Gromov's Positive Scalar Curvature Conjecture on Simplicial Volume

Guoliang Yu

Texas A&M University

Harvard Mathematical Picture Language Seminar
Monday, July 20, 2026

Joint work with Qiaochu Ma

Riemannian geometry: scalar curvature

Let M be an m-dimensional Riemannian manifold.

Definition by small balls

At a point $p \in M$, *scalar curvature* is characterized by

$$\frac{\mathrm{Vol}(B_M(p,r))}{\mathrm{Vol}(B_{\mathbb{R}^m}(r))} = 1 - \frac{\mathrm{Scal}(p)}{6(m+2)} r^2 + o(r^2).$$

Remarks:

(1) Positive scalar curvature means that very small balls have smaller volume than Euclidean balls.

(2) Scalar curvature plays an essential role in general relativity such as the positive mass problem.

The current state of scalar curvature geometry

Gromov: The emergent picture of spaces with scalar curvature ≥ 0, where topology and geometry are intimately intertwined, is reminiscent of the symplectic geometry, but the former has not reached yet the maturity of the latter.

The mystery of the scalar curvature remains unsolved.

(Four lectures on scalar curvature, 2023).

Simplicial volume

Let M be a closed oriented m-manifold.

A singular chain

$$c = \sum_i a_i \sigma_i$$

represents the fundamental class $[M] \in H_m(M; \mathbb{R})$ if $[c] = [M]$.

Definition

The *simplicial volume* of M is

$$\|M\| = \|[M]\|_1 = \inf_{[c]=[M]} \sum_i |a_i|.$$

It measures how efficiently the fundamental class can be built from singular simplices.

The sphere example and intuition

For $d \geq 1$, define

$$\sigma_d \colon [0,1] \to S^1, \qquad \sigma_d(t) = e^{2\pi i d t}.$$

We have

$$[\sigma_d] = d[S^1].$$

Thus $\frac{1}{d}\sigma_d$ represents the fundamental class $[S^1]$. Its ℓ^1-norm is

$$\left\| \frac{1}{d}\sigma_d \right\|_1 = \frac{1}{d}.$$

Hence

$$\boxed{\|S^1\| = 0.}$$

More generally

$$\boxed{\|S^n\| = 0.}$$

Example: Hyperbolic Surfaces

Let Σ_g be a closed oriented surface of genus $g \geq 2$. Equip Σ_g with a hyperbolic metric of curvature -1, and let ω be its area form. By Gauss–Bonnet,

$$\mathrm{Area}(\Sigma_g) = \int_{\Sigma_g} \omega = -2\pi\chi(\Sigma_g).$$

Let $c = \sum_i a_i \sigma_i$ be a singular fundamental cycle. Lift each σ_i to $\mathbb{H}^2$, replace it by the geodesic triangle with the same vertices, and project back to Σ_g. This gives a straightened cycle

$$\mathrm{st}(c) = \sum_i a_i \, \mathrm{st}(\sigma_i)$$

representing the same fundamental class.

Since every geodesic triangle in $\mathbb{H}^2$ has area at most π,

$$\mathrm{Area}(\Sigma_g) = \langle \omega, \mathrm{st}(c) \rangle \leq \pi \sum_i |a_i|.$$

Thus

$$\|\Sigma_g\| \geq \frac{\mathrm{Area}(\Sigma_g)}{\pi} = -2\chi(\Sigma_g) > 0.$$

Simplicial volume and rigidity

- (Gromov) For a closed hyperbolic manifold,

$$\|M\| = \frac{\mathrm{Vol}(M)}{v_m} > 0,$$

  where v_m is the volume of the regular ideal simplex in $\mathbb{H}^m$. This formula lies at the heart of Mostow rigidity: it shows that the hyperbolic volume is determined by the topology of M.

- Thurston further developed the straightening method to prove the non-vanishing of simplicial volume. For example, this technique can be used to establish the positivity of the simplicial volume of locally symmetric spaces of noncompact type, as in the work of Lafont and Schmidt.

- (Gromov) If M has amenable fundamental group, then $\|M\| = 0$.

Gromov's conjecture

Positive scalar curvature is a local geometric condition while simplicial volume is a global invariant measuring the topological complexity of the manifold.

Gromov's PSC conjecture on simplicial volume (1986)

Positive scalar curvature forces the simplicial volume $\|M\|$ to vanish.

Geometric consequences

Theorem (Ma-Yu)

There exists a universal constant $C > 0$ dependent only on the dimension m such that if M is an m-dimensional closed oriented Riemannian manifold satisfying

$$Scal \geq C \qquad \text{and} \qquad Ricc \geq -1,$$

then the simplicial volume vanishes:

$$||[M]|| = 0,$$

where Ricci curvature is the directional average of sectional curvature.

The Ricci curvature lower bound may be replaced by an upper bound 1 on the *volume entropy*:

$$h = \limsup_{R\to\infty} \frac{\ln \mathrm{Vol}\left(B_{\widetilde{M}}(x,R)\right)}{R}.$$

Exponential decay property

The fundamental group Γ is said to satisfy *exponential decay property* at the decay rate $d \geq 0$ if

$$\boxed{\|a\|^2_{C_r^*(\Gamma)} \leq C \exp(dR) \sum_{\gamma \in \Gamma} |a(\gamma)|^2}$$

for every $R > 0$ and every $a \in C_c(\Gamma)$ satisfying $\mathrm{supp}(a) \subseteq B_\Gamma(R)$, where the length function on Γ is inherited from the Riemannian metric on the universal cover.

Remarks: (1) If the exponential function is replaced by a polynomial function, then the decay property is the classical *Property RD* introduced by Uffe Haagerup.

(2) By Cauchy-Schwartz, the decay rate can be controlled by the volume entropy, which in turn can be estimated by the Ricci curvature lower bound.

Main theorem

Let M be a closed oriented manifold. Let $\widetilde{M} \longrightarrow M$ be its universal cover, and let $\Gamma = \pi_1(M)$.

Theorem (Ma–Yu)

There exists a universal constant $C > 0$ dependent only on the dimension m such that if M is a closed oriented manifold of dimension m whose universal cover $\widetilde{M}$ is spin and M admits a Riemannian metric with scalar curvature

$$\mathrm{Scal} \geq C \qquad \text{and} \qquad \text{the decay rate} \leq 1,$$

then the simplicial volume vanishes:

$$\|M\| = 0.$$

We want a first-order differential operator whose square is the Laplacian.

Let

$$D = c_1 \frac{\partial}{\partial x_1} + \cdots + c_m \frac{\partial}{\partial x_m},$$

where

$$c_i^2 = -1, \qquad c_i c_j + c_j c_i = 0 \quad (i \neq j).$$

Then

$$D^2 = -\sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2}.$$

The spin Dirac operator on a manifold

On a spin manifold, the Dirac operator is

$$D = c(e_1)\nabla_{e_1} + \cdots + c(e_m)\nabla_{e_m}.$$

Lichnerowicz formula

$$D^2 = \nabla^*\nabla + \frac{\mathrm{Scal}}{4}.$$

If $\mathrm{Scal} \geq \kappa > 0$, then

$$D^2 \geq \frac{\kappa}{4}.$$

Hence D is invertible.

Index theory: the first obstruction

For an elliptic operator D^+ on a closed manifold,

$$\mathrm{index}(D^+) = \dim\ker D^+ - \dim\ker D^-.$$

Atiyah–Singer index theorem

For the spin Dirac operator,

$$\mathrm{index}(D^+) = \widehat{A}(M).$$

Positive scalar curvature makes D invertible, so

$$\widehat{A}(M) = 0.$$

Small-propagation representative of the higher index

Let $\widetilde{D}$ be the Dirac operator on the universal cover of $\widetilde{M}$. Choose smooth functions $\phi_{\varepsilon,0}, \phi_{\varepsilon,1}$ such that

$$\phi_{\varepsilon,0}(\pm\infty) = \pm 1, \qquad \phi_{\varepsilon,1} = \phi_{\varepsilon,0}^2 - 1,$$

and their Fourier transforms have compact support contained in $[-\varepsilon/10, \varepsilon/10]$.

$$\boxed{I_\varepsilon = \begin{pmatrix} \phi_{\varepsilon,1}^+(\widetilde{D})^2 & -\phi_{\varepsilon,1}^+(\widetilde{D})(1-\phi_{\varepsilon,1}^+(\widetilde{D}))\phi_{\varepsilon,0}^-(\widetilde{D}) \\ -\phi_{\varepsilon,1}^-(\widetilde{D})\phi_{\varepsilon,0}^+(\widetilde{D}) & -\phi_{\varepsilon,1}^-(\widetilde{D})^2 \end{pmatrix}.}$$

Small propagation

$$\mathrm{prop}(I_\varepsilon) \leq \varepsilon.$$

This means $I_\varepsilon(x,y) = 0$ if $d(x,y) > \varepsilon$.

The world's most elementary higher index theorem

Quantitative higher index theorem (Ma-Yu)

$$[M] = \sum_{\sigma \in S_{m+1}} \operatorname{sgn}(\sigma)\, \mathrm{Tr}\Big(I_\varepsilon(x_{\sigma(0)}, x_{\sigma(1)}) \cdots I_\varepsilon(x_{\sigma(m)}, x_{\sigma(0)}) \Big) \cdot [x_0, \cdots, x_m],$$

where $m = \dim(M)$ and the right hand side is a cycle in ϵ-homology group $H_k^\epsilon(\widetilde{M}, \Gamma)$.

Remarks:

(1) This formula provides a direct bridge between scalar curvature and the fundamental class through the Dirac operator.

(2) The result is a consequence of a quantitative version of the Connes–Moscovici higher index theorem established by Ma and Yu (2025).

ε-homology theory

We define the ε-chain complex $C_k^\epsilon(M, \widetilde{\Gamma})$ to be:

$$\left\{\Gamma\text{-invariant measures on } \widetilde{M}^{k+1} \text{ supported where } \operatorname{diam}(x_0, \ldots, x_k) \leq \epsilon\right\}.$$

ε-homology group

$$H_k^\epsilon(\widetilde{M}, \Gamma) = H_k\left(C_*^\epsilon(\widetilde{M}, \Gamma), \partial\right).$$

Intuition: $H_k^\epsilon(\widetilde{M}, \Gamma)$ is homology generated by simplices of size at most ϵ.

When ε is small, $H_k^\epsilon(\widetilde{M}, \Gamma) = H_k(M)$.

The basic estimate

Applying Schatten–Hölder and the group decay inequality to the higher index formula for $[M]$ gives:

Fundamental class estimate

For sufficiently large i,

$$\|M\| \leq C\exp(d\varepsilon) \left\| (1+\widetilde{D})^i I_\varepsilon^* I_\varepsilon \right\| .$$

Here the factor $\exp(d\varepsilon)$ is the contribution from the exponential decay property of the fundamental group.

Why positive scalar curvature helps

Assume

$$\mathrm{Scal} \geq C > 0.$$

Then the Lichnerowicz formula gives

$$\widetilde{D}^2 \geq \frac{C}{4}.$$

So the spectrum of $\widetilde{D}$ has a gap around zero:

$$\mathrm{spec}(\widetilde{D}) \cap \left(-\frac{\sqrt{C}}{2}, \frac{\sqrt{C}}{2}\right) = \emptyset.$$

We can choose $\phi_{\varepsilon,0}$ such that index representative I_ε goes to 0 as $\varepsilon \to \infty$.

Norm and propagation estimate

There exists smooth function $\phi_{\varepsilon,0}$ such that

$$|\phi_{1,\varepsilon}(\lambda)| \leq ce^{-c_m \varepsilon}$$

for all

$$|\lambda| \geq \frac{\sqrt{C}}{2},$$

where c_m is a universal positive constant dependent only on m.

The operator norm in the estimate is bounded by

$$\left\|(1 + D^2_{\widetilde{M}})^i I_\varepsilon^* I_\varepsilon\right\| \leq c' \exp(-c_m \epsilon).$$

The final limiting argument

Combine the two estimates:

$$\|M\| \leq C\exp((d - c_m)\varepsilon).$$

Letting $\varepsilon \to \infty$, we obtain

$$\|M\| = 0.$$

References

- A. Connes and H. Moscovici, *Cyclic cohomology, the Novikov conjecture and hyperbolic groups.* Topology, 29, 345–388, 1990.
- M. Gromov, *Large Riemannian manifolds.* Lecture Notes in Mathematics, Springer-Verlag 1201, 108–122, 1986.
- M. Gromov, *Four lectures on scalar curvature.* In Perspectives in scalar curvature, Vol. 1. World Scientific. 2023.
- Q. Ma and G. Yu. *Small scale index theory, scalar curvature, and Gromov's simplicial norms.* arXiv: 2508.14791, 2025.
- Q. Ma and G. Yu, *Gromov's simplicial volume vanishing conjecture on positive scalar curvature and a fundamental group decay.* arXiv:2606.29135, 2026.
- J. Wang, Z. Xie and G. Yu, *Gromov's dihedral rigidity conjecture in dimension three.* arXiv:2606.30130, 2026.
- J. Roe, *Coarse cohomology and index theory on complete Riemannian manifolds.* Mem. Amer. Math. Soc., 1993.
- Not Only Scalar Curvature Seminar, a biweekly Zoom seminar organized by Gromov, Hanke, Sormani, and Yu. All past talks available on IHES Channels.