

# Higher Representation Theory, Lecture One

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The standard  $n$ -simplex  $\Delta^n, n \geq 0$  in  $\mathbb{R}_+^{n+1} = \{(x_0, \dots, x_n) | x_j \geq 0, j = 0, \dots, n\}$  is given by

$$\Delta^n = \{(x_0, \dots, x_n) \in \mathbb{R}_+^{n+1} : \sum_{i=0}^n x_i = 1\}.$$

**Example 0.1.** A 0-simplex is a point; a 1-simplex is a line segment; a 2-simplex is a triangle; a 3-simplex is a tetrahedron.

The  $n$ -simplex  $\Delta^n$  is a convex hull of  $n + 1$  vertices in  $\mathbb{R}_+^{n+1}$ . The convex hull of any nonempty subset of the  $n + 1$  vertices is the face of the simplex. If the size of the subset is  $k, k < n$ , the face is a  $k$ -face.

An  $n$ -**simplex**  $\sigma$  is a map from  $\Delta^n$  into a topological space  $\mathcal{X}$ , which is an homeomorphism on the interior of each  $k$ -face of  $\Delta^n$ . We sometimes identify  $\sigma$  with its image or support space if this does not lead to confusion.

**Exercise 0.2.** The sphere  $\mathbb{S}^{2n-1}, n \geq 1$  has a one simplex triangulation obtained by glueing pairwise the  $2n$  faces of a  $2n - 1$ -simplex.

A simplex  $\sigma$  together with a sign, or orientation,  $\varepsilon(\sigma) \in \{\pm 1\}$  is called an **oriented simplex**. The symmetric group  $\mathbb{S}^{n+1}$  acts on  $\Delta^n$  by permuting the coordinates, and thus the vertices of  $\Delta^n$ ; this action extends to an action, called **reordering** on  $n$ -simplices. For an oriented  $n$ -simplex  $\sigma$  and  $g \in \mathbb{S}^{n+1}$  the permuted simplex  $g(\sigma)$  has sign  $\varepsilon(g(\sigma)) = \varepsilon(g)\varepsilon(\sigma)$ , where  $\varepsilon(g)$  is the sign of  $g$ .

There is one  $-1$ -simplex  $\emptyset$  or, sometimes, several different  $-1$ -simplices  $\emptyset_i, i \in I$ , each having empty boundary. In the latter case there are accordingly several types of 0-simplices or points, a point of type  $i \in I$  having boundary  $\emptyset_i$ .

A **simplicial  $d$ -complex**  $\Sigma$  is a collection of simplices of dimensions  $0, 1, \dots, d$  such that for any  $\sigma \in \Sigma$  the boundary  $\partial\sigma$  of  $\sigma$  is in  $\Sigma$ , and the intersection of any two simplices in  $\sigma$  is in  $\Sigma$ . (We view the simplices as images in a topological space.) The simplices of dimension  $k$  in  $\Sigma$  are called  **$k$ -faces** and we denote the  $k$ -faces of  $\Sigma$  by  $\Sigma^k$ . The simplicial  $d$ -complex  $\Sigma$  is called **oriented** if each simplex of  $\Sigma^d$  is oriented. The boundary of an oriented  $n$ -simplex  $\Delta^n$  is an oriented  $n - 1$ -simplicial complex, in which the sign of a face  $\{x \in \Delta^n; x_k = 0\}$  is  $(-1)^k \varepsilon(\Delta^n)$ .

An  $n$ -dimensional oriented simplicial complex  $V$  in which each  $n - 1$ -simplex appears in the boundary of  $n$ -simplices, up to a reordering, either once or twice with opposite orientations. The support space [[ [what is "support space" here -Jinsong](#)]] of such a complex, up to piecewise linear (PL) homeomorphisms [[ [what does it mean -Jinsong](#)]], is called a **singular triangulated  $n$ -manifold**. The boundary  $\partial V$  of  $V$  is a singular triangulated  $n - 1$ -manifold consisting of the

$n - 1$ -simplices which appear only once in the boundary of  $n$ -simplices. Remark that in this definition each  $n$ -simplex is ordered independently of the others.

A triangulation of a topological space is **nonsingular**, if for each simplex the images of the vertices are distinct. In this case we can order the set of vertices of the triangulation linearly, and then give each simplex the induced order. [\[\[?? -Jinsong\]\]](#) For this ordering, the order induced on an inner hyperface by each of its two neighbors is the same.

**Definition 0.3 (Codimension simplicial complex).** A **codimension  $c$   $d$ -simplicial complex**  $\Sigma$  is a collection of simplices of dimensions  $d - c, \dots, d$  such that for any  $\sigma \in \Sigma^k$  with  $k > d - c$  we have  $\partial\sigma \subset \Sigma$ . In such a complex all simplices in dimension  $< d - c$  are neglected.

**Definition 0.4 (Frame).** A  **$k$ -frame**  $\mu$  of a complex  $\Sigma$  is the image, by a simplicial mapping, of the boundary  $\partial\Delta^k$  of the standard  $k$ -simplex into  $\Sigma$ , and consists of a collection of  $k - 1$  simplices. Let  $\Sigma^{[k]}$  denote the collection of the  $k$ -frames of  $\Sigma$ .

We say that  $\Sigma$  is  **$k$ -plain** if for any  $\mu \in \Sigma^{[k]}$  there is one and only one simplex  $\sigma \in \Sigma^k$  with  $\partial\sigma = \mu$ .

**Definition 0.5 (Order invariant).** A simplicial complex  $\Sigma$  is called **order invariant** if for any  $\sigma \in \Sigma^k$  and any permutation  $s \in S_{k+1}$  acting on its vertices, the reordered simplex  $s\sigma$  is in  $\Sigma$ . A simplicial map  $\phi : \Sigma \rightarrow \Sigma_1$  between order invariant complexes is called **order invariant** if for any  $\sigma \in \Sigma^k$  and any  $s \in S_{k+1}$  we have  $\phi \circ s = s \circ \phi$  on  $\sigma$ .

Suppose  $\Sigma$  is a complex, the  $k$ -skeleton of  $\Sigma$  is the complex consisting of all faces having dimension at most  $k$ .

**Definition 0.6 (State).** A **state**  $\lambda$  of the  $k$ -faces of an order invariant simplicial complex  $\Theta$  with values in an order invariant simplicial complex  $\Sigma$  is an order invariant simplicial map of the  $k$ -skeleton of  $\Theta$  into the  $k$ -skeleton of  $\Sigma$ . [\[\[?? Should it depend on  \$k\$ ? Or do we have examples for states? -Jinsong\]\]](#)

We now define the state space of a  $d$ -dimensional quantum cocycle as follows.

**Definition 0.7 (State space).** A  **$d$ -state space**  $\Sigma$  is a codimension 3 finite order invariant simplicial  $d$ -complex which is  $d$ -plain.

**Example 0.8.** Let  $G$  be a finite group. Construct a simplicial 3-complex  $\Sigma_G$  with  $\text{card}\Sigma_G^0 = 1$ ,  $\Sigma_G^1 = \{\sigma_g : g \in G\}$ , such that the simplex opposite to  $\sigma_g$  is  $\sigma_{g^{-1}}$ . For any 2-frame  $\nu : \partial\Delta^2 \rightarrow \Sigma$  with  $\nu(\{i, j\}) = \sigma_{g_{ij}}$  let  $\text{card}\{\sigma \in \Sigma_G^2 : \partial\sigma = \nu\} = 1$  if  $g_{01}g_{12} = g_{02}$  and 0 otherwise. Complete  $\Sigma_G$  to a 3-plain complex, i.e. add a 3-simplex for any 3-frame.

Then  $\pi_1(\Sigma_G) = G$  and if we further complete  $\Sigma_G$  to a  $d$ -plain complex in all dimensions  $d > 3$  we obtain a construction for the space  $K(G, 1)$ .[\[\[?? -Jinsong\]\]](#)

We associate finite dimensional Hilbert spaces to  $d - 1$ -frames in the  $d$ -state space  $\Sigma$ . For any frame  $\nu \in \Sigma^{[d-1]}$  let  $\Xi_\nu = \{\sigma \in \Sigma^{d-1} : \partial\sigma = \nu\}$  and associate to  $\nu$  a complex Hilbert space  $H_\nu$  with orthonormal basis  $\Xi_\nu$ . We denote by  $N_\nu$  the cardinality of  $\Xi_\nu$ .

We now construct Hilbert spaces associated to arbitrary triangulated hypersurfaces as follows. Let  $S$  be an oriented  $d$ -manifold with a triangulation  $\tau$ . Define the **relative orientation**  $\varepsilon_{S, \tau}(\nu)$  of a  $d - 1$ -face  $\nu$  as 1 or  $-1$  depending on whether  $S$  and  $\tau$  induce on  $\nu$  equal or opposite orientations.

For a Hilbert space  $H$  let  $H^1 = H$  and  $H^{-1} = \overline{H}$ , where  $\overline{H}$  is the conjugate Hilbert space of  $H$ . For a state  $\lambda$  of the  $d - 2$ -faces of  $\tau$ , we define the associated Hilbert space

$$H_{S,\tau,\lambda} = \bigotimes_{\nu \in \tau^{d-1}} H_{\nu}^{\varepsilon_{S,\tau}(\nu)}.$$

If  $(S, \tau)$  is a triangulated  $d$ -manifold with a fixed state  $\lambda_0$  of its  $d - 2$ -faces and boundary, we let

$$H_{S,\tau,\lambda_0} = \bigoplus_{\lambda} H_{S,\tau,\lambda}$$

where the direct sum is taken over all states  $\lambda$  of the  $d - 1$ -faces of  $S$  which extend  $\lambda_0$  (this gives a sum over the compatible states of the inner edges). The Hilbert space of an empty  $d - 1$ -manifold is taken to be  $\mathbf{C}$ . Remark that with the above definition if  $\overline{S}$  is the surface obtained from  $S$  by changing the orientation, then  $H_{\overline{S},\tau,\lambda} = \overline{H_{S,\tau,\lambda}}$ .

The Hilbert space  $H_{S,\tau,\lambda_0}$  has a canonical orthonormal basis consisting of all simplicial maps  $\phi : \tau \rightarrow \Sigma$  which restrict to  $\lambda_0$  on  $\tau^{d-2}$ . [\[\[ what is  \$\Sigma\$  here? -Jinsong\]\]](#) This basis can be viewed as consisting of random hypersurfaces with values in the state space.

Let  $\Delta = \Delta^d$  be the standard  $d$ -simplex, and denote by  $\tau$  the standard triangulation of its boundary  $\partial\Delta$ . Let  $H_{\partial\Delta} = \bigoplus_{\lambda} H_{\partial\Delta,\tau,\lambda}$  where  $\lambda$  is a state of the  $d - 1$ -faces of  $\Delta$ .

**Definition 0.9. Quantum chain.** *A quantum  $d$ -chain  $(\zeta, U, [[.]])$  of a complex  $\Delta$  is a triple consisting of a vector  $\zeta = \zeta_{\Delta} \in H_{\partial\Delta}$ , a unitary representation  $U$  of the symmetric group  $S_d$ , called the reordering representation, and a map  $[[.]] : \Delta^{d-2} \rightarrow \mathbf{R}_+$ , called the weight.*

We have for each state  $\lambda$  of the  $d - 1$ -faces of  $\Delta$  a vector  $\zeta_{\Delta,\lambda} \in H_{\partial\Delta,\tau,\lambda}$  such that  $\zeta_{\Delta} = \bigoplus_{\lambda} \zeta_{\Delta,\lambda}$ .

The role of the representation  $U$  is to make the Hilbert spaces associated to hypersurfaces independent of the order of the vertices of the simplices used in the triangulation.

**Definition 0.10 (Reordering).** *Suppose  $\Sigma$  is an order invariant simplicial complex  $\Sigma$ . For any permutation  $s \in S_d$  the unitary operator  $U_s$  acts on the sum  $\bigoplus_{\nu \in \Sigma^{[d-1]}} H_{\nu}$ , with  $U_s(H_{\nu}) = H_{s\nu}$ , such that  $U_s$  is linear for  $s$  even and conjugate linear for  $s$  odd, and  $s \mapsto U_s$  is a representation of  $S_d$ .*

A local change of order  $s$  of the triangulation  $\tau$  consists of a permutation  $s_{\nu}$  of the vertices of each simplex of  $\tau$ , such that the resulting simplicial complex  $s\tau$  is again a triangulation.

If  $(S, \tau)$  is a triangulated  $d - 1$ -manifold with a state  $\lambda$  of its  $d - 2$ -skeleton, a local change of order  $s$  of  $\tau$  induces a unitary  $U_s = \bigotimes_{\nu \in \tau^{[d-1]}} U_{s_{\nu},\lambda} : H_{S,\tau,\lambda} \rightarrow H_{S,s\tau,s\lambda}$ , where  $U_{s_{\nu},\lambda} = U_{s_{\nu}}^{\varepsilon_{S,\tau}(s\nu)} : H_{\nu}^{\varepsilon_{S,\tau}(\nu)} \rightarrow H_{s\nu}^{\varepsilon_{S,s\tau}(s\nu)}$ . The map  $s \mapsto U_s$  is functorial, and so the Hilbert space associated to  $(S, \tau)$  is independent, up to a natural isomorphism, of the ordering of the simplices of the triangulation.

A permutation  $s$  of the vertices of  $\Delta$  induces a change of orientation on  $\tau$  and a corresponding unitary  $U_s : H_{\partial\Delta,\tau} \rightarrow H_{\partial\Delta,s\tau}$ . There is a natural isomorphism between  $(\partial\Delta, s\tau)$  and  $(\partial\Delta, \tau)$ , which is orientation preserving or reversing depending on the parity of  $s$ , and induces a linear or conjugate linear unitary identification map  $I_s : H_{\partial\Delta,s\tau} \rightarrow H_{\partial\Delta,\tau}$ .

**Definition 0.11 (Symmetry).** *The cochain  $\zeta$  is called symmetric if for any change of orientation  $s$  of the triangulation  $\tau$  we have*

$$\zeta_{\Delta} = I_s U_s(\zeta_{\Delta}).$$

We extend the definition of the weight to an arbitrary triangulated  $d - 2$ -manifold  $(S, \tau)$  with a fixed state  $\lambda$  by

$$[[S]] = \prod_{\mu} [[\lambda(\mu)]]$$

over all  $d - 2$ -simplices  $\mu$  of  $\tau$ .

We now extend the definition of the cochain  $\zeta$  to arbitrary oriented triangulated  $d$ -manifolds.

Let  $V$  be an oriented singular  $d$ -manifold with boundary,  $\tau$  a triangulation of  $V$ ,  $\tau_0$  a triangulation of  $\partial V$  which coincides, up to reordering, with the restriction of  $\tau$  to  $\partial V$  and let  $\lambda_0$  be a state of the  $d - 3$ -faces of  $\tau$  and of the boundary  $d - 1$ -faces of  $\tau$ . A  $d$ -simplex  $\sigma$  of  $\tau$  has orientations coming from  $V$  and from  $\tau$ , and we let  $\varepsilon_{V, \tau}(\sigma) = \pm 1$  be their product. We define

$$\zeta_{V, \tau_0, \lambda_0} = \sum_{\lambda} \kappa_{\lambda} U_{\text{outer}} \otimes Tr_{\text{inner}} \left( \bigotimes_{\sigma} \zeta_{\sigma}^{\varepsilon_{V, \tau}(\sigma)} \right) \in H_{\partial V, \tau_0}.$$

Here  $\lambda$  is a state of  $\tau^{d-1}$  extending  $\lambda_0$  and  $\sigma$  is a  $d$ -simplex of  $\tau$ . The normalization coefficient  $\kappa_{\lambda}$  is a product over  $d - 2$ -simplices  $v$  of  $\tau$

$$\kappa_{\lambda} = \prod_v [[\lambda(v)]]^{n(v)-2}$$

where  $n(v)$  is the number of  $d - 1$ -simplices neighboring  $v$ , counted with multiplicities.

The trace  $Tr_{\text{inner}}$  is a tensor product of traces  $\bigotimes_{\nu} Tr_{U, H_{\nu}}$  taken over all inner  $d - 1$ -simplices  $\nu$ . For any such  $d - 1$ -simplex  $\nu$  the two neighboring  $d$ -simplices  $\sigma$  and  $\sigma'$  induce two opposite orientations, say  $+$  and  $-$ , and possibly different orders of its vertices,  $o$  and  $o'$ . Let  $s$  be a permutation with  $so = o'$ . Then  $\zeta_{\sigma} \in H_o \otimes H_{\partial\sigma-o}$  and  $\zeta_{\sigma'} \in \overline{H}_{o'} \otimes H_{\partial\sigma'-o'}$ . Thus  $U_s H_o = H_{o'}$  and for  $\xi \in H_o, \xi' \in H_{o'}$  we let  $Tr_{U, H_{\nu}}(\xi \otimes \overline{\xi'}) = \langle U_s \xi, \xi' \rangle$ . After applying the traces over all inner faces  $\zeta_{\text{outer}} = Tr_{\text{inner}} \bigotimes_{\sigma} \zeta_{\sigma}^{\varepsilon_{V, \tau}(\sigma)}$ , the result is in the Hilbert space of the boundary. The boundary triangulation  $\tau_0$  may induce on each boundary face  $\nu$  an order  $o'$  which is different from the order  $o$  induced by the neighboring simplex, and we use the unitary  $U_{\text{outer}} = \prod_{\nu} U_s$  to obtain  $U_{\text{outer}} \zeta_{\text{outer}} \in H_{\partial V, \tau_0}$ . Remark that if the boundary of  $V$  is empty, then the result is a complex number.

**Proposition 0.12.** *The value  $\zeta_{V, \tau_0, \lambda_0}$  of the cochain is invariant to the reordering of the vertices of each  $d$ -simplex  $\sigma \in \tau$ .*

*Proof.* The Hilbert space of each  $d - 1$ -face of  $\sigma$  is connected to its pair Hilbert space (coming from the neighboring simplex or from the boundary triangulation) by the reordering unitaries  $U_s$ . If  $\sigma$  is reordered, from the symmetry property of  $\zeta$  we obtain an extra unitary  $U_{s'}$  for each face of  $\sigma$ . The invariance results from the group property  $U_s U_{s'} = U_{ss'}$ .  $\square$

Remark that we have used the reordering unitaries  $U$  as a buffer, or glue, between any two  $n$ -simplices, and between an  $n$ -simplex and the boundary. The formula simplifies considerably when the triangulation is nonsingular, in which case none of the reordering unitaries are needed (except in the proof of the invariance to reordering).

From the definition we have the following property.

**Proposition 0.13** (glueing property). *Let  $V = V_1 \cup_S V_2$  be a partition of the singular  $d$ -manifold  $V$  into two singular submanifolds  $V_1$  and  $V_2$  with intersection a singular  $d - 1$ -manifold  $S$ . Suppose that  $\tau$  is a triangulation of  $V$  compatible with the partition  $V = V_1 \cup V_2$ , a state  $\lambda_0$  of the  $d - 3$ -faces of*

$V$  is chosen, and  $\tau_0$  is a reordering of the boundary triangulation of  $\tau$ . Express the boundary Hilbert spaces of  $V, V_1, V_2$  as direct sums over all states  $\lambda_1$  of  $\partial S$  which extend  $\lambda_0$ .

Then

$$\zeta_{V, \tau_0, \lambda_0} = \bigoplus_{\lambda_1} [[\partial S]] \text{Tr}_{H_S} (\zeta_{V_1, \tau_0, \lambda_0} \otimes \zeta_{V_2, \tau_0, \lambda_0}).$$

*Proof.* The formula follows immediately from the local form of the definition of  $\zeta_{V, \tau_0, \lambda_0}$ . The only thing that has to be checked is the contribution to the weight of a  $d - 2$ -face  $v$  of  $S$  in the term corresponding to a state  $\lambda$  of the  $d - 1$ -faces of  $V$ . If  $\sigma$  is a face of  $S$ , then the power of  $[[\lambda(v)]]$  in  $V_1, V_2$  and  $V$  is  $n_1(v) - 2, n_2(v) - 2$  and  $n(v) - 2$ , where  $n_i(v)$  is the number of  $d - 1$ -faces which neighbor  $v$  in  $V_i$ . For an inner, respectively outer, face  $v$  of  $S$ , we have  $n(v) = n_1(v) + n_2(v) - 2$  respectively  $n(v) = n_1(v) + n_2(v) - 1$ , since 2, respectively 1, neighbor  $d - 1$  faces of  $v$  in  $S$  are glued, and so  $(n(v) - 2) - (n_1(v) - 2) - (n_2(v) - 2)$  equals 0, respectively 1.  $\square$

Partition the boundary of the standard  $d + 1$ -simplex  $\Delta^{d+1}$  into two nonempty complexes  $D$  and  $D'$ , with  $k$  and respectively  $k' = d + 2 - k$   $d$ -faces, sharing a common  $d - 1$ -dimensional boundary  $S$ . Then  $D$  and  $D'$  have isomorphic state spaces and the same boundary, but their inner triangulations differ. A change of triangulation  $D \rightarrow D'$  inside a triangulated manifold is called a **shell move**.

In classical algebraic topology, invariance under this type of move around the boundary of a  $d + 1$ -simplex characterizes  $d$ -cocycles. Analogously we have the following definition.

**Definition 0.14 (Cocycles).** *The quantum  $d$ -cochain  $\zeta$  is called a  $d$ -cocycle if it is symmetric and for any shell move  $D \rightarrow D'$ , any state  $\lambda_0$  of the  $d - 3$  faces of  $S = \partial D = \partial D'$  and any states  $\lambda$  and  $\lambda'$  of the  $d - 3$  faces of  $D$ , respectively  $D'$  which coincide with  $\lambda_0$  on  $S$  we have*

$$\zeta_D = \zeta_{D'} \in H_{S, \lambda_0}.$$

Let  $V$  be a singular  $d$ -manifold and partition its boundary into two nonempty hypersurfaces  $D$  and  $D' = \partial\Delta \setminus D$  sharing a (possibly empty) common  $d - 2$ -dimensional boundary  $S = \partial D = \partial D'$ . Let  $\lambda_0$  be a state of the  $d - 3$  faces of  $\Delta$  and of the boundary  $S$ , and let  $\zeta_{V, D, D', \lambda_0} \in H_{D, \lambda_0} \otimes H_{\overline{D'}, \lambda_0}$  be the projection of the cochain value  $\zeta_{V, \lambda_0} \in H_{\partial V, \lambda_0}$  onto the subspace  $H_{D, \lambda_0} \otimes H_{\overline{D'}, \lambda_0} \subset H_{\partial V, \lambda_0}$ . Then

$$[[S]]^{-1} \zeta_{V, D, D', \lambda_0} \in H_{D, \lambda_0} \otimes H_{\overline{D'}, \lambda_0} \simeq \text{Hom}[H_{D, \lambda_0}, H_{D', \lambda_0}]$$

induces a map  $T_{V, D, D', \lambda_0} \in \text{Hom}[H_{D, \lambda_0}, H_{D', \lambda_0}]$ , called the **transition operator** from  $D$  to  $D'$  associated to  $V$ .

**Definition 0.15 (Isometry).** *The quantum  $d$ -cochain  $\zeta$  is called **isometric** if for any partition of the boundary of the elementary  $d$ -simplex  $\partial\Delta = D \cup_S \overline{D'}$  into surfaces having  $k$  and respectively  $k'$  faces with  $k \leq k'$ , and any state  $\lambda_0$  of  $S$  and the  $d - 3$ -faces of  $\Delta$ , the transition operator  $T_{\Delta, D, D', \lambda_0} : H_{D, \lambda_0} \rightarrow H_{D', \lambda_0}$  is isometric.*

Remark that for  $d$  odd and  $k = \frac{d+1}{2}$  the transition operator  $T$  results unitary.

**Definition 0.16 (Equivalence).** *Two quantum  $d$ -cocycles  $\zeta$  and  $\zeta'$  having supports  $\Sigma$  and  $\Sigma'$  are called **equivalent** if there is an order invariant isomorphism  $\phi$  of the  $d - 1$ -skeletons of  $\Sigma$  and  $\Sigma'$ , and unitaries  $V_\nu : H_\nu \rightarrow H_{\nu'}$  between the Hilbert spaces of the corresponding  $d - 1$ -frames, such that with  $V = \bigoplus_\nu V_\nu$ , the reordering unitaries satisfy  $U'_s = V U_s V^{-1}$ , and the cocycles have the property  $\zeta' = V \zeta$ .*

Thus the analogue of the coboundary perturbation in classical algebraic topology is a perturbation with values in the unitary group  $U(N_\nu)$  for each  $d - 1$ -frame  $\nu$ . In the group example we recover the usual group cohomology with values in the multiplicative complex numbers of module 1.

## References