

# Some Inequalities in Locally Compact Quantum Groups

Part of Quantum Fourier Analysis

Presented by

**Jinsong Wu**

**Institute for Advanced Study in Mathematics, HIT**

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# Quantum Fourier Analysis

Symmetries	Algebras	Dimensions	Measures
Subfactors	Noncommutative	Finite	Tracial
Fusion Ring	Comm & Noncomm	Finite	Tracial
Groups	Comm & Noncomm	Infinite	Tracial
LCQG	Noncommutative	Infinite	Non-tracial

## Locally Compact Quantum Groups: Definition [Kustermans-Vaes 00,03]

A locally compact quantum group  $\mathbb{G} = (\mathcal{M}, \Delta, \varphi, \psi)$  consists of

- (1) A von Neumann algebra  $\mathcal{M}$ ;
- (2) A **comultiplication**  $\Delta : \mathcal{M} \rightarrow \mathcal{M} \bar{\otimes} \mathcal{M}$  is a unital normal \*-homomorphism satisfying

$$(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta,$$

where  $\iota : \mathcal{M} \rightarrow \mathcal{M}$  is the identity.

- (3) A **left Haar weight**  $\varphi$  and a right Haar weight  $\psi$  on  $\mathcal{M}$ :

$$(\iota \otimes \varphi)\Delta(x) = \varphi(x)1, \quad (\psi \otimes \iota)\Delta(x) = \psi(x)1;$$

## Special Cases

1.  $\mathbb{G}$  is a locally compact group if  $\mathcal{M}$  is abelian.
2.  $\mathbb{G}$  is **unimodular** if  $\varphi = \psi$ .
3.  $\mathbb{G}$  is a **unimodular Kac algebra** if  $\varphi = \psi$  is tracial. [Kac Vainermann , Enock Schwartz 70s]
4.  $\mathbb{G}$  is a **compact quantum group** if  $\varphi = \psi$  is a state. [Woronowicz 92]
5.  $\mathbb{G}$  is a **discrete quantum group** if  $\widehat{\mathbb{G}}$  is a compact quantum group. [van Daele 96]
6. Compact Matrix Quantum Groups [Woronowicz 87] : Quantum  $SU_q$ , Quantum Permutation Groups [Wang 98] etc.
7. Quantum  $E(2)$  Group [Woronowicz 91], Quantum  $ax + b$  Group [Woronowicz Zakrzewski 02] , Quantum  $az + b$  Group [Woronowicz 01, Soltan 05] etc.
8. Matrix Algebras, Quantum Torus are not LCQGs,

## Notations and Setups (Tomita-Takesaki Theory is involved)

We begin with the Gelfand-Naimark-Segal (semi-cycle) construction:

(1)  $\mathfrak{N}_\varphi = \{x \in \mathcal{M} : \varphi(x^*x) < \infty\}$

(2)  $\mathcal{H}_\varphi = \overline{\mathfrak{N}_\varphi}^{\langle \cdot, \cdot \rangle}$  is the underlying Hilbert space;

(3)  $\Lambda_\varphi : \mathfrak{N}_\varphi \rightarrow \mathcal{H}_\varphi$  is the inclusion map.

(4)  $\sigma_t^\varphi, \sigma_t^\psi$  are modular automorphisms;

(5)  $\delta$  is the modular element and  $\psi = \varphi_\delta$  (formally  $\psi = \varphi(\delta^{1/2} \cdot \delta^{1/2})$ );

(6)  $J_\varphi$  is the modular conjugation;

(7)  $\nabla_\varphi$  is the modular operator.

## Notations and Setups: Multiplicative Unitary

(1) The **multiplicative unitary**  $W \in \mathcal{B}(\mathcal{H}_\varphi \otimes \mathcal{H}_\varphi)$ :

$$W^*(\Lambda_\varphi(x) \otimes \Lambda_\varphi(y)) = (\Lambda_\varphi \otimes \Lambda_\varphi)(\Delta(y)x \otimes 1).$$

(2)  $\Delta(x) = W^*(1 \otimes x)W$  for any  $x \in \mathcal{M}$ .

(3)  $\widehat{\mathbb{G}}$  the dual locally compact quantum group  $(\widehat{\mathcal{M}}, \widehat{\Delta}, \widehat{\varphi}, \widehat{\psi})$

▶  $\widehat{\mathcal{M}} = \overline{\{(\omega \otimes \iota)W : \omega \in \mathcal{B}(\mathcal{H}_\varphi)_*\}}^{SOT}$

▶  $\widehat{\Delta} = \widehat{W}^* \cdot \widehat{W}$ , where  $\widehat{W} = \Sigma W^* \Sigma$ ,  $\Sigma$  flips the tensor.

▶  $\widehat{\varphi}$  satisfies  $\widehat{\Lambda}((x\varphi \otimes \iota)W) = \Lambda_\varphi(x)$ ,  $x \in \mathfrak{N}_\varphi$ .

▶  $\widehat{J}, \widehat{\nabla}$  modular conjugation and operator.

▶  $\widehat{\psi}$  will be given later.

(4)  $W \in \widehat{\mathcal{M}} \overline{\otimes} \widehat{\mathcal{M}}$ .

(5) If  $G$  is a group,  $(Wf)(t, s) = f(ts)$  for any  $t, s \in G$ .

$$(\mathcal{M} = L^\infty(G))$$

**It is critical to construct a multiplicative unitary to obtain a quantum group or a hopf algebra !**

## Notations and Setups: Antipode

(1)  $S$  is **antipode**:

$$S((\iota \otimes \varphi)(\Delta(x^*)(1 \otimes y))) = (\iota \otimes \varphi)((1 \otimes x^*)\Delta(y))$$
$$S(\iota \otimes \omega)W = (\iota \otimes \omega)W^*$$

(2) Polar decomposition  $S = R\tau_{-i/2}$ , where  $R$  is the **unitary antipode**,  $\tau$  is the **scaling automorphism**.

(3)  $\psi$  is taken to be  $\varphi R$

(4)  $\widehat{R}(\lambda(\omega)) = \lambda(\omega R)$ , where  $\lambda(\omega) = (\omega \otimes \iota)W$

(5)  $\widehat{\tau}_t(\lambda(\omega)) = \lambda(\omega\tau_t)$

(6)  $\widehat{S} = \widehat{R}\widehat{\tau}_{-i/2}$

(7)  $\widehat{\psi}$  is taken to be  $\widehat{\varphi}\widehat{R}$ .

►  $\mathbb{G}$  is a **Kac algebra** if  $\tau$  is trivial and  $\delta$  is affiliated with the center.

## Notations and Setups

(1) An automorphism group  $\rho_t$ :

$$\rho_t(\omega)(x) = \omega(\delta^{-it}\tau_{-t}(x)), \quad \omega \in \mathcal{M}_*$$

(2)  $\tau_t = \widehat{\nabla}^{it} \cdot \widehat{\nabla}^{-it}$ .

(3)  $\widehat{\tau}_t = \nabla_{\varphi}^{it} \cdot \nabla_{\varphi}^{-it}$ .

(4)  $\sigma_t^{\psi} = \delta^{it}\sigma_t^{\varphi}\delta^{-it}$ .

(5)  $(\widehat{\nabla}^{it} \otimes \nabla_{\varphi}^{it})W = W(\widehat{\nabla}^{it} \otimes \nabla_{\varphi}^{it})$ .

(6)  $\widehat{\sigma}_t(\lambda(\omega)) = \lambda(\rho_t(\omega))$ .



## Noncommutative $L^p$ Space

- ▶ Interpolated  $L^p$  Space (Izumi's  $L^p$  Space) [Terp 82, Izumi 97]
- ▶ Spatial  $L^p$  Space (Hilsum's  $L^p$  Space) [Connes , Hilsum 81]
- ▶ Haagerup's  $L^p$  Space [Haagerup 79, Terp 81, HJX 08 ]

## Interpolated $L^p$ Space

$$\begin{array}{ccc} & x\varphi = \varphi(\cdot x) & \\ & \nearrow & \searrow \\ x \mathcal{L}_\varphi & \xrightarrow{\iota^1} & \mathcal{M}_* & \xrightarrow{(r^\infty)^*} & \mathcal{R}_\varphi^* \\ & \searrow & \nearrow & & \\ & \mathcal{M} & \xrightarrow{(r^1)^*} & \mathcal{R}_\varphi^* & \\ & x & & & \end{array}$$

Figure: Embedding with parameter  $z = -1/2$

## Interpolated $L^p$ Space

By Complex Interpolation Method, we have

(1)  $L^1(\mathbb{G}) = \mathcal{M}_*$ ,  $L^\infty(\mathbb{G}) = \mathcal{M}$ ;

(2)  $L^1(\mathbb{G}) \cap L^\infty(\mathbb{G}) = \mathcal{L}_\varphi$ ;

(3)  $L^p(\mathbb{G}) = (L^1(\mathbb{G}), L^\infty(\mathbb{G}))_{[1/p]}$ ;

(4)  $L^2(\mathbb{G}) = \mathcal{H}_\varphi$ ;

(5)  $L^2(\mathbb{G}) \cap L^\infty(\mathbb{G}) = \mathfrak{N}_\varphi$ ;

(6)  $L^1(\mathbb{G}) \cap L^2(\mathbb{G}) = \mathcal{I}_\varphi = \{\omega \in L^1(\mathbb{G}) : |\omega(x^*)| \leq C\|A_\varphi(x)\|\}$

(7)  $\xi_t : L^1(\mathbb{G}) \cap L^t(\mathbb{G}) \rightarrow L^t(\mathbb{G})$ ;

(8)  $l^t : L^\infty(\mathbb{G}) \cap L^t(\mathbb{G}) \rightarrow L^t(\mathbb{G})$ ;

## Spatial $L^p$ Space

Let  $\phi$  be a normal semifinite faithful weight on the commutant  $\mathcal{M}'$  acting on  $\mathcal{H}_\phi$ .

(1) **Spatial derivative:**  $d_\phi = \frac{d\phi}{d\phi}$ .

(2) **Dense subset:**  $\left\{ x d_\phi^{1/p} : \text{some proper } x \in \mathcal{M} \right\}$ ;

(3)  **$t$ -Homogeneous operators:**  $ax \subseteq x\sigma_{it}^\phi(a)$ , where  $a \in \mathcal{M}'$ .

(4) **Measure:**  $\int |x|^p d\phi$  for  $-\frac{1}{p}$ -homogeneous  $x$ ,  $\int \frac{d\omega}{d\phi} d\phi = \omega(\mathbf{1})$ ;

(5)  $L^p(\phi) = \left\{ x : -\frac{1}{p}\text{-homogeneous and } \int |x|^p d\phi < \infty \right\}$ .

(6)  $\Phi_p : L^p(\mathbb{G}) \rightarrow L^p(\phi)$  is the isometric isomorphism satisfying

$$\Phi_p(\xi_p(x\varphi)) = x d_\phi^{1/p}$$

## Haagerup's $L^p$ Space

- (1)  $\mathcal{M} \rtimes_{\sigma}^{\varphi} \mathbb{R} (= \mathcal{N})$  the cross product von Neumann algebra with tracial weight  $\tau$ . Let  $\theta$  be the dual action of  $\mathbb{R}$  in  $\mathcal{N}$ .
- (2)  $\tau$ -measurable operators: complete Hausdorff topological space, an algebra w. r. t. strong product and strong sum.
- (3)  $L_h^p(\mathcal{M}) = \{x : \tau\text{-measurable}, \theta_s(x) = e^{-s/p}x, s \in \mathbb{R}\}$
- (4) **Measure:**  $tr(x) = \tau(\chi_{(1,\infty)}(|x|^p))$ .
- (5) There is an isometric isomorphism between Haagerup's  $L^p$  space and Hilsum's  $L^p$  space.

We shall identify the three noncommutative  $L^p$  spaces properly.



## Hölder's Inequality

For any  $x \in L^t(\phi)$ ,  $y \in L^s(\phi)$ ,

$$\|xy\|_{r,\phi} \leq \|x\|_{t,\phi} \|y\|_{s,\phi}, \quad \frac{1}{r} = \frac{1}{t} + \frac{1}{s}.$$

Moreover  $\|xy\|_{r,\phi} = \|x\|_{t,\phi} \|y\|_{s,\phi}$  if and only if  $\frac{|x|^t}{\|x\|_{t,\phi}^t} = \frac{|y^*|^s}{\|y\|_{s,\phi}^s}$

## Fourier Transform [van Daele 07]

1. **Fourier transform:**  $\mathcal{F}_t : L^t(\phi) \rightarrow L^{t'}(\hat{\phi}), \frac{1}{t} + \frac{1}{t'} = 1, 1 \leq t \leq 2$   
satisfying

$$\mathcal{F}_t(\Phi_t^{-1}(\xi_t(\omega))) = \hat{\Phi}_t^{-1}(\iota^{t'}(\lambda(\omega))), \quad \omega \in L^1(\mathbb{G}) \cap L^t(\mathbb{G})$$

$$\mathcal{F}_t(x) = \int W(x d_\phi^{1/t'} \otimes \hat{d}_{\hat{\phi}}^{1/t'}) d\phi \otimes \iota, \quad x \in L^t(\phi)$$

2. **Plancherel's formula:**  $\|\mathcal{F}_2(x)\|_{2,\hat{\phi}} = \|x\|_{2,\phi}$ .

3.  $\int \mathcal{F}_t(x) y^* d\phi = \int x \hat{\mathcal{F}}_t(y)^* d\phi$ , where  $\hat{\mathcal{F}}_t$  is the Fourier transform from  $L^t(\hat{\phi})$  to  $L^{t'}(\phi)$ .

4. **Hausdorff-Young inequality:** [Cooney 10, Caspers 12]

$$\|\mathcal{F}_t(x)\|_{t',\hat{\phi}} \leq \|x\|_{t,\phi}$$

## Bi-shifts of Group-like Projection [LWW 17, JLW 18]

(1) **Group-like projection**  $B$  is a projection and

$$\Delta(B)(1 \otimes B) = B \otimes B, \quad B \neq 0$$

(2) **Biprojection**  $B$  is a projection and  $\lambda(B\varphi)$  is a multiple of a projection.

(3) **Left shift of group-like projection**: A projection  $B_g$  is a left shift of a group-like projection  $B$  if  $\varphi(B_g) = \varphi(B)$  and

$$\Delta(B_g)(1 \otimes B) = B_g \otimes B, \quad \Delta(B)(1 \otimes B_g) = R(B_g) \otimes B_g.$$

(4) **Bi-shift of group-like projection**  $x \in L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$  such that

$$x\varphi = (yB_g\varphi) * (\hat{\lambda}(\tilde{B}_h\hat{\varphi})\varphi),$$

where  $B_g$  is a left shift of a group-like projection  $B$ ,  $\tilde{B}_h$  is a left shift of the group-like projection  $\tilde{B} = \mathcal{R}(\lambda(B\varphi))$ .

**It acts like translation and modulation of open compact subgroups.**

## Hausdorff-Young Inequality [LW 17, W 20]

Suppose  $\mathbb{G}$  is a locally compact quantum group. The following are equivalent:

- (1)  $\|\mathcal{F}_t(x)\|_{\frac{t}{t-1}, \hat{\phi}} = \|x\|_{t, \phi}$  for some  $1 < t < 2$ ;
- (2)  $\|\mathcal{F}_t(x)\|_{\frac{t}{t-1}, \hat{\phi}} = \|x\|_{t, \phi}$  for all  $1 \leq t \leq 2$ ;
- (3)  $x$  is a bi-shift of a group-like projection.

This result was only proved for locally compact groups but not for their duals. [Russo 1974, Fournier 1977]

$$C_t = \sup_{0 \neq x \in L^t(\phi)} \frac{\|\mathcal{F}_t(x)\|_{\frac{t}{t-1}, \hat{\phi}}}{\|x\|_{t, \phi}} \leq 1.$$

# Convolutions

$$(\omega_1 \otimes \omega_2)\Delta, \quad (\omega \otimes \iota)\Delta(a), \quad (\iota \otimes \omega)\Delta(a)$$

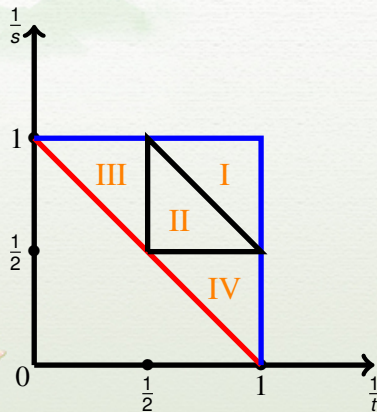


Figure: The Regions of Young's inequality

## Convolutions

- (1) [LWW 17]  $\frac{1}{r} + 1 = \frac{1}{t} + \frac{1}{s}$ ,  $1 \leq r, t, s \leq 2$ , i.e. Region "I", the convolution is defined as

$$x * \rho_{-i/t'}(y) \in L^r(\phi), \quad \frac{1}{t} + \frac{1}{t'} = 1.$$

- (2) Fourier Transform Interchanges Convolution and Product

$$\mathcal{F}_r(x * \rho_{-i/t'}(y)) = \mathcal{F}_t(x)\mathcal{F}_s(y).$$

- (3) [JLW 18]  $\frac{1}{r} + 1 = \frac{1}{t} + \frac{1}{s}$ ,  $1 \leq t, s \leq 2, r \geq 2$ , i.e. Region "II", the convolution is defined as

$$\widehat{\mathcal{F}}_{r'}(\mathcal{F}_t(x)\mathcal{F}_s(y)) \in L^r(\phi), \quad \frac{1}{r} + \frac{1}{r'} = 1.$$

## Convolutions

Pull back of Involution # (hash):

1. For  $1 \leq t \leq 2$ ,  $\mathcal{F}_t(x^\#) = \mathcal{F}_t(x)^*$ ,  $x \in L^t(\phi)$ .
2. For  $2 \leq 2 \leq \infty$ ,  $x^\# = \widehat{\mathcal{F}}_{t'}(x_0^*)$ , where  $x = \widehat{\mathcal{F}}_{t'}(x_0)$  for some  $x_0 \in L^{t'}(\hat{\phi})$ .

Some Facts:

1.  $x^{*\#} \neq x^{\#\#}$
2.  $(xy)^\# \neq x^\#y^\# (\neq y^\#x^\#)$
3.  $(x \star y)^\# = x^\# \star y^\#$

4.  $(x^\#)^\# = x$

5.  $\int x^* y^\# d\phi = \int x^\# y^* d\phi$

## Convolutions [W 20]

$$x \star y = \begin{cases} x * \rho_{-i/t'}(y) & (1/t, 1/s) \text{ in the Region "I"} \\ \widehat{\mathcal{F}}_{t'}(\mathcal{F}_t(x)\mathcal{F}_s(y)) & (1/t, 1/s) \text{ in the Region "II"} \\ x * \rho_{-i/t'}(y^{*\#}) & (1/t, 1/s) \text{ in the Region "III"} \\ x^{*\#} * \rho_{-i/t'}(y) & (1/t, 1/s) \text{ in the Region "IV"} \end{cases}$$

$$\left(\frac{d\omega}{d\phi}\right)^{*\#} * a = (\omega \otimes \iota)\Delta(a), \quad a * \rho_{-i} \left(\frac{d\omega}{d\phi}\right)^{*\#} = (\iota \otimes \omega)\Delta(a).$$

## Young's Inequality [LWW 17, JLW 18, W 20]

For  $(1/t, 1/s)$  in the Region "I", we have

$$\|x * \rho_{-i/t'}(y)\|_{r,\phi} \leq \|x\|_{t,\phi} \|y\|_{s,\phi}.$$

For  $(1/t, 1/s)$  in the Region "II", we have

$$\left\| \widehat{\mathcal{F}}_{r'}(\mathcal{F}_t(x)\mathcal{F}_s(y)) \right\|_{r,\phi} \leq \|x\|_{t,\phi} \|y\|_{s,\phi}$$

For  $(1/t, 1/s)$  in the Region "III", we have

$$\left\| x * \rho_{-i/t'}(y^{*\#}) \right\|_{r,\phi} \leq \|x\|_{t,\phi} \|y\|_{s,\phi}.$$

For  $(1/t, 1/s)$  in the Region "IV", we have

$$\left\| x^{*\#} * \rho_{-i/t'}(y) \right\|_{r,\phi} \leq \|x\|_{t,\phi} \|y\|_{s,\phi}.$$

## Young's Inequality [LWW 17, W 20]

Let  $\mathbb{G}$  be a unimodular locally compact quantum group.  
For  $(1/t, 1/s)$  in the Region "IV",

$$\|\tau_{i/t-i/2}(x) * \tau_{-i/t'}(y)\|_{r,\phi} \leq \|x\|_{t,\phi} \|y\|_{s,\phi},$$

For  $(1/t, 1/s)$  in the Region "III",

$$\|x * \tau_{i/r-i/2}(y)\|_{r,\phi} \leq \|x\|_{t,\phi} \|y\|_{s,\phi}.$$

Let  $\mathbb{G}$  be a Kac algebra.

$$\|x * y\delta^{-1/t'}\|_{r,\phi} \leq \|x\|_{t,\phi} \|y\|_{s,\phi}.$$

## Young's Inequality [LW 18, W 20]

Suppose  $\mathbb{G}$  is a locally compact quantum group. Then the following are equivalent:

- (1)  $\|x \star y\|_{r,\phi} = \|x\|_{t,\phi} \|y\|_{s,\phi}$  for **some**  $1 < r, t, s < \infty$  such that  $\frac{1}{r} + 1 = \frac{1}{t} + \frac{1}{s}$ ;
- (2)  $\|x \star y\|_{r,\phi} = \|x\|_{t,\phi} \|y\|_{s,\phi}$  for **all**  $1 < r, t, s < \infty$  such that  $\frac{1}{r} + 1 = \frac{1}{t} + \frac{1}{s}$ ;
- (3)  $x, y$  are bi-shifts of group-like projections satisfying certain conditions.

Suppose  $\mathbb{G}$  is a unimodular Kac algebra, we have

$$\|x \star y\|_r \leq \| |x| \star |y| \|_r^{1/2} \| |x^*| \star |y^*| \|_r^{1/2}.$$

## Young's Inequality [LW 18, W 20]

$$C_{t,s} = \sup_{0 \neq x \in L^t(\phi), 0 \neq y \in L^s(\phi)} \frac{\|x \star y\|_{r,\phi}}{\|x\|_{t,\phi} \|y\|_{s,\phi}} \leq 1$$

(1)  $C_{1,1} = C_{1,\infty} = C_{\infty,1} = 1.$

(2)  $C_{t,s} = C_{t,r'} = C_{r',s}$ , where  $\frac{1}{r'} + \frac{1}{t} + \frac{1}{s} = 2.$

(3)  $C_{t,s} \leq C_t C_s C_{r'}$  when  $(1/t, 1/s)$  is in the Region "II".

## Entropic Convolution Inequality [W 20]

For any  $\omega \in L^1(\mathbb{G})$ , the entropy  $H_\phi \left( \frac{d\omega}{d\phi} \right)$  is defined as

$$H_\phi \left( \frac{d\omega}{d\phi} \right) = - \int \frac{d\omega}{d\phi} \left( \ln \frac{d\omega}{d\phi} - \ln d_\varphi \right) d\phi.$$

Suppose that  $\omega_2 \tau_\theta = \omega_1$ ,  $0 \leq \theta \leq 1$ . Then

$$\begin{aligned} & H_\phi \left( \frac{d\omega_1}{d\phi} * \frac{d\omega_2}{d\phi} \right) \\ & \geq (1 - \theta) H_\phi \left( \frac{d\omega_1}{d\phi} \right) + \theta H_\phi \left( \frac{d\omega_2}{d\phi} \right) + (1 - \theta) \omega_2(\ln \delta). \end{aligned}$$

The extremizers of the inequality for unimodular Kac algebras are left shifts of group-like projections.

## Donoho-Stark Uncertainty Principle [JLW 18]

$\varphi$ -value of support projection:

$$\mathcal{S}(x) = \varphi(\mathcal{R}(\Phi_t^{-1}(x)^*))$$

where  $\mathcal{R}$  is taking the range projection.

Suppose  $\mathbb{G}$  is a locally compact quantum group.

Then for any  $\omega$  in  $L^1(\mathbb{G}) \cap L^2(\mathbb{G})$ ,  $1 \leq t \leq 2$ ,  $2 \leq s \leq \infty$ , we have

$$\mathcal{S}(\xi_t(\omega))\mathcal{S}(\iota^s(\lambda(\omega))) \geq 1.$$

Moreover for any  $x \in L^t(\phi)$

$$\mathcal{S}(x)\mathcal{S}(\mathcal{F}_t(x)) \geq 1.$$

## Donoho-Stark uncertainty principle [JLW 18]

The following are equivalent:

1.  $\omega \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$  is a minimizer.
2.  $\omega$  is an extremal bi-partial isometry such that  $|\omega| \sigma_t^\varphi = |\omega|$ ,  $\hat{\sigma}_t(|\lambda(\omega)|) = |\lambda(\omega)|$ ,  $\forall t \in \mathbb{R}$ .
3.  $\omega$  is a bi-partial isometry,  $|\omega| \sigma_t^\varphi = |\omega|$ ,  $\forall t \in \mathbb{R}$ , and  $\lambda(\omega)$  is in  $L^1(\hat{\mathbb{G}})$  such that  $\|\hat{\lambda}(\lambda(\omega)\hat{\varphi})\|_\infty = \|\lambda(\omega)\hat{\varphi}\|$ .
4.  $\omega \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$  satisfies that  $\mathcal{S}(\omega)\mathcal{S}(\lambda(\omega)) = 1$  and  $\hat{\sigma}_t(|\lambda(\omega)|) = |\lambda(\omega)|$ .
5.  $\omega \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$  satisfies that

$$\mathcal{S}(\omega)\mathcal{S}(\lambda(\omega)) = 1, \quad \mathcal{S}(\xi(\omega))\mathcal{S}(\hat{\Lambda}(\lambda(\omega))) = 1.$$

6.  $\omega$  is a bi-shift of a group-like projection  $B \in L^1(\mathbb{G})$ .

## Hirschman-Beckner uncertainty principle [JLW 18, W 19]

Entropy for  $L^2$  space

$$H_o(\xi) = -\langle (\log |\Phi_2^{-1}(\xi)|^2 - \log d_\varphi) J_\varphi \xi, J_\varphi \xi \rangle.$$

Suppose  $\mathbb{G}$  is a locally compact quantum group and  $\|\xi\| = 1$ .

$$H_o(\xi) + H_o(\mathcal{F}_2(\xi)) \geq 0.$$

The following are equivalent:

- (1)  $\omega$  is a minimizer of Donoho-Stark uncertainty principle;
- (2)  $\omega$  is a minimizer of Hirschman-Beckner uncertainty principle;
- (3)  $\omega$  is a bi-shift of a group-like projection.

## Hardy's uncertainty principle [JLW 18]

Suppose  $\mathbb{G}$  is a locally compact quantum group with a bi-shift  $w$  of a group-like projection. Let  $x \in L^1(\mathbb{G}) \cap L^\infty(\mathbb{G})$  be such that

$$|x^*| \leq C|w^*|, \quad |\lambda(x\varphi)| \leq C'|\lambda(w\varphi)|,$$

for some  $C, C' > 0$ .

Then  $x$  is a multiple of  $w$ .

## Sum Set Estimate [LW 17]

Suppose  $\mathbb{G}$  is a unimodular Kac algebra with a Haar tracial weight  $\varphi$ . Let  $p, q$  be projections in  $L^\infty(\mathbb{G})$ . Then

$$\max\{\varphi(p), \varphi(q)\} \leq \mathcal{S}(p * q).$$

The following are equivalent:

- (1)  $\mathcal{S}(p * q) = \varphi(p) < \infty$ ;
- (2)  $\varphi(q)^{-1}p * q$  is a projection in  $L^1(\mathbb{G})$
- (3)  $\mathcal{S}(p * (q * R(q)^{* (m)})) * q^{* (j)}) = \varphi(p)$  for some  $m \geq 0, j \in \{0, 1\}, m + j > 0, q^{* (0)}$  means  $q$  vanishes.
- (4) there exists a biprojection  $B$  such that  $q$  is a right subshift of  $B$  and  $p = \mathcal{R}(x * B)$  for some  $x > 0$ .

## Nikoski's Inequality [W 20]

Let  $\mathbb{G}$  be a locally compact quantum group. Suppose that  $1 \leq s \leq \infty$  and  $1 \leq t < \min\{2, s\}$  and  $\omega \in L^t(\mathbb{G}) \cap L^s(\mathbb{G})$ . Assume that  $\widehat{\mathcal{S}}(\xi_t(\omega)) = \mathcal{S}(\mathcal{F}_t(\xi_t(\omega))) < \infty$ . Then

$$\|\xi_s(\omega)\|_s \leq \widehat{\mathcal{S}}(\xi(\omega))^{\frac{1}{t} - \frac{1}{s}} \|\xi_t(\omega)\|_t.$$

Moreover, if  $t \neq s$ ,  $s \neq \infty$ , then  $\|\xi_s(\omega)\|_s = \widehat{\mathcal{S}}(\xi(\omega))^{\frac{1}{t} - \frac{1}{s}} \|\xi_t(\omega)\|_t$  if and only if  $\omega$  is a bi-shift of a group-like projection.

This is proved for locally compact groups in 2019 and earlier by Nikoski for compact groups.

## Positivity

(I) **Schur Product Theorem** Let  $\mathbb{G}$  be a Kac algebra. Suppose that  $x \in L^t(\phi), y \in L^s(\phi)$  such that  $x \nabla_{\varphi}^{1/t'} \geq 0$  and  $y \nabla_{\varphi}^{-1/t'} \geq 0$ ,

$$x * y \delta^{-1/t'} \geq 0.$$

(II) Let  $\mathbb{G}$  be a locally compact quantum group. An element  $x \in L^t(\phi)$  is positive definite if

(1)  $\mathcal{F}_t(x) \geq 0$  when  $1 \leq t \leq 2$ .

(2)  $\int (y^{\#} * \rho_{-i/2t}(y)) x^* d\phi \geq 0$  for any  $y \in L_{\#}^{\frac{2t}{2t-1}}(\phi)$  when  $2 \leq t \leq \infty$ .

## More

- (1) Positive definite [Daws Salmi 13, Runde Viselter 14]
- (2) Amenability [Many]
- (3) Idempotent states [Salmi, Pal, Franz, Skalski etc]
- (4) Sharp Hausdorff-Young inequality (Lie groups) [Klein Russo 78, Cowling Martini Muller Parcet 19]
- (5) Sidon sets (compact quantum groups) [Wang 16]
- (6) .....

## Questions

- (1) Find the norm of Fourier transform for locally compact quantum groups.
- (2) Find the best constant  $C_t$  for proper locally compact quantum groups.
- (3) Is it true that  $C_t = \hat{C}_t$ , where  $\hat{C}_t$  is the best constant for the dual?
- (4) Find the best constant  $C_{t,s}$  for proper locally compact quantum groups.
- (5) Is it true that  $C_{t,s} = \hat{C}_{t,s}$ , where  $\hat{C}_{t,s}$  is the best constant for the dual?
- (6) Is it true that  $C_{t,s} = C_t C_s C_{r'}$ ?
- (7) Find the reverse Young inequality.
- (8) Prove the fractional Young inequality.

## Questions

- (9) Find the formulation of the Brascamp-Lieb inequalities which generalize Young's inequality.
- (10) Find the right formulation of sunset estimation for locally compact quantum groups.
- (11) Are the minimizers of entropy convolution inequality left shift of group-like projection?
- (12) Does the amenability imply the co-amenable of the dual?
- (13) Suppose  $x \in L^t(\phi)$  is positive definite, where  $t > 2$ . Is there  $x_0 > 0$  such that  $\widehat{\mathcal{F}}_t(x_0) = x$ ?
- (14) Fourier multipliers on locally compact quantum groups?  
<https://www.researchgate.net/project/Fourier-multipliers-on-quantum-groups-and-noncommutative-Lp-spaces>
- (15) Perturbation theory?
- (16) Relation between Fourier transform and (quantum) differentiation?



**Thank You for Attention!**