,, \quad \{H, \Pi\} = 0$$

Symmetry protected Hamiltonians

- Symmetries act on quasi-momenta

$$\mathbb{T}^d \to \mathbb{T}^d , \quad k \mapsto \tau k := \begin{cases} -k, & (S = \Theta, \Sigma) , \\ k, & (S = \Pi) \end{cases}$$

- Given a combination X of symmetries, picked among Θ, Σ, Π, the Hamiltonian H is symmetric if

$$[H, \Theta] = 0 , \quad \{H, \Sigma\} = 0 , \quad \{H, \Pi\} = 0$$

compatibly with $\Pi = \Theta\Sigma$.

Symmetry protected Hamiltonians

- Symmetries act on quasi-momenta

$$\mathbb{T}^d \to \mathbb{T}^d \,, \quad k \mapsto \tau k := \begin{cases} -k, & (S = \Theta, \Sigma)\,, \\ k, & (S = \Pi) \end{cases}$$

- Given a combination X of symmetries, picked among Θ, Σ, Π, the Hamiltonian H is symmetric if

$$[H, \Theta] = 0\,, \quad \{H, \Sigma\} = 0\,, \quad \{H, \Pi\} = 0$$

compatibly with $\Pi = \Theta\Sigma$. Spelled out:

$$H(-k)\Theta = \Theta H(k)\,, \quad H(-k)\Sigma + \Sigma H(k) = 0\,, \quad H(k)\Pi + \Pi H(k) = 0$$

Symmetry protected Hamiltonians

- Symmetries act on quasi-momenta

$$\mathbb{T}^d \to \mathbb{T}^d\,, \quad k \mapsto \tau k := \begin{cases} -k, & (S = \Theta, \Sigma)\,, \\ k, & (S = \Pi) \end{cases}$$

- Given a combination X of symmetries, picked among Θ, Σ, Π, the Hamiltonian H is symmetric if

$$[H, \Theta] = 0\,, \quad \{H, \Sigma\} = 0\,, \quad \{H, \Pi\} = 0$$

compatibly with $\Pi = \Theta\Sigma$. Spelled out:

$$H(-k)\Theta = \Theta H(k)\,, \quad H(-k)\Sigma + \Sigma H(k) = 0\,, \quad H(k)\Pi + \Pi H(k) = 0$$

- The objects classified by the table are the above Hamiltonians,
  - for one class (X, d) at a time,

Symmetry protected Hamiltonians

- Symmetries act on quasi-momenta

$$\mathbb{T}^d \to \mathbb{T}^d , \quad k \mapsto \tau k := \begin{cases} -k, & (S = \Theta, \Sigma) , \\ k, & (S = \Pi) \end{cases}$$

- Given a combination X of symmetries, picked among Θ, Σ, Π, the Hamiltonian H is symmetric if

$$[H, \Theta] = 0 , \quad \{H, \Sigma\} = 0 , \quad \{H, \Pi\} = 0$$

compatibly with $\Pi = \Theta\Sigma$. Spelled out:

$$H(-k)\Theta = \Theta H(k) , \quad H(-k)\Sigma + \Sigma H(k) = 0 , \quad H(k)\Pi + \Pi H(k) = 0$$

- The objects classified by the table are the above Hamiltonians,
  - for one class (X, d) at a time,
  - up to homotopy,

Symmetry protected Hamiltonians

- Symmetries act on quasi-momenta

$$\mathbb{T}^d \to \mathbb{T}^d \,, \quad k \mapsto \tau k := \begin{cases} -k, & (S = \Theta, \Sigma)\,, \\ k, & (S = \Pi) \end{cases}$$

- Given a combination X of symmetries, picked among Θ, Σ, Π, the Hamiltonian H is symmetric if

$$[H, \Theta] = 0\,, \quad \{H, \Sigma\} = 0\,, \quad \{H, \Pi\} = 0$$

compatibly with $\Pi = \Theta\Sigma$. Spelled out:

$$H(-k)\Theta = \Theta H(k)\,, \quad H(-k)\Sigma + \Sigma H(k) = 0\,, \quad H(k)\Pi + \Pi H(k) = 0$$

- The objects classified by the table are the above Hamiltonians,
  - for one class (X, d) at a time,
  - up to homotopy,
  - and some details to be supplied later.

The periodic table of topological matter

Plan of the proof

Derivation

Rough plan of the proof

Constancy of index groups along diagonals

Rough plan of the proof

Constancy of index groups along diagonals to be explained by diagonal maps (group isomorphisms)

CAZ Class	Symmetry Θ	Σ	Π	d = 0	1	2	3	4	5	6	7
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	1	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Rough plan of the proof

Constancy of index groups along diagonals to be explained by diagonal maps (group isomorphisms)

CAZ Class	Symmetry Θ	Σ	Π	d = 0	1	2	3	4	5	6	7
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	1	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Compute 0th column as induction base.

Rough plan of the proof

Group isomorphisms to be constructed for any pair of adjacent rows

	Symmetry			d							
CAZ Class	Θ	Σ	Π	0	1	2	3	4	5	6	7
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$ ↖	$\mathbb{Z}$ ↖	0 ↖	0 ↖	0 ↖	$\mathbb{Z}$ ↖	0 ↖	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	1	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Rough plan of the proof

Group isomorphisms to be constructed for any pair of adjacent rows

	Symmetry			d							
CAZ Class	Θ	Σ	Π	0	1	2	3	4	5	6	7
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	1	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$

For a given pair of rows, a same construction applies to all d

Rough plan of the proof

Group isomorphisms to be constructed for any pair of adjacent rows

	Symmetry			d							
CAZ Class	Θ	Σ	Π	0	1	2	3	4	5	6	7
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	1	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$

For a given pair of rows, a same construction applies to all d, up to two exceptions (D $\rightarrow$ BDI, C $\rightarrow$ CII).

Time to supply some details I: Minimal objects

Fibered Hamiltonian $H = H(k)$, ($k \in \mathbb{T}^d$) and symmetries $S = \Theta, \Sigma, \Pi$

$$H(-k)\Theta = \Theta H(k)\,, \quad \underline{H(-k)\Sigma = -\Sigma H(k)\,,} \quad H(k)\Pi = -\Pi H(k)$$

Time to supply some details I: Minimal objects

Fibered Hamiltonian $H = H(k)$, $(k \in \mathbb{T}^d)$ and symmetries $S = \Theta, \Sigma, \Pi$

$$H(-k)\Theta = \Theta H(k)\,, \quad \underline{H(-k)\Sigma = -\Sigma H(k)\,,} \quad H(k)\Pi = -\Pi H(k)$$

- Spectral subspaces $E_k^{\pm} = \operatorname{ran} P^{\pm}(k)$ have constant dimension $N^{\pm}$ (gap!), $N^{+} + N^{-} = N$.

Time to supply some details I: Minimal objects

Fibered Hamiltonian $H = H(k)$, ($k \in \mathbb{T}^d$) and symmetries $S = \Theta, \Sigma, \Pi$

$$H(-k)\Theta = \Theta H(k)\,, \quad \underline{H(-k)\Sigma = -\Sigma H(k)\,,} \quad H(k)\Pi = -\Pi H(k)$$

- Spectral subspaces $E_k^\pm = \operatorname{ran} P^\pm(k)$ have constant dimension $N^\pm$ (gap!), $N^+ + N^- = N$.
- For $S = \Sigma, \Pi$, we have $N^+ = N^-$. Not so for $S = \Theta$.

Time to supply some details I: Minimal objects

Fibered Hamiltonian $H = H(k)$, $(k \in \mathbb{T}^d)$ and symmetries $S = \Theta, \Sigma, \Pi$

$$H(-k)\Theta = \Theta H(k)\,, \quad \underline{H(-k)\Sigma = -\Sigma H(k)\,,} \quad H(k)\Pi = -\Pi H(k)$$

- Spectral subspaces $E_k^{\pm} = \operatorname{ran} P^{\pm}(k)$ have constant dimension $N^{\pm}$ (gap!), $N^{+} + N^{-} = N$.
- For $S = \Sigma, \Pi$, we have $N^{+} = N^{-}$. Not so for $S = \Theta$.

For each X there is a minimal, trivial Hamiltonian H_0 in $d = 0$ on $\mathfrak{h} = \mathbb{C}^N$ (extending fiberwise to $d \neq 0$).

Time to supply some details I: Minimal objects

Fibered Hamiltonian $H = H(k)$, ($k \in \mathbb{T}^d$) and symmetries $S = \Theta, \Sigma, \Pi$

$$H(-k)\Theta = \Theta H(k)\,, \quad \underline{H(-k)\Sigma = -\Sigma H(k)\,,} \quad H(k)\Pi = -\Pi H(k)$$

- Spectral subspaces $E_k^{\pm} = \operatorname{ran} P^{\pm}(k)$ have constant dimension $N^{\pm}$ (gap!), $N^+ + N^- = N$.
- For $S = \Sigma, \Pi$, we have $N^+ = N^-$. Not so for $S = \Theta$.

For each X there is a minimal, trivial Hamiltonian H_0 in $d = 0$ on $\mathfrak{h} = \mathbb{C}^N$ (extending fiberwise to $d \neq 0$). (Need for them: next slide)

Time to supply some details I: Minimal objects

Fibered Hamiltonian $H = H(k)$, ($k \in \mathbb{T}^d$) and symmetries $S = \Theta, \Sigma, \Pi$

$$H(-k)\Theta = \Theta H(k)\,, \quad \underline{H(-k)\Sigma = -\Sigma H(k)\,,} \quad H(k)\Pi = -\Pi H(k)$$

- Spectral subspaces $E_k^\pm = \operatorname{ran} P^\pm(k)$ have constant dimension $N^\pm$ (gap!), $N^+ + N^- = N$.
- For $S = \Sigma, \Pi$, we have $N^+ = N^-$. Not so for $S = \Theta$.

For each X there is a minimal, trivial Hamiltonian H_0 in $d = 0$ on $\mathfrak{h} = \mathbb{C}^N$ (extending fiberwise to $d \neq 0$). Examples:

- For $X =$ A (no symmetry)

$$H_0 = \sigma_3\,, \qquad (N = 2, N^\pm = 1)$$

Time to supply some details I: Minimal objects

- Spectral subspaces $E_k^{\pm} = \operatorname{ran} P^{\pm}(k)$ have constant dimension $N^{\pm}$ (gap!), $N^{+} + N^{-} = N$.
- For $S = \Sigma, \Pi$, we have $N^{+} = N^{-}$. Not so for $S = \Theta$.

For each X there is a minimal, trivial Hamiltonian H_0 in $d = 0$ on $\mathfrak{h} = \mathbb{C}^N$ (extending fiberwise to $d \neq 0$). Examples:

- For $X =$ A (no symmetry)

$$H_0 = \sigma_3 \,, \qquad (N = 2, N^{\pm} = 1)$$

- For $X =$ AIII (just $S = \Pi$)

$$\Pi = \sigma_3 \,, \quad H_0 = \sigma_1 \,, \quad (N = 2, N^{\pm} = 1)$$

Time to supply some details I: Minimal objects

- Spectral subspaces $E_k^\pm = \operatorname{ran} P^\pm(k)$ have constant dimension $N^\pm$ (gap!), $N^+ + N^- = N$.
- For $S = \Sigma, \Pi$, we have $N^+ = N^-$. Not so for $S = \Theta$.

For each X there is a minimal, trivial Hamiltonian H_0 in $d = 0$ on $\mathfrak{h} = \mathbb{C}^N$ (extending fiberwise to $d \neq 0$). Examples:

- For $X = \mathrm{A}$ (no symmetry)

$$H_0 = \sigma_3\,, \qquad (N = 2, N^\pm = 1)$$

- For $X = \mathrm{AIII}$ (just $S = \Pi$)

$$\Pi = \sigma_3\,, \quad H_0 = \sigma_1\,, \quad (N = 2, N^\pm = 1)$$

In general: For each class (X, d), there is

- a normal form of its symmetries
- a trivial (minimal, but non-zero) Hamiltonian H_0

Time to supply some details II: Equivalence

- Let $C(\mathbb{T}^d, M_N(\mathbb{C}))$ be the continuous functions $k \mapsto H(k)$ taking values in the matrices of order N
- For each class X, let

$$M(X, d, N) := \{H \in C(\mathbb{T}^d, M_N(\mathbb{C})) \mid H \text{ Hamiltonian of class } X\},$$

$$(X, d) := \bigsqcup_{N}^{\infty} M(X, d, N).$$

Define an equivalence rel. $\sim$ by homotopy and stacking by a trivial Hamiltonian H_0,

$$H \sim H \oplus H_0.$$

Time to supply some details II: Equivalence

- Let $C(\mathbb{T}^d, M_N(\mathbb{C}))$ be the continuous functions $k \mapsto H(k)$ taking values in the matrices of order N
- For each class X, let

$$M(X, d, N) := \{H \in C(\mathbb{T}^d, M_N(\mathbb{C})) \mid H \text{ Hamiltonian of class } X\},$$

$$(X, d) := \bigsqcup_{N}^{\infty} M(X, d, N).$$

Define an equivalence rel. $\sim$ by homotopy and stacking by a trivial Hamiltonian H_0,

$$H \sim H \oplus H_0.$$

Homotopy works at fixed N, stacking works between N's.

Time to supply some details II: Equivalence

- Let $C(\mathbb{T}^d, M_N(\mathbb{C}))$ be the continuous functions $k \mapsto H(k)$ taking values in the matrices of order N
- For each class X, let

$$M(X, d, N) := \{H \in C(\mathbb{T}^d, M_N(\mathbb{C})) \mid H \text{ Hamiltonian of class } X\},$$

$$(X, d) := \bigsqcup_{N}^{\infty} M(X, d, N).$$

  Define an equivalence rel. $\sim$ by homotopy and stacking by a trivial Hamiltonian H_0,

$$H \sim H \oplus H_0.$$

  Homotopy works at fixed N, stacking works between N's.
- Rationale for stacking: For $(X, d) =$(A, 0)

$$H_0 = \sigma_3$$

Time to supply some details II: Equivalence

- Let $C(\mathbb{T}^d, M_N(\mathbb{C}))$ be the continuous functions $k \mapsto H(k)$ taking values in the matrices of order N
- For each class X, let

$$M(X, d, N) := \{H \in C(\mathbb{T}^d, M_N(\mathbb{C})) \mid H \text{ Hamiltonian of class } X\},$$

$$(X, d) := \bigsqcup_{N}^{\infty} M(X, d, N).$$

Define an equivalence rel. $\sim$ by homotopy and stacking by a trivial Hamiltonian H_0,

$$H \sim H \oplus H_0.$$

Homotopy works at fixed N, stacking works between N's.
- Rationale for stacking: For $(X, d) =$(A, 0)

$$H_0 = \sigma_3$$

is homotopic to $-H_0 = -\sigma_3$

Time to supply some details II: Equivalence

- Let $C(\mathbb{T}^d, M_N(\mathbb{C}))$ be the continuous functions $k \mapsto H(k)$ taking values in the matrices of order N
- For each class X, let

$$M(X, d, N) := \{H \in C(\mathbb{T}^d, M_N(\mathbb{C})) \mid H \text{ Hamiltonian of class } X\} ,$$

$$(X, d) := \bigsqcup_N^\infty M(X, d, N) .$$

Define an equivalence rel. $\sim$ by homotopy and stacking by a trivial Hamiltonian H_0,

$$H \sim H \oplus H_0 .$$

Homotopy works at fixed N, stacking works between N's.
- Rationale for stacking: For $(X, d) =$(A, 0)

$$H_0 = \sigma_3$$

is homotopic to $-H_0 = -\sigma_3$, so its equivalence class is

Time to supply some details II: Equivalence

- Let $C(\mathbb{T}^d, M_N(\mathbb{C}))$ be the continuous functions $k \mapsto H(k)$ taking values in the matrices of order N
- For each class X, let

$$M(X, d, N) := \{H \in C(\mathbb{T}^d, M_N(\mathbb{C})) \mid H \text{ Hamiltonian of class } X\} ,$$

$$(X, d) := \bigsqcup_{N}^{\infty} M(X, d, N) .$$

  Define an equivalence rel. $\sim$ by homotopy and stacking by a trivial Hamiltonian H_0,

$$H \sim H \oplus H_0 .$$

  Homotopy works at fixed N, stacking works between N's.
- Rationale for stacking: For $(X, d) =$(A, 0)

$$H_0 = \sigma_3$$

  is homotopic to $-H_0 = -\sigma_3$, so its equivalence class is

$$[H_0] = 0$$

  as $[-H_0] = -[H_0]$ is wished for. Despite H_0 not homotopic to 0.

Time to supply some details II: Equivalence

- Let $C(\mathbb{T}^d, M_N(\mathbb{C}))$ be the continuous functions $k \mapsto H(k)$ taking values in the matrices of order N
- For each class X, let

$$M(X,d,N) := \{H \in C(\mathbb{T}^d, M_N(\mathbb{C})) \mid H \text{ Hamiltonian of class } X\},$$

$$(X,d) := \bigsqcup_N^\infty M(X,d,N).$$

Define an equivalence rel. $\sim$ by homotopy and stacking by a trivial Hamiltonian H_0,

$$H \sim H \oplus H_0.$$

Homotopy works at fixed N, stacking works between N's.

- Rationale for stacking: For (X,d) =(A, 0)

$$H_0 = \sigma_3$$

is homotopic to $-H_0 = -\sigma_3$, so its equivalence class is

$$[H_0] = 0$$

as $[-H_0] = -[H_0]$ is wished for. Despite H_0 not homotopic to 0.

Time to supply some details II: Equivalence

- Let $C(\mathbb{T}^d, M_N(\mathbb{C}))$ be the continuous functions $k \mapsto H(k)$ taking values in the matrices of order N
- For each class X, let

$$M(X, d, N) := \{H \in C(\mathbb{T}^d, M_N(\mathbb{C})) \mid H \text{ Hamiltonian of class } X\},$$

$$(X, d) := \bigsqcup_N^\infty M(X, d, N).$$

  Define an equivalence rel. $\sim$ by homotopy and stacking by a trivial Hamiltonian H_0,

$$H \sim H \oplus H_0 .$$

  Homotopy works at fixed N, stacking works between N's.
- Rationale for stacking: ✓ (most classes)
- We obtain

$$[X, d] := (X, d)/\sim$$

  as an Abelian group.

Time to supply some details II: Equivalence

- Let $C(\mathbb{T}^d, M_N(\mathbb{C}))$ be the continuous functions $k \mapsto H(k)$ taking values in the matrices of order N
- For each class X, let

$$M(X, d, N) := \{H \in C(\mathbb{T}^d, M_N(\mathbb{C})) \mid H \text{ Hamiltonian of class } X\},$$

$$(X, d) := \bigsqcup_N^{\infty} M(X, d, N).$$

  Define an equivalence rel. $\sim$ by homotopy and stacking by a trivial Hamiltonian H_0,

$$H \sim H \oplus H_0 .$$

  Homotopy works at fixed N, stacking works between N's.
- Rationale for stacking: ✓ (most classes)
- We obtain

$$[X, d] := (X, d)/\sim$$

  as an Abelian group. It is the group G shown in the table.

Reformulations of objects I: As bundles

Spectral decomposition of Hamiltonian H and bundle decomposition of ambient bundle B

$$H = P^+ - P^- , \quad \underline{E_k^\pm = \operatorname{ran} P^\pm(k)} , \quad B = E^+ \oplus E^-$$

Reformulations of objects I: As bundles

Spectral decomposition of Hamiltonian H and bundle decomposition of ambient bundle B

$$H = P^+ - P^- \,, \quad E_k^{\pm} = \operatorname{ran} P^{\pm}(k)\,, \quad B = E^+ \oplus E^-$$

Bundle decomposition as an equivariant bundle:

$$\begin{array}{ccc} E^{\pm} & \xrightarrow{S} & E^{\pm s} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathbb{T}^d & \xrightarrow{\tau} & \mathbb{T}^d \end{array} \qquad s = \begin{cases} +1, & (S = \Theta)\,, \\ -1, & (S = \Sigma, \Pi)\,. \end{cases}$$

Reformulations of objects I: As bundles

Spectral decomposition of Hamiltonian H and bundle decomposition of ambient bundle B

$$H = P^{+} - P^{-}\,, \quad E_k^{\pm} = \operatorname{ran} P^{\pm}(k)\,, \quad B = E^{+} \oplus E^{-}$$

Bundle decomposition as an equivariant bundle:

$$\begin{array}{ccc} E^{\pm} & \xrightarrow{S} & E^{\pm s} \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{T}^d & \xrightarrow{\tau} & \mathbb{T}^d \end{array} \qquad s = \begin{cases} +1, & (S = \Theta)\,, \\ -1, & (S = \Sigma, \Pi)\,. \end{cases}$$

Thus, Hamiltonians are reformulated

$$H \leftrightarrow E^{+} \oplus E^{-} \leftrightarrow (P^{+}, P^{-})$$

Reformulations of objects I: As bundles

Spectral decomposition of Hamiltonian H and bundle decomposition of ambient bundle B

$$H = P^+ - P^- \,, \quad \underline{E_k^\pm = \mathrm{ran}\, P^\pm(k)\,,} \quad B = E^+ \oplus E^-$$

Bundle decomposition as an equivariant bundle:

$$\begin{array}{ccc} E^\pm & \xrightarrow{S} & E^{\pm s} \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{T}^d & \xrightarrow{\tau} & \mathbb{T}^d \end{array} \qquad s = \begin{cases} +1, & (S = \Theta)\,, \\ -1, & (S = \Sigma, \Pi)\,. \end{cases}$$

Thus, Hamiltonians are reformulated

$$H \leftrightarrow E^+ \oplus E^- \leftrightarrow (P^+, P^-)$$

- Equivalence of Hamiltonians is passed down

Reformulations of objects I: As bundles

Spectral decomposition of Hamiltonian H and bundle decomposition of ambient bundle B

$$H = P^+ - P^- \,, \quad E_k^\pm = \mathrm{ran}\, P^\pm(k) \,, \quad B = E^+ \oplus E^-$$

Bundle decomposition as an equivariant bundle:

$$\begin{array}{ccc} E^\pm & \xrightarrow{S} & E^{\pm s} \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{T}^d & \xrightarrow{\tau} & \mathbb{T}^d \end{array} \qquad s = \begin{cases} +1, & (S = \Theta)\,, \\ -1, & (S = \Sigma, \Pi)\,. \end{cases}$$

Thus, Hamiltonians are reformulated

$$H \leftrightarrow E^+ \oplus E^- \leftrightarrow (P^+, P^-)$$

- Equivalence of Hamiltonians is passed down
- For $S = \varnothing, \Theta$ (no or lone symmetry), bundle E^- ($\leftrightarrow P^-$) is trivial (w.l.o.g.), so

$$(P^+, P^-) \sim (P, 1)$$

Reformulations of objects II: As unitaries

Chiral classes X contain the symmetry Π.

Reformulations of objects II: As unitaries

Chiral classes X contain the symmetry Π. Every second class is chiral:

	Symmetry			d							
CAZ Class	Θ	Σ	Π	0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	1	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Reformulations of objects II: As unitaries

Chiral classes X contain the symmetry Π.

$$\Pi = \sigma_3 \otimes 1\,, \quad H = \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix}$$

with U unitary, and likewise fibered $U = U(k)$, ($k \in \mathbb{T}^d$).

Reformulations of objects II: As unitaries

Chiral classes X contain the symmetry Π.

$$\Pi = \sigma_3 \otimes 1\,, \quad H = \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix}$$

with U unitary, and likewise fibered $U = U(k)$, ($k \in \mathbb{T}^d$). So

$$H \leftrightarrow U$$

Time to supply some details III: Strong indices

- There are strong and weak indices.

Time to supply some details III: Strong indices

- There are strong and weak indices.
- Weak indices can be detected from the restriction of H (equivalently of E, U) to some sub-torus of dimension $d-1$

$$\mathbb{T}_i^{d-1} = \{k = (k_1, \ldots, k_d) \in \mathbb{T}^d \mid k_i = 0\}$$

Time to supply some details III: Strong indices

- There are strong and weak indices.
- Weak indices can be detected from the restriction of H (equivalently of E, U) to some sub-torus of dimension $d-1$

$$\mathbb{T}_i^{d-1} = \{k = (k_1, \ldots, k_d) \in \mathbb{T}^d \mid k_i = 0\}$$

- We consider objects that are weakly trivial. Focus on strong indices.

Time to supply some details III: Strong indices

- There are strong and weak indices.
- Weak indices can be detected from the restriction of H (equivalently of E, U) to some sub-torus of dimension $d-1$

$$\mathbb{T}_i^{d-1} = \{k = (k_1, \ldots, k_d) \in \mathbb{T}^d \mid k_i = 0\}$$

- We consider objects that are weakly trivial. Focus on strong indices.
- An analogy (d as number of rings):

Time to supply some details III: Strong indices

- There are strong and weak indices.
- Weak indices can be detected from the restriction of H (equivalently of E, U) to some sub-torus of dimension $d-1$

$$\mathbb{T}_i^{d-1} = \{k = (k_1, \ldots, k_d) \in \mathbb{T}^d \mid k_i = 0\}$$

- We consider objects that are weakly trivial. Focus on strong indices.
- An analogy (d as number of rings): The three Borromean rings are entangled, but any two of them are not:

Time to supply some details III: Strong indices

- There are strong and weak indices.
- Weak indices can be detected from the restriction of H (equivalently of E, U) to some sub-torus of dimension $d-1$

$$\mathbb{T}_i^{d-1} = \{k = (k_1, \ldots, k_d) \in \mathbb{T}^d \mid k_i = 0\}$$

- We consider objects that are weakly trivial. Focus on strong indices.
- An analogy (d as number of rings): The three Borromean rings are entangled, but any two of them are not:

"Their only index is strong."

The periodic table of topological matter

Plan of the proof

Derivation

Complex classes

Symmetry		d							
CAZ Class	Π	0	1	2	3	4	5	6	7
A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

Complex classes

Symmetry		d							
CAZ Class	Π	0	1	2	3	4	5	6	7
A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

Derivation will serve as blueprint for real classes.

Complex classes: From non-chiral to chiral classes

Symmetry		d							
CAZ Class	Π	0	1	2	3	4	5	6	7
A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

Need isomorphism

$$(P, 1) \mapsto U$$

Complex classes: From non-chiral to chiral classes

Symmetry		d							
CAZ Class	Π	0	1	2	3	4	5	6	7
A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

Need isomorphism

$$(P, 1) \mapsto U$$

Equivalently, $E \mapsto U$ ($E_k = \operatorname{ran} P(k)$)

Complex classes: From non-chiral to chiral classes

Symmetry		d							
CAZ Class	Π	0	1	2	3	4	5	6	7
A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
A	0	$\mathbb{Z}$ $\nwarrow$	0 $\nwarrow$	$\mathbb{Z}$ $\nwarrow$	0 $\nwarrow$	$\mathbb{Z}$ $\nwarrow$	0 $\nwarrow$	$\mathbb{Z}$ $\nwarrow$	0

Need isomorphism

$$(P, 1) \mapsto U$$

Equivalently, $E \mapsto U$ ($E_k = \operatorname{ran} P(k)$)

(We'll give definition of map, not proof of isomorphism)

The idea you all know (analogy)

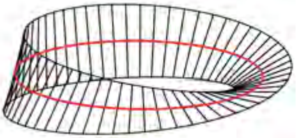
Real line bundles over the circle ($k \in \mathbb{R} \bmod 2\pi\mathbb{Z}$).

The idea you all know (analogy)

Real line bundles over the circle ($k \in \mathbb{R} \bmod 2\pi\mathbb{Z}$). Two kinds thereof:

The idea you all know (analogy)

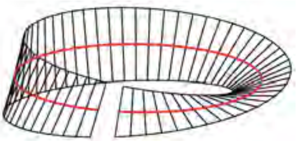
Real line bundles over the circle ($k \in \mathbb{R} \bmod 2\pi\mathbb{Z}$). Two kinds thereof:



How to tell them apart?

The idea you all know (analogy)

Real line bundles over the circle ($k \in \mathbb{R} \bmod 2\pi\mathbb{Z}$). Two kinds thereof:

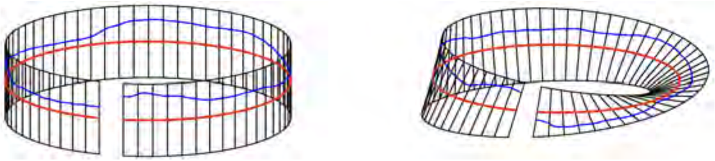


How to tell them apart?

- Cut.

The idea you all know (analogy)

Real line bundles over the circle ($k \in \mathbb{R} \bmod 2\pi\mathbb{Z}$). Two kinds thereof:

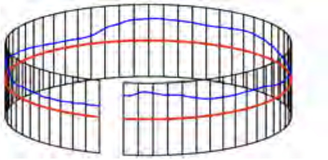


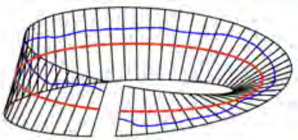
How to tell them apart?

- Cut. Then draw a continuous section V_k never crossing the circle; relate its restrictions V_π and $V_{-\pi}$ at endpoints by a transition scalar $T \neq 0$: $V_\pi = V_{-\pi} T$.

The idea you all know (analogy)

Real line bundles over the circle ($k \in \mathbb{R} \bmod 2\pi\mathbb{Z}$). Two kinds thereof:



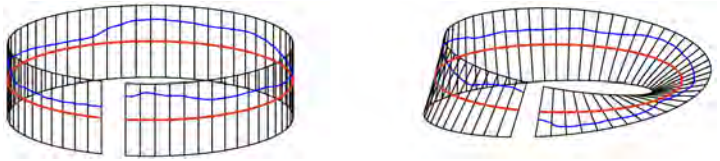


How to tell them apart?

- Cut. Then draw a continuous section V_k never crossing the circle; relate its restrictions V_π and $V_{-\pi}$ at endpoints by a transition scalar $T \neq 0$: $V_\pi = V_{-\pi} T$.
- If untwisted, $T > 0$; if twisted, $T < 0$.

The idea you all know (analogy)

Real line bundles over the circle ($k \in \mathbb{R} \bmod 2\pi\mathbb{Z}$). Two kinds thereof:

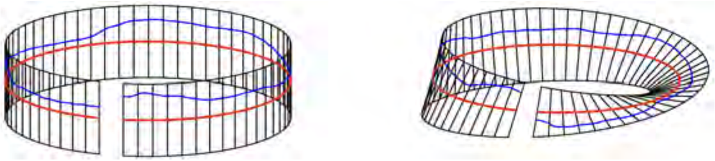


How to tell them apart?

- Cut. Then draw a continuous section V_k never crossing the circle; relate its restrictions V_π and $V_{-\pi}$ at endpoints by a transition scalar $T \neq 0$: $V_\pi = V_{-\pi}T$.
- If untwisted, $T > 0$; if twisted, $T < 0$. Yields $\mathbb{Z}_2$ index, since $\mathbb{R} \setminus \{0\}$ has two connected components.

The idea you all know (analogy)

Real line bundles over the circle ($k \in \mathbb{R} \bmod 2\pi\mathbb{Z}$). Two kinds thereof:



How to tell them apart?

- Cut. Then draw a continuous section V_k never crossing the circle; relate its restrictions V_π and $V_{-\pi}$ at endpoints by a transition scalar $T \neq 0$: $V_\pi = V_{-\pi} T$.
- If untwisted, $T > 0$; if twisted, $T < 0$. Yields $\mathbb{Z}_2$ index, since $\mathbb{R} \setminus \{0\}$ has two connected components.
- For a complex line bundle: No index, since $\mathbb{C} \setminus \{0\}$ is connected; cf. [A,$d = 1$] $= 0$.

The diagonal maps from A to AIII

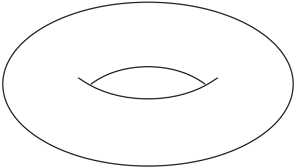
Symmetry		d							
CAZ Class	Π	0	1	2	3	4	5	6	7
A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

Need map $E \mapsto U$

The diagonal maps from A to AIII

Symmetry		*d*							
CAZ Class	Π	0	1	2	3	4	5	6	7
A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

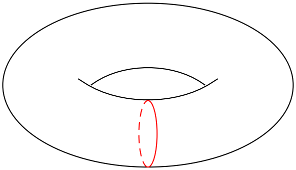
Need map $E \mapsto U$



The diagonal maps from A to AIII

Symmetry		d							
CAZ Class	Π	0	1	2	3	4	5	6	7
A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

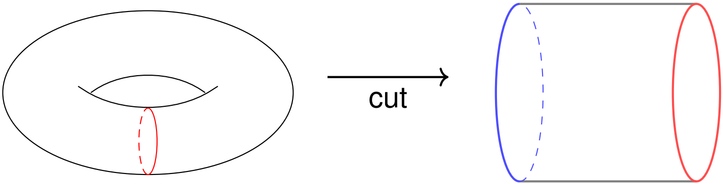
Need map $E \mapsto U$



The diagonal maps from A to AIII

Symmetry		d							
CAZ Class	Π	0	1	2	3	4	5	6	7
A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

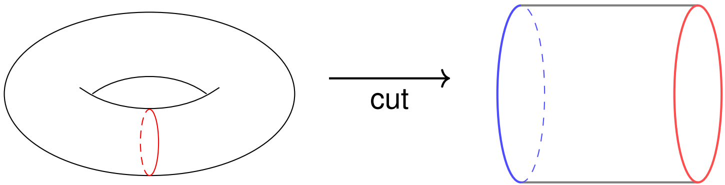
Need map $E \mapsto U$



The diagonal maps from A to AIII

Symmetry		d							
CAZ Class	Π	0	1	2	3	4	5	6	7
A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

Need map $E \mapsto U$

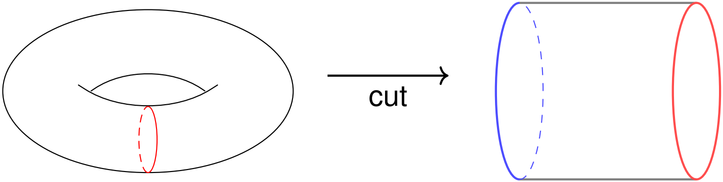


1. **Cut:** View torus $\mathbb{T}^d$ as cylinder $\dot{\mathbb{T}}^d$ with identified ends

The diagonal maps from A to AIII

Symmetry		d							
CAZ Class	Π	0	1	2	3	4	5	6	7
A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

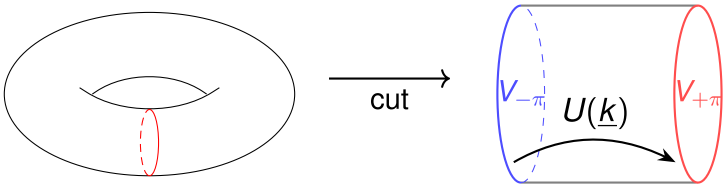
Need map $E \mapsto U$



1. **Cut:** View torus as cylinder with identified ends: $\mathbb{T}^d \cong \dot{\mathbb{T}}^d / \sim$.

The diagonal maps from A to AIII

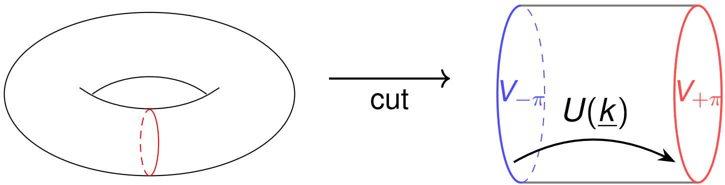
Need map $E \mapsto U$



1. **Cut:** View torus as cylinder with identified ends: $\mathbb{T}^d \cong \dot{\mathbb{T}}^d / \sim$.
   Let $E \to \mathbb{T}^d$ and $\dot{E} \to \dot{\mathbb{T}}^d$ the bundles before and after cutting.

The diagonal maps from A to AIII

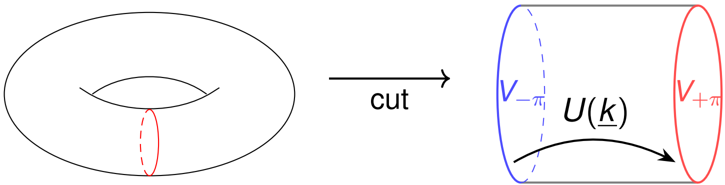
Need map $E \mapsto U$



1. **Cut:** View torus as cylinder with identified ends: $\mathbb{T}^d \cong \dot{\mathbb{T}}^d / \sim$.
   Let $E \to \mathbb{T}^d$ and $\dot{E} \to \dot{\mathbb{T}}^d$ the bundles before and after cutting.
2. **Global frame:** There is a frame V of $\dot{E}$

The diagonal maps from A to AIII

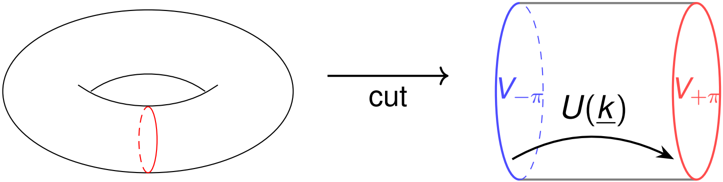
Need map $E \mapsto U$



1. **Cut:** View torus as cylinder with identified ends: $\mathbb{T}^d \cong \dot{\mathbb{T}}^d / \sim$. Let $E \to \mathbb{T}^d$ and $\dot{E} \to \dot{\mathbb{T}}^d$ the bundles before and after cutting.
2. **Global frame:** There is a frame V of $\dot{E}$ (by: no weak index; but not extending to a frame of E, as a rule).

The diagonal maps from A to AIII

Need map $E \mapsto U$



1. **Cut:** View torus as cylinder with identified ends: $\mathbb{T}^d \cong \dot{\mathbb{T}}^d / \sim$. Let $E \to \mathbb{T}^d$ and $\dot{E} \to \dot{\mathbb{T}}^d$ the bundles before and after cutting.
2. **Global frame:** There is a frame V of $\dot{E}$ (by: no weak index; but not extending to a frame of E, as a rule).
3. **Clutching:** $k = (\underline{k}, k_d) \in \mathbb{T}^{d-1} \times [-\pi, \pi]$. The restriction of V at $k_d = +\pi$ differs from $k_d = -\pi$ by a unitary $U(\underline{k})$

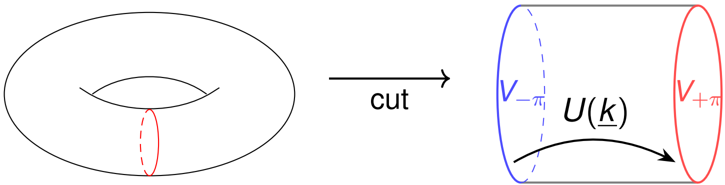
$$V(\underline{k}, +\pi) = V(\underline{k}, -\pi)U(\underline{k})$$

The transition map is

$$U : \mathbb{T}^{d-1} \to U(N)\,, \quad \underline{k} \mapsto U(\underline{k})\,.$$

The diagonal maps from A to AIII

Need map $E \mapsto U$



1. **Cut:** View torus as cylinder with identified ends: $\mathbb{T}^d \cong \dot{\mathbb{T}}^d / \sim$. Let $E \to \mathbb{T}^d$ and $\dot{E} \to \dot{\mathbb{T}}^d$ the bundles before and after cutting.
2. **Global frame:** There is a frame V of $\dot{E}$ (by: no weak index; but not extending to a frame of E, as a rule).
3. **Clutching:** $k = (\underline{k}, k_d) \in \mathbb{T}^{d-1} \times [-\pi, \pi]$. The restriction of V at $k_d = +\pi$ differs from $k_d = -\pi$ by a unitary $U(\underline{k})$

   $$V(\underline{k}, +\pi) = V(\underline{k}, -\pi)U(\underline{k})$$

   The transition map is

   $$U : \mathbb{T}^{d-1} \to U(N)\,, \quad \underline{k} \mapsto U(\underline{k})\,.$$
4. This is the sought unitary U in $E \mapsto U$

Complex classes: From chiral to non-chiral classes

Symmetry		d							
CAZ Class	Π	0	1	2	3	4	5	6	7
A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

Complex classes: From chiral to non-chiral classes

Symmetry		d							
CAZ Class	Π	0	1	2	3	4	5	6	7
A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

- Need map

$$U \mapsto (P, 1)$$

Complex classes: From chiral to non-chiral classes

Symmetry		d							
CAZ Class	Π	0	1	2	3	4	5	6	7
A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

- Need map

$$U \mapsto (P, 1)$$

- Relax to $U \in GL(N)$ and to P a non-orthogonal projection.

Complex classes: From chiral to non-chiral classes

Symmetry CAZ Class	Π	d = 0	1	2	3	4	5	6	7
A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0

- Need map

$$U \mapsto (P, 1)$$

- Relax to $U \in GL(N)$ and to P a non-orthogonal projection.
- In $d = 1$: Replace $k \in S^1$ by $z = \mathrm{e}^{\mathrm{i}k}$
- In $d > 1$: Do the same for k_d in $k = (\underline{k}, k_d) \in \mathbb{T}^d = \mathbb{T}^{d-1} \times S^1$

The diagonal maps from AIII to A

Step 1. From $d = 1$ to $d = 0$: Given an affine linear map $U : S^1 \to GL(N)$, $z \mapsto U(z)$, define a projection P by the integral

$$P := \frac{1}{2\pi \mathrm{i}} \oint_{S^1} U(z)^{-1} \partial_z U(z) dz \,.$$

The diagonal maps from AIII to A

Step 1. From $d = 1$ to $d = 0$: Given an affine linear map $U : S^1 \to GL(N)$, $z \mapsto U(z)$, define a projection P by the integral

$$P := \frac{1}{2\pi \mathrm{i}} \oint_{S^1} U(z)^{-1} \partial_z U(z) dz \,.$$

- By Cauchy's integral formula: $P^2 = P$. Well-known for $U(z) = z - H$, $\partial_z U(z) = 1$ (Riesz projector)

The diagonal maps from AIII to A

Step 1. From $d = 1$ to $d = 0$: Given an affine linear map $U : S^1 \to GL(N)$, $z \mapsto U(z)$, define a projection P by the integral

$$P := \frac{1}{2\pi \mathrm{i}} \oint_{S^1} U(z)^{-1} \partial_z U(z) dz \,.$$

- By Cauchy's integral formula: $P^2 = P$. Well-known for $U(z) = z - H$, $\partial_z U(z) = 1$ (Riesz projector)
- Generalized to arbitrary d as $U(\underline{k}, z) \mapsto P(\underline{k})$, where $\underline{k} \in \mathbb{T}^{d-1}$ are bystanders.

The diagonal map from AIII to A (continued)

Step 2. Given a polynomial map $U : S^1 \to GL(N)$, with $U(z) = u_0 + u_1 z + \ldots + u_n z^n$, ($u_i \in M_N(\mathbb{C})$), we define its *linearization* $L^{(n)}U$ by

$$L^{(n)}U := \begin{pmatrix} u_0 & u_1 & u_2 & \cdots & u_n \\ -z & \mathbb{1} & 0 & \cdots & 0 \\ 0 & -z & \mathbb{1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -z & \mathbb{1} \end{pmatrix} .$$

The diagonal map from AIII to A (continued)

Step 2. Given a polynomial map $U : S^1 \to GL(N)$, with $U(z) = u_0 + u_1 z + \ldots + u_n z^n$, $(u_i \in M_N(\mathbb{C}))$, we define its *linearization* $L^{(n)}U$ by

$$L^{(n)}U := \begin{pmatrix} u_0 & u_1 & u_2 & \cdots & u_n \\ -z & \mathbb{1} & 0 & \cdots & 0 \\ 0 & -z & \mathbb{1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -z & \mathbb{1} \end{pmatrix} .$$

Remark: $L^{(n)}U$ is homotopic to

$$\begin{pmatrix} U(z) & 0 & 0 & \cdots & 0 \\ 0 & \mathbb{1} & 0 & \cdots & 0 \\ 0 & 0 & \mathbb{1} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \mathbb{1} \end{pmatrix} = U \oplus \mathbb{1}_n ,$$

which is equal to U by stacking. Thus, we can use **step 1**.

The diagonal map from AIII to A (continued)

Step 2. Given a polynomial map $U : S^1 \to GL(N)$, with $U(z) = u_0 + u_1 z + \ldots + u_n z^n$, ($u_i \in M_N(\mathbb{C})$), we define its *linearization* $L^{(n)}U$ by

$$L^{(n)}U := \begin{pmatrix} u_0 & u_1 & u_2 & \cdots & u_n \\ -z & \mathbb{1} & 0 & \cdots & 0 \\ 0 & -z & \mathbb{1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -z & \mathbb{1} \end{pmatrix}.$$

Remark: $L^{(n)}U$ is homotopic to

$$\begin{pmatrix} U(z) & 0 & 0 & \cdots & 0 \\ 0 & \mathbb{1} & 0 & \cdots & 0 \\ 0 & 0 & \mathbb{1} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \mathbb{1} \end{pmatrix} = U \oplus \mathbb{1}_n,$$

which is equal to U by stacking. Thus, we can use **step 1**.

Example: $U(z) = \mathbb{1} z^n$ maps to $P = \mathbb{1}_n \oplus 0_1 \sim \mathbb{1}_n$

The diagonal map from AIII to A (continued)

Step 3. Any continuous map $U : \mathbb{T}^d \to GL(N)$ is approximated by a Laurent polynomial:

$$U(\underline{k}, z) \approx \hat{U}(\underline{k}, z) z^{-n} \,,$$

where $\hat{U}$ is a polynomial in z and $z^{-n} = (z^n)^{-1}$

The diagonal map from AIII to A (continued)

Step 3. Any continuous map $U : \mathbb{T}^d \to GL(N)$ is approximated by a Laurent polynomial:

$$U(\underline{k}, z) \approx \hat{U}(\underline{k}, z)z^{-n} ,$$

where $\hat{U}$ is a polynomial in z and $z^{-n} = (z^n)^{-1}$

Thus

- $\hat{U} \mapsto P$ by Steps 1, 2.
- $z^n \mapsto \mathbb{1}_n$ by the Example
- So $\hat{U}(\cdot, z)z^{-n} \mapsto (P(\cdot), \mathbb{1}_n)$

That defines the map $U \mapsto (P, 1)$, as required.

The diagonal map from AIII to A (continued)

Step 3. Any continuous map $U : \mathbb{T}^d \to GL(N)$ is approximated by a Laurent polynomial:

$$U(\underline{k}, z) \approx \hat{U}(\underline{k}, z) z^{-n},$$

where $\hat{U}$ is a polynomial in z and $z^{-n} = (z^n)^{-1}$

Thus

- $\hat{U} \mapsto P$ by Steps 1, 2.
- $z^n \mapsto \mathbb{1}_n$ by the Example
- So $\hat{U}(\cdot, z) z^{-n} \mapsto (P(\cdot), \mathbb{1}_n)$

That defines the map $U \mapsto (P, 1)$, as required.

It also completes the discussion of complex classes.

Real classes: From chiral to non-chiral classes

4 pairs (chiral $\to$ non-chiral), grouped into 2 cases

Real classes: From chiral to non-chiral classes

- First case: BDI $\to$ AI and CII $\to$ AII

CAZ Class	Symmetry Θ	Σ	Π	d = 0	1	2	3	4	5	6	7
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	1	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Real classes: From chiral to non-chiral classes

- First case: BDI $\to$ AI and CII $\to$ AII

	Symmetry			d							
CAZ Class	Θ	Σ	Π	0	1	2	3	4	5	6	7
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	1	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Classes AI and AII have lone symmetry Θ, $(\sigma_\Theta = \pm 1)$

Real classes: From chiral to non-chiral classes

- First case: BDI $\to$ AI and CII $\to$ AII, while $d \to d - 1$

Real classes: From chiral to non-chiral classes

- First case: BDI $\rightarrow$ AI and CII $\rightarrow$ AII, while $d \rightarrow d-1$

Normal forms for Θ

$$\Theta = \begin{cases} K, \ (\sigma_\Theta = +1) \\ \varepsilon K, \ (\sigma_\Theta = -1) \end{cases} , \ \varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \quad K = \text{complex conjugation}$$

Real classes: From chiral to non-chiral classes

- First case: BDI $\to$ AI and CII $\to$ AII, while $d \to d-1$

Normal forms for Θ

$$\Theta = \begin{cases} K, \ (\sigma_\Theta = +1) \\ \varepsilon K, \ (\sigma_\Theta = -1) \end{cases}, \quad \varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad K = \text{complex conjugation}$$

yield

BDI	$U(k) = \overline{U(-k)}$
CII	$U(k)\varepsilon = \varepsilon\overline{U(-k)}$

AI	$P^\pm(k) = \overline{P^\pm(-k)}$
AII	$P^\pm(k)\varepsilon = \varepsilon\overline{P^\pm(-k)}$

yet with $k \in \mathbb{T}^d$ (left), $k \in \mathbb{T}^{d-1}$ (right)

Real classes: From chiral to non-chiral classes

- First case: BDI $\to$ AI and CII $\to$ AII, while $d \to d-1$

Normal forms for Θ

$$\Theta = \begin{cases} K, \ (\sigma_\Theta = +1) \\ \varepsilon K, \ (\sigma_\Theta = -1) \end{cases}, \ \varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad K = \text{complex conjugation}$$

yield

BDI	$U(\underline{k}, z) = \overline{U(-\underline{k}, \bar{z})}$
CII	$U(\underline{k}, z)\varepsilon = \varepsilon\overline{U(-\underline{k}, \bar{z})}$

AI	$P^{\pm}(\underline{k}) = \overline{P^{\pm}(-\underline{k})}$
AII	$P^{\pm}(\underline{k})\varepsilon = \varepsilon\overline{P^{\pm}(-\underline{k})}$

with $k = (\underline{k}, z)$

Real classes: From chiral to non-chiral classes

- First case: BDI $\to$ AI and CII $\to$ AII, while $d \to d-1$

Normal forms for Θ

$$\Theta = \begin{cases} K, \ (\sigma_\Theta = +1) \\ \varepsilon K, \ (\sigma_\Theta = -1) \end{cases} , \ \varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \quad K = \text{complex conjugation}$$

yield

BDI	$U(z) = \overline{U(-\bar{z})}$
CII	$U(z)\varepsilon = \varepsilon\overline{U(\bar{z})}$

AI	$P^{\pm} = \overline{P^{\pm}}$
AII	$P^{\pm}\varepsilon = \varepsilon\overline{P^{\pm}}$

with $\underline{k}$ absent in $d = 1$; bystander otherwise

Real classes: From chiral to non-chiral classes

- First case: BDI $\to$ AI and CII $\to$ AII, while $d \to d-1$

Normal forms for Θ

$$\Theta = \begin{cases} K, \ (\sigma_\Theta = +1) \\ \varepsilon K, \ (\sigma_\Theta = -1) \end{cases}, \quad \varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad K = \text{complex conjugation}$$

yield

BDI	$U(z) = \overline{U(-\bar{z})}$
CII	$U(z)\varepsilon = \varepsilon\overline{U(\bar{z})}$

AI	$P^{\pm} = \overline{P^{\pm}}$
AII	$P^{\pm}\varepsilon = \varepsilon\overline{P^{\pm}}$

with $\underline{k}$ absent in $d = 1$; bystander otherwise

Recall AIII $\to$ A (Riesz projector)

$$P = \frac{1}{2\pi\mathrm{i}} \oint_{S^1} U(z)^{-1} \partial_z U(z) dz\,.$$

Real classes: From chiral to non-chiral classes

- First case: BDI $\to$ AI and CII $\to$ AII, while $d \to d-1$

Normal forms for Θ

$$\Theta = \begin{cases} K, \ (\sigma_\Theta = +1) \\ \varepsilon K, \ (\sigma_\Theta = -1) \end{cases} , \ \varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \quad K = \text{complex conjugation}$$

yield

BDI	$U(z) = \overline{U(-\bar{z})}$
CII	$U(z)\varepsilon = \varepsilon\overline{U(\bar{z})}$

AI	$P^\pm = \overline{P^\pm}$
AII	$P^\pm\varepsilon = \varepsilon\overline{P^\pm}$

with $\underline{k}$ absent in $d = 1$; bystander otherwise

Recall AIII $\to$ A (Riesz projector)

$$P = \frac{1}{2\pi \mathrm{i}} \oint_{S^1} U(z)^{-1} \partial_z U(z) dz \, .$$

Properties passed down from U to $P = P^+$; (P^- is complementary).

Real classes: From chiral to non-chiral classes

- Second case: DIII $\to$ D and CI $\to$ C

CAZ Class	Symmetry Θ	Σ	Π	d = 0	1	2	3	4	5	6	7
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	1	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Real classes: From chiral to non-chiral classes

- Second case: DIII $\to$ D and CI $\to$ C

CAZ Class	Symmetry Θ	Σ	Π	d = 0	1	2	3	4	5	6	7
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	1	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Classes D and C have lone symmetry Σ, $(\sigma_\Sigma = \pm 1)$

Real classes: From chiral to non-chiral classes

- Second case: DIII $\to$ D and CI $\to$ C

	Symmetry			d							
CAZ Class	Θ	Σ	Π	0	1	2	3	4	5	6	7
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	1	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Classes D and C have lone symmetry Σ, $(\sigma_\Sigma = \pm 1)$

Similar to first case; to be skipped.

Real classes: From non-chiral to chiral classes

- First case: AI $\to$ CI and AII $\to$ DIII.

	Symmetry			d							
CAZ Class	Θ	Σ	Π	0	1	2	3	4	5	6	7
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	1	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Real classes: From non-chiral to chiral classes

- First case: AI $\to$ CI and AII $\to$ DIII.

	Symmetry			d							
CAZ Class	Θ	Σ	Π	0	1	2	3	4	5	6	7
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	1	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Classes AI and AII have lone symmetry Θ, ($\sigma_\Theta = \pm 1$).

Real classes: From non-chiral to chiral classes

- First case: AI $\to$ CI and AII $\to$ DIII, while $d \to d - 1$

Real classes: From non-chiral to chiral classes

- First case: AI $\to$ CI and AII $\to$ DIII, while $d \to d - 1$

  - Need map $E \mapsto U$

Real classes: From non-chiral to chiral classes

• First case: AI $\to$ CI and AII $\to$ DIII, while $d \to d - 1$

- Need map $E \mapsto U$
- Like for A $\to$ AIII, consider
  - any vector bundle $E \to \mathbb{T}^d$ and its cut $\dot{E} \to \dot{\mathbb{T}}^d$ placed at $k_d = \pm\pi$
  - frames V on $\dot{E}$ and their transition maps U

Real classes: From non-chiral to chiral classes

• First case: AI $\to$ CI and AII $\to$ DIII, while $d \to d-1$

- Need map $E \mapsto U$
- Like for A $\to$ AIII, consider
  - any vector bundle $E \to \mathbb{T}^d$ and its cut $\dot{E} \to \dot{\mathbb{T}}^d$ placed at $k_d = \pm\pi$
  - frames V on $\dot{E}$ and their transition maps U
- But now frames V (unitaries) need to be adapted to symmetry AI [or AII] (compatibility $k \leftrightarrow -k$)

Real classes: From non-chiral to chiral classes

• First case: AI $\to$ CI and AII $\to$ DIII, while $d \to d-1$

- Need map $E \mapsto U$
- Like for A $\to$ AIII, consider
  - any vector bundle $E \to \mathbb{T}^d$ and its cut $\dot{E} \to \dot{\mathbb{T}}^d$ placed at $k_d = \pm\pi$
  - frames V on $\dot{E}$ and their transition maps U
- But now frames V (unitaries) need to be adapted to symmetry AI [or AII] (compatibility $k \leftrightarrow -k$)
- As a result, transition map U has symmetry CI [or DIII]

An aside: On adding a symmetry requirement

An aside: On adding a symmetry requirement

Consider classes before adding a symmetry. What happens to them after adding it?

- Collection of objects gets restricted.

An aside: On adding a symmetry requirement

Consider classes before adding a symmetry. What happens to them after adding it?

- Collection of objects gets restricted. Classes loose elements, possibly disappearing.

An aside: On adding a symmetry requirement

Consider classes before adding a symmetry. What happens to them after adding it?

- Collection of objects gets restricted. Classes loose elements, possibly disappearing.
- Remaining (symmetric) objects, even within in a same class, may no longer be connected by an alike deformation.

An aside: On adding a symmetry requirement

Consider classes before adding a symmetry. What happens to them after adding it?

- Collection of objects gets restricted. Classes loose elements, possibly disappearing.
- Remaining (symmetric) objects, even within in a same class, may no longer be connected by an alike deformation. Classes possibly split.

An aside: On adding a symmetry requirement

Consider classes before adding a symmetry. What happens to them after adding it?

- Collection of objects gets restricted. Classes loose elements, possibly disappearing.
- Remaining (symmetric) objects, even within in a same class, may no longer be connected by an alike deformation. Classes possibly split.
- Combination of both changes.

An aside: On adding a symmetry requirement

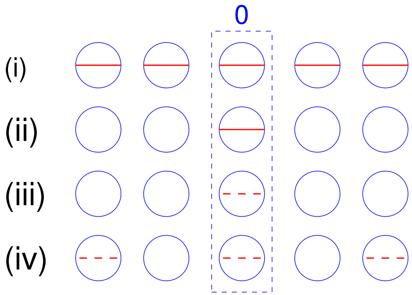
Consider classes before adding a symmetry. What happens to them after adding it?

- Collection of objects gets restricted. Classes loose elements, possibly disappearing.
- Remaining (symmetric) objects, even within in a same class, may no longer be connected by an alike deformation. Classes possibly split.
- Combination of both changes. Conceivable scenarios:

An aside: On adding a symmetry requirement

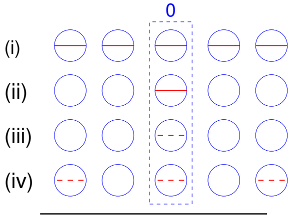
Consider classes before adding a symmetry. What happens to them after adding it?

- Collection of objects gets restricted. Classes loose elements, possibly disappearing.
- Remaining (symmetric) objects, even within in a same class, may no longer be connected by an alike deformation. Classes possibly split.
- Combination of both changes. Conceivable scenarios:



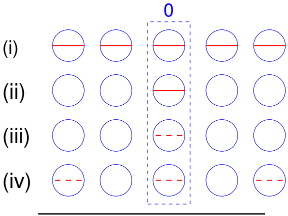
An aside: On adding a symmetry requirement

Consider classes before adding a symmetry. What happens to them after adding it?



An aside: On adding a symmetry requirement

Consider classes before adding a symmetry. What happens to them after adding it?

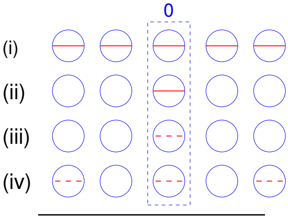


Let's formalize that:

- Let $X_0 \subset X$ be the symmetry classes before and after adding the symmetry requirement.

An aside: On adding a symmetry requirement

Consider classes before adding a symmetry. What happens to them after adding it?

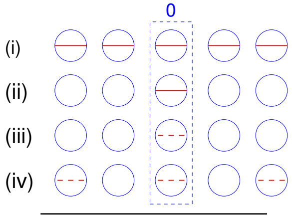


Let's formalize that:

- Let $X_0 \subset X$ be the symmetry classes before and after adding the symmetry requirement.
- Foregoing the symmetry, $X \supset X_0$, induces map $\iota : G \to G_0$ between index groups.

An aside: On adding a symmetry requirement

Consider classes before adding a symmetry. What happens to them after adding it?

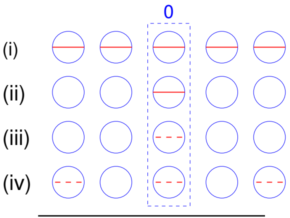


Let's formalize that:

- Let $X_0 \subset X$ be the symmetry classes before and after adding the symmetry requirement.
- Foregoing the symmetry, $X \supset X_0$, induces map $\iota : G \to G_0$ between index groups.
- Cases:
  - (a): $\iota = 0$; i.e. ι is trivial
  - (b): ι is non-trivial, but has trivial kernel; i.e. ι is a non-trivial injection
  - (c) ι is a non-trivial non-injection

An aside: On adding a symmetry requirement

Consider classes before adding a symmetry. What happens to them after adding it?

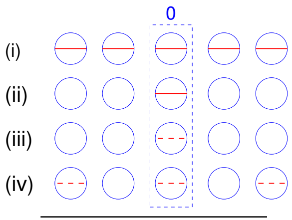


Let's formalize that:

- Let $X_0 \subset X$ be the symmetry classes before and after adding the symmetry requirement.
- Foregoing the symmetry, $X \supset X_0$, induces map $\iota : G \to G_0$ between index groups.
- Cases:
  - (a): $\iota = 0$; i.e. ι is trivial, cf. (ii, iii)
  - (b): ι is non-trivial, but has trivial kernel; i.e. ι is a non-trivial injection, cf. (i)
  - (c) ι is a non-trivial non-injection, cf. (iv)
- Application to $X =$BDI, CII and $X_0 =$AIII

An aside: On adding a symmetry requirement

Consider classes before adding a symmetry. What happens to them after adding it?



Let's formalize that:

- Let $X_0 \subset X$ be the symmetry classes before and after adding the symmetry requirement.
- Foregoing the symmetry, $X \supset X_0$, induces map $\iota : G \to G_0$ between index groups.
- Cases:
  - (a): $\iota = 0$; i.e. ι is trivial, cf. (ii, iii)
  - (b): ι is non-trivial, but has trivial kernel; i.e. ι is a non-trivial injection, cf. (i)
  - (c) ι is a non-trivial non-injection, cf. (iv)
- Application to X =BDI, CII and X_0 =AIII: case (c) does not occur.

Real classes: From non-chiral to chiral classes

- Second case: D $\rightarrow$ BDI and C $\rightarrow$ CII.

	Symmetry			d							
CAZ Class	Θ	Σ	Π	0	1	2	3	4	5	6	7
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	1	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Real classes: From non-chiral to chiral classes

- Second case: D $\to$ BDI and C $\to$ CII.

	Symmetry			d							
CAZ Class	Θ	Σ	Π	0	1	2	3	4	5	6	7
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	1	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Classes D and C have lone symmetry Σ, ($\sigma_\Sigma = \pm 1$).

Real classes: From non-chiral to chiral classes

For a given pair of rows, a same construction applies to all d, up to two exceptions ($\mathrm{D} \to \mathrm{BDI}$, $\mathrm{C} \to \mathrm{CII}$)

Real classes: From non-chiral to chiral classes

For a given pair of rows, a same construction applies to all d, up to two exceptions ($D \to BDI$, $C \to CII$)

- We'll just pursue $D \to BDI$, while $d \to d-1$

CAZ Class	Symmetry Θ	Σ	Π	d = 0	1	2	3	4	5	6	7
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	1	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AI	1	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Real classes: From non-chiral to chiral classes

For a given pair of rows, a same construction applies to all d, up to two exceptions ($D \to BDI$, $C \to CII$)

- We'll just pursue $D \to BDI$, while $d \to d-1$

CAZ Class	Θ	Σ	Π	0	1	2	3	4	5	6	7
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0

- Cases for maps ι: BDI $\to$ AIII upon foregoing the symmetry
  - (a): $\iota = 0$
  - (b): ι is a non-trivial injection

Real classes: From non-chiral to chiral classes

CAZ Class	Θ	Σ	Π	0	1	2	3	4	5	6	7
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0

- Cases for maps ι: BDI $\to$ AIII upon foregoing the symmetry
  - (a): $\iota = 0$
  - (b): ι is a non-trivial injection
- There are frames V (unitaries) on $\dot{E}$ adapted to symmetry D (compatibility $k \leftrightarrow -k$).

Real classes: From non-chiral to chiral classes

CAZ Class	Θ	Σ	Π	0	1	2	3	4	5	6	7
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0

- Cases for maps ι: BDI $\to$ AIII upon foregoing the symmetry
  - (a): $\iota = 0$
  - (b): ι is a non-trivial injection
- There are frames V (unitaries) on $\dot{E}$ adapted to symmetry D (compatibility $k \leftrightarrow -k$). They satify the BDI symmetry ($\overline{V(k)} = V(-k)$).

Real classes: From non-chiral to chiral classes

CAZ Class	Θ	Σ	Π	0	1	2	3	4	5	6	7
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0

- Cases for maps ι: BDI $\to$ AIII upon foregoing the symmetry
  - (a): $\iota = 0$
  - (b): ι is a non-trivial injection
- There are frames V (unitaries) on $\dot{E}$ adapted to symmetry D (compatibility $k \leftrightarrow -k$). They satify the BDI symmetry ($\overline{V(k)} = V(-k)$).
- Expect map $E \mapsto U$ to finish the job. But ...

Real classes: From non-chiral to chiral classes

CAZ Class	Θ	Σ	Π	0	1	2	3	4	5	6	7
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0

- Cases for maps ι: BDI $\to$ AIII upon foregoing the symmetry
  - (a): $\iota = 0$
  - (b): ι is a non-trivial injection
- There are frames V (unitaries) on $\dot{E}$ adapted to symmetry D (compatibility $k \leftrightarrow -k$). They satify the BDI symmetry ($\overline{V(k)} = V(-k)$).
- Expect map $E \mapsto U$ to finish the job. But ...

Concluding lemma

- If case (a) occurs in $d-1$,

Real classes: From non-chiral to chiral classes

CAZ Class	Θ	Σ	Π	0	1	2	3	4	5	6	7
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0

- Cases for maps ι: BDI $\to$ AIII upon foregoing the symmetry
  - (a): $\iota = 0$
  - (b): ι is a non-trivial injection
- There are frames V (unitaries) on $\dot{E}$ adapted to symmetry D (compatibility $k \leftrightarrow -k$). They satify the BDI symmetry ($\overline{V(k)} = V(-k)$).
- Expect map $E \mapsto U$ to finish the job. But ...

Concluding lemma

- If case (a) occurs in $d-1$, then in dimension d the frame V extends to E: $V|_{k_d=\pi} = V|_{k_d=-\pi}$ (i.e. trivial transition $U = 1$);

Real classes: From non-chiral to chiral classes

CAZ Class	Θ	Σ	Π	0	1	2	3	4	5	6	7
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0

- Cases for maps ι: BDI $\to$ AIII upon foregoing the symmetry
  - (a): $\iota = 0$
  - (b): ι is a non-trivial injection
- There are frames V (unitaries) on $\dot{E}$ adapted to symmetry D (compatibility $k \leftrightarrow -k$). They satify the BDI symmetry ($\overline{V(k)} = V(-k)$).
- Expect map $E \mapsto U$ to finish the job. But ...

Concluding lemma

- If case (a) occurs in $d-1$, then in dimension d the frame V extends to E: $V|_{k_d=\pi} = V|_{k_d=-\pi}$ (i.e. trivial transition $U = 1$); retain BDI index of restrictions $V|_{k_d=\pi}$ (better: difference to the trivial index of $V|_{k_d=0}$).

Real classes: From non-chiral to chiral classes

CAZ Class	Θ	Σ	Π	0	1	2	3	4	5	6	7
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0

- Cases for maps ι: BDI $\to$ AIII upon foregoing the symmetry
  - (a): $\iota = 0$
  - (b): ι is a non-trivial injection
- There are frames V (unitaries) on $\dot{E}$ adapted to symmetry D (compatibility $k \leftrightarrow -k$). They satify the BDI symmetry ($\overline{V(k)} = V(-k)$).
- Expect map $E \mapsto U$ to finish the job. But . . .

Concluding lemma

- If case (a) occurs in $d-1$, then in dimension d the frame V extends to E: $V|_{k_d=\pi} = V|_{k_d=-\pi}$ (i.e. trivial transition $U = 1$); retain BDI index of restrictions $V|_{k_d=\pi}$ (better: difference to the trivial index of $V|_{k_d=0}$).
- If case (b) occurs in $d-1$, then in dimension d do (essentially):

Real classes: From non-chiral to chiral classes

CAZ Class	Θ	Σ	Π	0	1	2	3	4	5	6	7
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0

- Cases for maps ι: BDI $\to$ AIII upon foregoing the symmetry
  - (a): $\iota = 0$
  - (b): ι is a non-trivial injection
- There are frames V (unitaries) on $\dot{E}$ adapted to symmetry D (compatibility $k \leftrightarrow -k$). They satify the BDI symmetry ($\overline{V(k)} = V(-k)$).
- Expect map $E \mapsto U$ to finish the job. But ...

Concluding lemma

- If case (a) occurs in $d-1$, then in dimension d the frame V extends to E: $V|_{k_d=\pi} = V|_{k_d=-\pi}$ (i.e. trivial transition $U = 1$); retain BDI index of restrictions $V|_{k_d=\pi}$ (better: difference to the trivial index of $V|_{k_d=0}$).
- If case (b) occurs in $d-1$, then in dimension d do (essentially): Take the index g from clutching in AIII, retain $\iota^{-1}(g)$ in BDI.

Summary

- A derivation of the Kitaev table has been given.
- It proceeds by diagonal maps
- They are of different kinds, the main difference being whether they map (a) non-chiral to chiral classes, or (b) viceversa.
- Essentially, $k = (\underline{k}, k_d) \in \mathbb{T}^d$ and
  - (a) $U = U(\underline{k})$ is the transition matrix of a bundle $E = (E_k)$
  - (b) $P = P(\underline{k})$ is the projection of a unitary $U = U(k)$

Summary

- A derivation of the Kitaev table has been given.
- It proceeds by diagonal maps
- They are of different kinds, the main difference being whether they map (a) non-chiral to chiral classes, or (b) viceversa.
- Essentially, $k = (\underline{k}, k_d) \in \mathbb{T}^d$ and
  - (a) $U = U(\underline{k})$ is the transition matrix of a bundle $E = (E_k)$
  - (b) $P = P(\underline{k})$ is the projection of a unitary $U = U(k)$

Thank you all for your attention!