Lectures 1-24 of Adrian Ocneanu's Course "Higher Representation Theory"

Notes by the Harvard group

1 1-dimensional Topological Quantum Field Theory

The plan is to get mathematical laws from the structure of space and build math objects satisfying those laws to fill the space. The usual mathematical law are filled in two dimensional space-time, one-dimensional space and one-dimensional time. So we shall build higher mathematics to full the higher dimensions. Mathematical objects are based on symmetry: internal symmetry of matter.

QFT like magma, you can cut it into pieces with boundaries

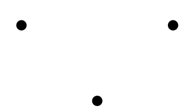


Figure 1: 3 positions in a room, particle is in a linear combination of positions

For any $\vec{x} = (c_1, c_2, c_3) \in \mathbb{C}, |\vec{x}| = 1$, $\mathbb{C}^\times = V$

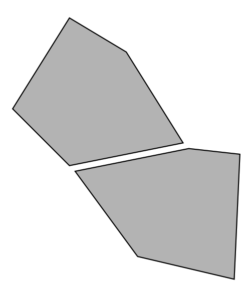
$$e^V = \mathbb{C} \oplus V \oplus \frac{V \otimes V}{2!} \oplus \cdots$$

where V is space for one particle and $V \otimes V$ is space for two particles, etc

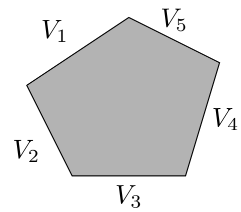
Example 1.1. *Define* e^ξ, $\xi \in V$, $\langle e^\xi, e^\eta \rangle = e^{\langle \xi, \eta \rangle} = e^V_{symm}$

We need to build mathematics which is exponential behaviour in nature, it will have "$\otimes$" instead of "$\oplus$".

An inner product is to glue them together



Choose a basis and $\otimes$ gives $\times$ (Cartesian product) and $\oplus$ gives $\sqcup$ (disjoint union) Now we consider the boundary. The Hilbert space on the boundary $H_{\partial M} = \overline{V} \otimes_A V_1 \otimes_{A_1} V_2 \otimes_{A_2} \cdots$, where $A, A_1, A_2, \ldots$ are abelian algebras and the endpoints match.

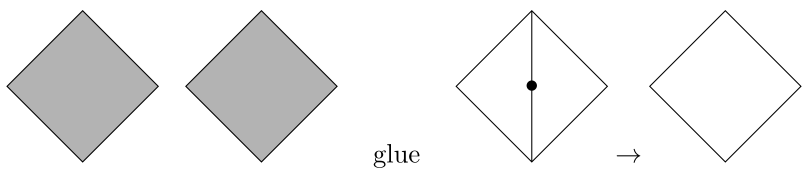


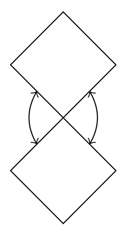
QFT dictionary between topology and algebra. If the inside shrinks to a point, algebraically it is $\mathbb{C}$. Inside gives a map $Z_M : H_{\partial M} \to \mathbb{C}$, which is the partition function. $Z_M = \langle \cdot | \zeta_M \rangle$

Main axiom: Result does not depend on cutting.

G is a finite group and we have the functions $V = \mathbb{C}^G$ on G. V has two laws: $(V, \cdot)$ pointwise multiplication (involution); $(V, *)$ convolution $f(g) \mapsto \overline{f(g^{-1})}$.

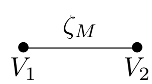
It can be made into a coproduct (linear dual) by taking $V \otimes V \to V$ or $V \to V \otimes V$. This gives hopf algebras which may be nonabelian.





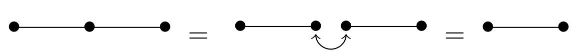
glue similarly.

Take 1 dimension the manifold M



where V_1, V_2 vector space and $\zeta_M \in V_1 \otimes V_2 = V_{\partial M}$ and $p : \overline{V_2} \to V_1$ is the operator.

Given 2 segments:



same manifold $p^2 = p^* = p$ (also flipping is the same), i.e.

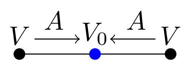
$$\sum_{v \text{ basis}} \langle p v_1 | v \rangle \langle v | v_2 \rangle = \langle p^2 v_1 | v_2 \rangle$$

How can we solve the projection?

$$p = \begin{bmatrix} & \\ & \end{bmatrix}$$

Columns generate the image. We can solve by Gram-Schmidt $A^* : V_0 \to V_0$ some new space, A^* is an isometry embeds V_0 into V. $p = A^*A$, $AA^* = 1_{V_0}$ cut the projection in two.

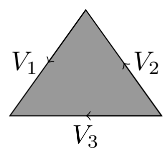
$$V \xrightarrow{A} V_0 \xleftarrow{A} V$$



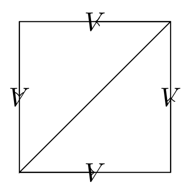
Now may not have many labels? These are called **TYPES** labels which need to match and can be forgotten if they are inside.

$E_{ij}E_{jk} = E_{ik}$ for any j not matrix units $E_{ij}E_{j'k} = \delta_{ij'}E_{ik}$ Then $E_{ij} = E^*_{ji}$. Take $1 \in I$ and $0 \notin I$ then E_{11} is a projection. New $0 : E_{10}E_{01} = E_{11}$, $E_{ij} = E_{11}E_{10}E_{01}E_{1j}$.

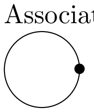
In the $2D$ sketch, the basic unit is a triangle



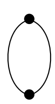
$V_1 \otimes V_2 \to V_3$



Associativity gives algebras

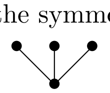


gives the trace from V into $\mathbb{C}$.

We have $tr(ab) = tr(ba)$ because  is symmetric.

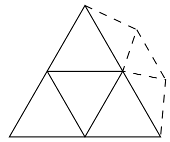
2 point suspension of triangle by composing 2 squares. Associative algebra with a trace, 2 on the same square

$SU(2)$ has subgroups, one of which is the symmetries

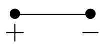
The Dynkin diagram for D_4 looks like 

The classification of subgroups of $SU(2)$, $SU(3)$. How to build higher symmetries? (Higher Dynkin diagrams?)

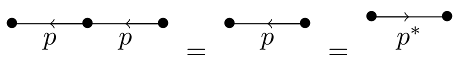
The Wigner $3j$ symbol gives the Clebsch-Gordon coefficients. Computed by Wigner in a very complicated manner.



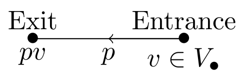
There are six factorization in the denominator of the formula, corresponding to the three triangles and three squares in the rotation of the parts of the pyramid.

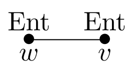
One dimensional manifold 

with orientation for the boundary. It has the property that



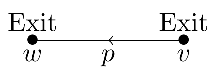
p is a projection from $V_\bullet$ to $V_\bullet$.





$= \langle pv, w\rangle$

is the inner product



$\in V_\bullet \otimes \overline{V_\bullet} = V_{\partial M}$

$\sum_{v \text{ in orthogonormal basis of } V} pv \otimes \overline{v}$ is a Dirac bracket notation

The usual notation

$$|p| = \sum_{v \in V} |p|v\rangle \otimes \langle v|, \quad |p|p| = |p|.$$

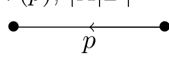
You can have different types of vacua. $p_{ij} \Rightarrow |_i p|_j$, ${}_i|p|_j p|_k =_i |p|_k$ takes you to different vacua.

We solved the one-dimensional TQFT by finding a basis in $p(V)$ called w_i

$$|p| = \sum_{w \in p(V)} |w\rangle \otimes \langle w|.$$

What is the interpretation of the identity here? $\langle v|id|w\rangle = \langle v|w\rangle \Rightarrow |id| = 1$.

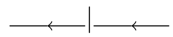
How do we glue things? $\sum_v \langle v|p|v\rangle = Tr(p)$; $|A|B| = \sum_{v\in V} |A|v\rangle\langle v|B\rangle$

What kind of matter does p describe? 

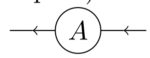
matter isotopic

$$\longrightarrow = \longleftarrow : p = p^*$$

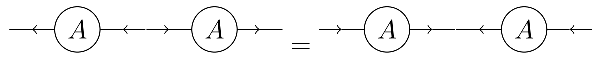
infinitely divisible

 : $p^2 = p$

permeable wall, only some (a linear subspace) vectors v pass through.

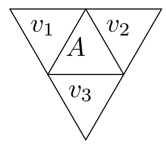
an operator A can be described as 

If

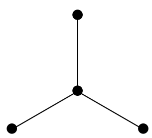


then A can be recovered by $Tr(A)$ which is the coefficients of the characteristic polynomial.

One dimension higher would be polygons

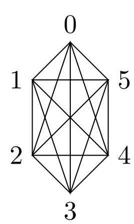


higher dimensional Dirac bracket

 has some quantum symmetries, an insight into quiver theory. Has exactly 6 quantum symmetries.

Framework of the course: The structure of low-dimensional space gives the most useful algebraic laws with QFT as a dictionary. Fill these laws (or space) with rich examples based on symmetry (internal symmetries of matter).

On a surface you can keep changing the triangulation but the vertices always stay the same.

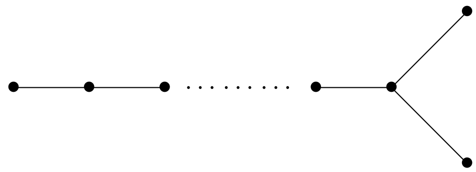


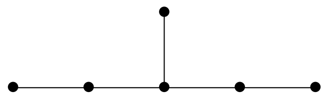
Codimension 1 faces $\hat{0}, \hat{1}, \ldots, \hat{5}$; Codimension 2 faces $\widehat{01}, \widehat{02}, \ldots,$ Codimension 3 faces $\widehat{012}, \ldots,$

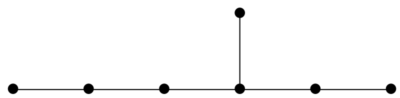
2 Dynkin Diagrams

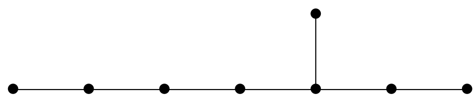
The following is the list of finite simply-laced Dynkin diagrams, together with the tadpole graph T_n (List A):

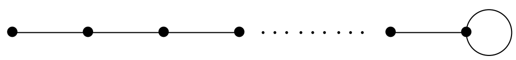
A_n $n \geq 1$.

D_n  $n \geq 4.$

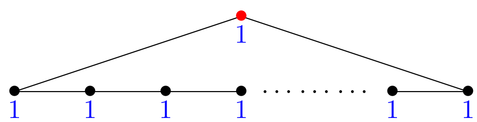
E_6 

E_7 

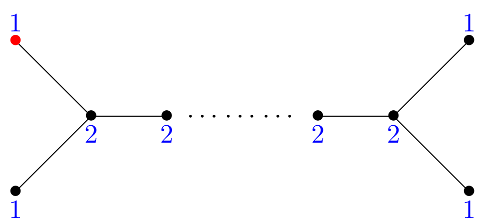
E_8 

T_n  $n \geq 1.$

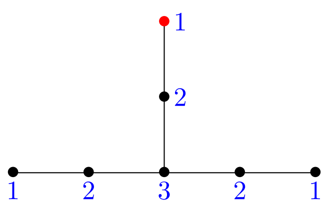
The following is the list of simply-laced affine Dynkin diagrams together with the two tadpole graphs $\widetilde{T_n^{(1)}}$ and $\widetilde{T_n^{(2)}}$ (List B): (The graph $\widetilde{A_1}$ here refers to the graph with two vertices and two edges homeomorphiic to a circle.)

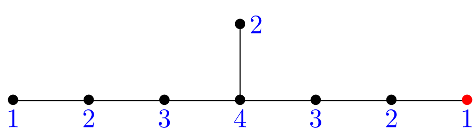
$\widetilde{A_n}$ 

$n \geq 1.$

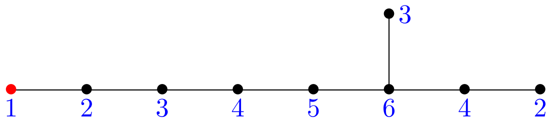
$\widetilde{D_n}$ 

$n \geq 4.$

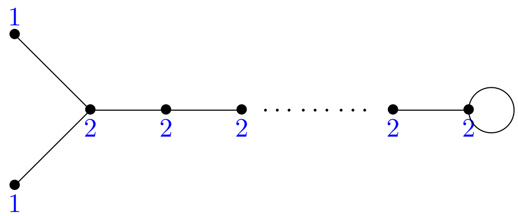
$\widetilde{E_6}$ 

$\widetilde{E_7}$ 

$\widetilde{E_8}$

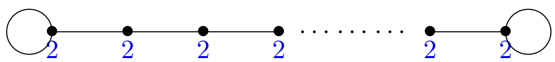


$\widetilde{T_n^{(1)}}$



$n \geq 2.$

$\widetilde{T_n^{(2)}}$



$n \geq 1.$

Let G be a unoriented finite graph with vertices $v_1, \ldots, v_n$, $n \geq 1$. The adjacency matrix of G is a symmetric matrix $\mathbb{A}_G = [a_{ij}]_{i,j=1}^n$ with $a_{ij} = 1$ if there is an edge between vertices v_i and v_j and with $a_{ij} = 0$ otherwise. In particular, we have $a_{ii} = 1$ if there is a loop at v_i. The norm $\|G\|$ of the graph G is the operator norm $\|\mathbb{A}_G\|$ of $\mathbb{A}_G$:

$$\|\mathbb{A}_G\| = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathbb{A}_G\}.$$

For example, the adjacency matrix of A_4 is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

the adjacency matrix of E_6 is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

and the adjacency matrix of $\widetilde{T_2^{(2)}}$ is

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

An $n \times n$ matrix $\mathbb{A}$ is irreducible if for any $1 \leq i, j \leq n$, there exists $k > 0$ such that the (i, j) entry of $\mathbb{A}^k$ is not zero. The adjacency matrices of the graphs in List A and List B are irreducible (apart from A_1). To see this, note that the (i, j) entry of $\mathbb{A}^k$ is the number of paths with length k from i to j and the graphs in List A and List B are connected.

Corollary 2.1. *Every graph in the list B has norm equal to* 2.

Proof. One can check that the vector indicated in blue is the Perron-Frobenius eigenvector for the corresponding adjacency matrix. The effect of multiplying a vector by $\mathbb{A}_G$ is to take the sum of neighbours. In each of the graphs in the list B, the sum of neighbours of a vertex is two times the value of the vertex. Therefore the Perron-Frobenius eigenvalue is 2. Then by the Perron-Frobenius Theorem, the norm of the graphs are also 2. $\square$

Example 2.2. *For $\widetilde{T_2^{(2)}}$, we have*

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}.$$

For $\widetilde{E_6}$, we have

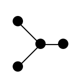
$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 4 \\ 4 \\ 2 \\ 2 \end{bmatrix}.$$

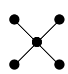
Theorem 2.3. *The only connected unoriented multi-graphs with $\|\mathbb{A}_G\| < 2$ are in the list A and with $\|\mathbb{A}_G\| = 2$ are in the list B.*

Proof. Suppose that G is a finite graph with norm < 2. By a direct computation on adjacency matrix and the Perron-Frobenius Theorem, we have G does not contain any graph in List B and

(1) There is at most one edge between two vertices;

(2) There is no loop in G, i.e. G does not contain $\widetilde{A_n}$;

(3) There is at most one triple point  or one tadpole  i.e. G does not contain $\widetilde{D_n}$ or $\widetilde{T_n^{(1)}}$ or $\widetilde{T_n^{(2)}}$;

(4) There is no quadruple point  i.e. G does not contain $\widetilde{D_4}$.

(5) G with one triple point has no longer legs than E_6, E_7, E_8, otherwise G contains graphs $\widetilde{E_6}, \widetilde{E_7}, \widetilde{E_8}$.

Hence G must be a graph in List A.

Suppose G is a graph with norm 2. Then G is a graph in List A by adding a point with an edge or a taphole and G will be in List B. $\square$

Exercise 2.4. *Determine all eigenvalues of the adjacency matrix of graphs in List A and List B.*

3 Quantum Numbers

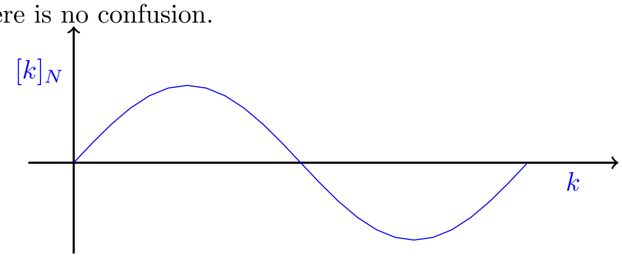
Let $q \neq 0$ be a number in $\mathbb{C}$. For any $k \geq 1$, we denote by

$$[k] = [k]_q = \frac{q^k - q^{-k}}{q - q^{-1}} = \sum_{j=1}^{k} q^{k-(2j+1)}.$$

The number $[k]$ is called a quantum number with respect to q. For instance,

$$[1] = 1, \quad [2] = q + q^{-1}, \quad [3] = q^2 + q^{-2} + 1, \quad \ldots .$$

Let N be an integer such that $N \geq 1$. When $q = e^{i\pi/N}$, we denote by $[k]_N = [k]_{e^{i\pi/N}} = \frac{\sin\frac{k\pi}{N}}{\sin\frac{\pi}{N}}$ if there is no confusion.



Proposition 3.1. *For $k, n \geq 1$, $n \geq k$ we have*

$$[k][n] = [n-k+1] + [n-k+3] + \cdots + [n+k-1].$$

In particular, $[2][n] = [n-1] + [n+1]$.

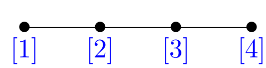
Proof.

$$\begin{aligned}
[k][n] &= \sum_{j=1}^{k} q^{k-(2j+1)} \sum_{l=1}^{n} q^{n-(2l+1)} \\
&= q^{n+k-2} + 2q^{n+k-4} + \cdots + kq^{n-k} + \cdots + kq^{k-n} + \cdots + 2q^{-n-k+4} + q^{-n-k+2} \\
&= [n-k+1] + [n-k+3] + \cdots + [n+k-1].
\end{aligned}$$

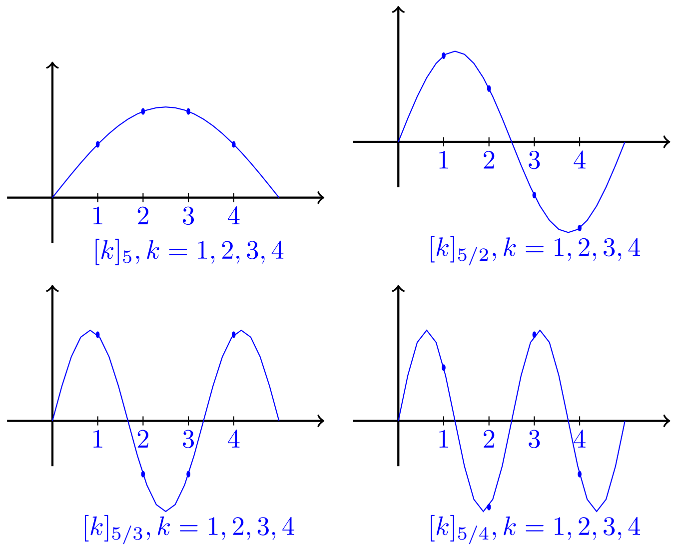
□

The eigenvalues and the eigenvectors of an adjacency matrix of (affine) Dynkin diagrams can be calculated using quantum integers.

Example 3.2. *Suppose λ is a nonzero eigenvalue in $sp(A_4)$. Then $\lambda = [2]$ for some q. An eigenvector of $\mathbb{A}_{A_4}$ corresponding to the eigenvalue $[2]$ would be:*

A_4 

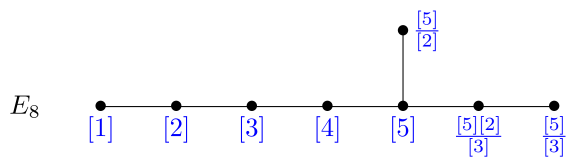
and $[3][2] = [4]$. *By Proposition 3.1, we have* $[5] = 0$. *Solving for* q, *we have* $q = e^{\frac{j\pi}{5}}$, $j = 1, 2, 3, 4$ *and* $\lambda = 2\cos\frac{j\pi}{5}$, $j = 1, 2, 3, 4$. *One can also see the eigenvectors in the following graphs:*



$[k]_5, k = 1, 2, 3, 4$ $\quad$ $[k]_{5/2}, k = 1, 2, 3, 4$

$[k]_{5/3}, k = 1, 2, 3, 4$ $\quad$ $[k]_{5/4}, k = 1, 2, 3, 4$

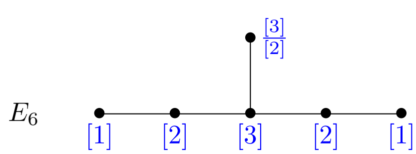
Example 3.3. *For the Dynkin diagram* E_8, *we have*



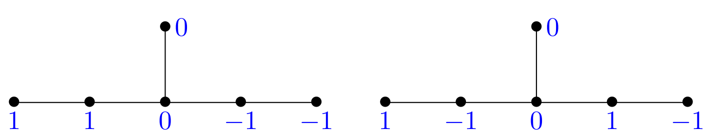
with

$$[5][2] = [4] + \frac{[5]}{[2]} + \frac{[5][2]}{[3]}.$$

Example 3.4. *For the Dynkin diagram* E_6, *we have*



with $[3][2] = [2] + [2] + \frac{[3]}{[2]}$ *and two exceptions when* $[3] = 0$:



Theorem 3.5. *Any graph G in the A-D-E list has (adjacency matrix) norm $[2]_N = 2\cos\frac{\pi}{N}$ and eigenvalues $[2]_{N/k} = 2\cos\frac{k\pi}{N}$ with k in the exponet list, where N is the coxeter number. These are among the eigenvalues of the graph A with the same norm. The eigenvalues of $E_n = E_6, E_7, E_8$ are given by the equation*

$$[n-5] + [n-3] = [n+1].$$

Proof. For A_n, the equation for the quantum integer is

$$[n+1] = 0.$$

For D_n, the equations for the quantum integer are

$$[n] - 2[n-1] + [n-2] = 0 \text{ or } [2] = 0.$$

For E_6, the equations for the quantum integer are

$$[5] = [3] + [1] \text{ or } [3] = 0,$$

or equivalently,

$$[7] = [3] + [1].$$

For E_7, the equation for the quantum integer is

$$[8] = [4] + [2].$$

For E_8, the equation for the quantum integer is

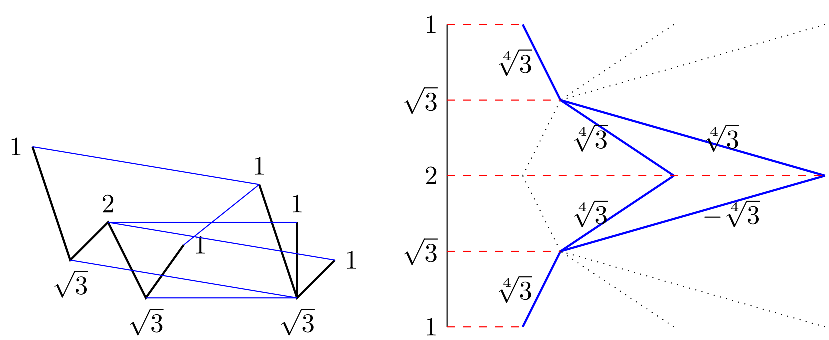
$$[9] = [5] + [3].$$

In a word, for E_n, the equation is $[n+1] = [n-3] + [n-5]$. $\square$

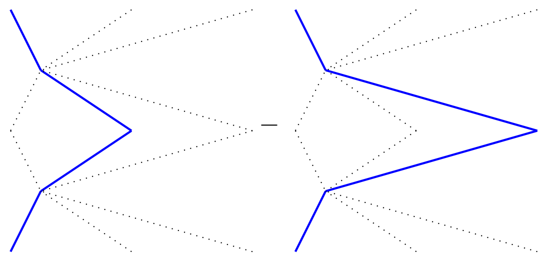
Remark 3.6. *One might use software "Mathematica" to guess the coxeter numbers of the graphs. For E_8, one could run the following command in "Mathematica":*

$$60a \ /. \ NSolve\ [\ \{Sin[3aPi]+Sin[5aPi] == Sin[9aPi], 0 <= a <= 1\},\ \{a\}]$$

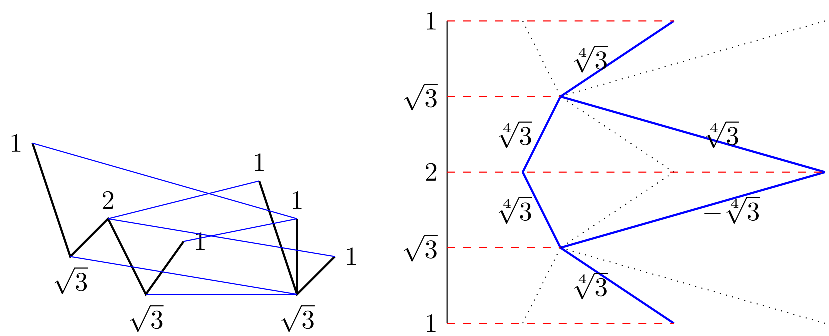
In this subsection, we will find all connections between A_5 and D_4.

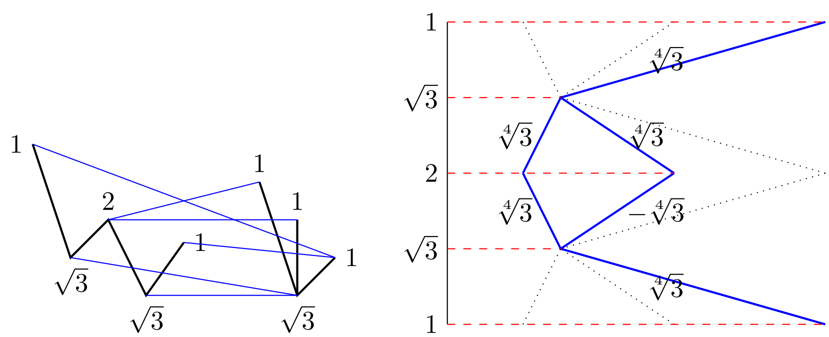


By the graph above, we could read the connection $1, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, 1$ for the diagonal direction and $1,1,1,1,1,1$ for the anti diagonal direction and the essential paths on D_4:

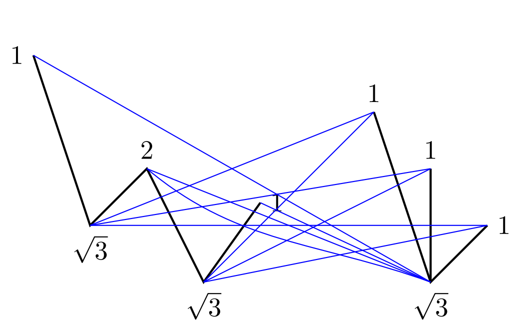


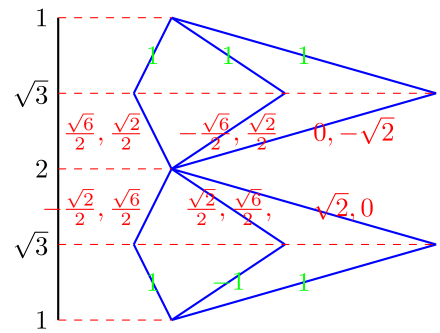
In the following, we could do the similar work for the next two connections between A_5 and D_4.



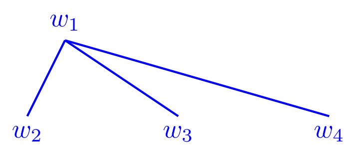


Now we will focus on the following connection (note that it is not difficult to obtain this connection by its definition).





We denote by $\{v_1, v_2, v_3, v_4, v_5\}$ the set of all vertices of A_5 and $\{w_1, w_2, w_3, w_4\}$ the set of all vertices of D_4.



Now we determine the unitary matrices step by step

(1) Consider the unitary matrices with the pairs of fixed diagonal vertices (v_1, w_2), (v_1, w_3), (v_1, w_4). They are all 1×1 matrices. We label green 1s up to $U(1)$ gauge on the edges of D_4 in the first row.

(2) Consider the unitary matrix with the pair of fixed anti diagonal vertices (v_2, w_1). It is a 3×3 matrix given by $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix}$. By the orthogonality of vectors and the cross product of vectors, we obtain that $x_1 + y_1 + z_1 = 0$ and (x_2, y_2, z_3) is a multiple of $(y_1 - z_1, z_1 - x_1, x_1 - y_1)$.

(3) Consider the unitary matrices with pairs of fixed diagonal vertices (v_3, w_2), (v_3, w_3), (v_4, w_4). They are all 2×2 matrices: $\frac{1}{\sqrt{2}} \begin{bmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{bmatrix}$, $\frac{1}{\sqrt{2}} \begin{bmatrix} y_1 & y_2 \\ -y_2 & y_1 \end{bmatrix}$, $\frac{1}{\sqrt{2}} \begin{bmatrix} z_1 & z_2 \\ -z_2 & z_1 \end{bmatrix}$. Up to $U(2)$ gauge, we assume that $z_1 = 0, z_1 = -\sqrt{2}$. Then solving it for x_1, y_1, we obtain that $y_1 = -x_1 = -\frac{\sqrt{6}}{2}$, and find that $x_2 = y_2 = \frac{\sqrt{2}}{2}$. Now we label the edges in the middle two rows in red.

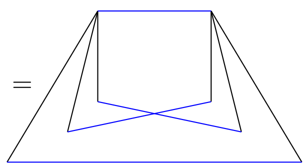
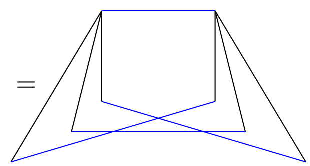
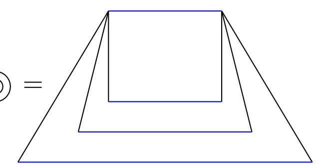
(4) Consider the unitary matrix with the pair of fixed anti diagonal vertices (v_4, w_1). It is a 3×3 matrix given by $\frac{1}{\sqrt{3}} \begin{bmatrix} \frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \sqrt{2} \\ a_1 & a_2 & a_3 \end{bmatrix}$. One can solve it for $a_1 = a_3 = 1, a_2 = -1$ by orthogonality of vectors. Finally, we label the edges in the last row in green.

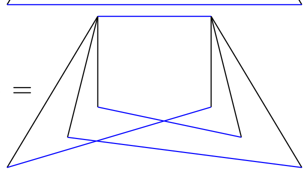
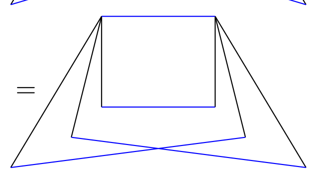
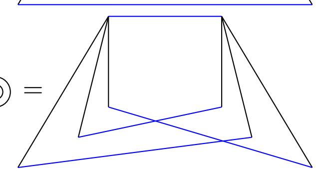
This machinery is good for finding connections between A_n graph and any $A - D - E$ graphs with the same coxeter number (note that any $A - D - E$ graphs have at most one triple point). Hence we have the following theorem:

Theorem 3.7. *For any $A-D-E$ graph G and any vertices v of G, there is a unique (up to gauge) q-map (or biunitary connection): $A \to G$, where A is the A_n graph with the same norm as $\|G\|$, i.e. the same Coxeter number.*

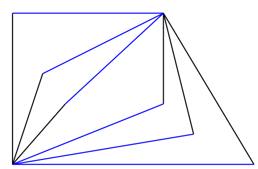
3.1 $D_4 - D_4$

By the definition of the biunitary connection, we see that there are seven possible connections between D_4 graphs.

$\odot =$  $\odot =$  $\circledcirc =$ 

$\circledcirc =$  $\odot =$  $\circledcirc =$ 

and



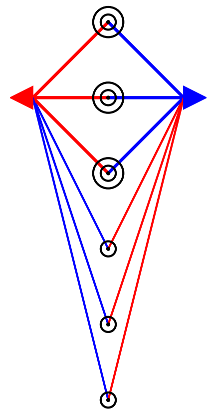
The unitary matrices for the first six connections are 1s. There is only one non-trivial unitary matrix for the last connection. Up to $U(1)$ gauge, we write it as

$$\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & q & -1-q \\ 1 & -1-q & q \end{bmatrix}.$$

Note that $|q| = |1+q| = 1$. Solving it for q, we have $q = e^{2\pi i/3}$ or $q = e^{-2\pi i/3}$. Hence there are two possible ways to assign unitary matrices, we denote one of them by $\blacktriangleleft$ and the other one by $\blacktriangleright$.

The gluing of connections becomes the multiplication and the eight connections form an algebra.

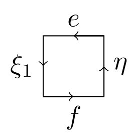
The following graph indicates the relations.



where the red line means a multiplication with  and the blue line means a multiplication with .

Now we will show that the composition of two biunitary connections is a direct sum of finitely many biunitary connections.

Suppose G_1, G_2 are bipartite graphs and G_3, G_4 connects G_1, G_2. We define a map Φ from an edge e_1 in G_1 to a matrix whose entries are linear combinations of edges in G_2 by

$\Phi(\xi_1) = \Big[\sum_{\eta}$  $\eta \Big]_{e,f}$

where $e \in G_3(G_4)$, $f \in G_4(G_3)$ and $\eta \in G_2$. The (e, f)-entry of $\Phi(\xi_1)$ is denoted by $\Phi(\xi_1)_{e,f}$.

Remark 3.8. *If G_2 is an $A - D - E$ graph, then the entry in the matrix is not a sum. It could be a sum if G_2 the Dynkin diagrams B_4, C_4.*

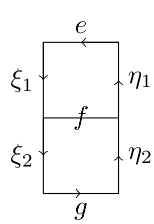
Then the map Φ can be defined on paths of G_1 as

$$\Phi(\xi_1 \xi_2 \cdots \xi_n) = \Phi(\xi_1)\Phi(\xi_2) \cdots \Phi(\xi_n),$$

and

$$(\Phi(\xi_1)\Phi(\xi_2))_{e,g} = \sum_f \Phi(\xi_1)_{e,f} \Phi(\xi_2)_{f,g}$$

If $n = 2$, the graph of (e, g) entry is

$\sum_{\eta_1, \eta_2, f}$ 

By the definition of biunitary connection, we see that the connection between G_1 and G_2 is a biunitary connection if and only if

$$C_1(\Phi(\xi_1)\Phi(\xi_2)) = \Phi(C_1(\xi_1\xi_2))$$

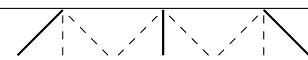
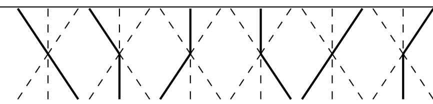
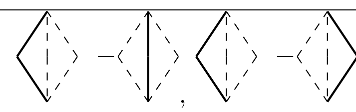
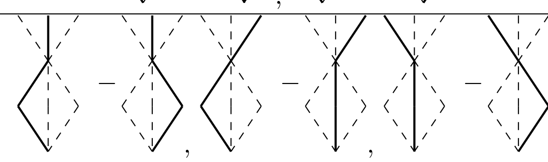
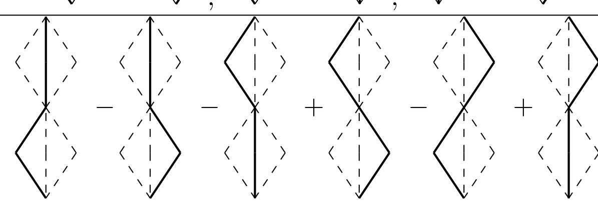
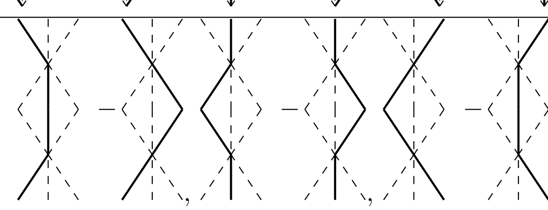
for any path $\xi_1\xi_2$, where C_1 is the contraction operator defined before.

Theorem 3.9 (Gabriel's Theorem). *(1) A (connected) quiver is of finite type if and only if its underlying graph is one of ADE Dynkin diagrams : A_n, D_n, E_6, E_7, E_8.*

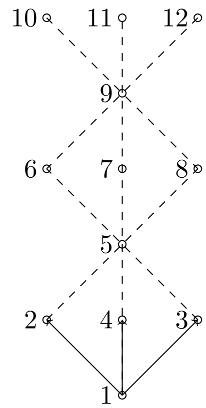
(2) The indecomposable representations are in a one-to-one correspondence with the positive roots of the root system of Dynkin diagram.

Theorem 3.10. *Any $A - D - E$ quiver indecomposable representations is cannonically obtained by essential path from a vertex v.*

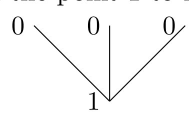
Sketch of Proof. We only consider the graph D_4. The rest of $A - D - E$ graphs will have similar arguments. Recall that the essential paths of D_4 are in the following table.

Length of Ess Path	All Ess Paths
0	$\circ$
1	
2	
2	
3	
4	
4	

Now we draw the following graph and label each point as shown in the graph.



Let the point 1 be a starting point and consider the essential path from the point 1 to the points $1, 2, 3, 4$ respectively. We find the essential path from the top to the bottom. Then there is one essential path from the point 1 to itself and no essential path to the points $2, 3, 4$. Hence the corresponding quiver is



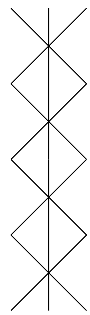
By using the essential paths for D_4, we have that following table for quivers and positive roots.

Starting Points	Quivers	Positive Roots
1	$\begin{smallmatrix}0&0&0\\&1&\end{smallmatrix}$	(0,1,0,0)
2	$\begin{smallmatrix}1&0&0\\&1&\end{smallmatrix}$	(1,1,0,0)
3	$\begin{smallmatrix}0&0&1\\&1&\end{smallmatrix}$	(0,1,0,1)
4	$\begin{smallmatrix}0&1&0\\&1&\end{smallmatrix}$	(0,1,1,0)
5	$\begin{smallmatrix}1&1&1\\&2&\end{smallmatrix}$	(1,2,1,1)
6	$\begin{smallmatrix}0&1&1\\&1&\end{smallmatrix}$	(0,1,1,1)
7	$\begin{smallmatrix}1&0&1\\&1&\end{smallmatrix}$	(1,1,0,1)
8	$\begin{smallmatrix}1&1&0\\&1&\end{smallmatrix}$	(1,1,1,0)
9	$\begin{smallmatrix}1&1&1\\&1&\end{smallmatrix}$	(1,1,1,1)
10	$\begin{smallmatrix}1&0&0\\&0&\end{smallmatrix}$	(1,0,0,0)
11	$\begin{smallmatrix}0&1&0\\&0&\end{smallmatrix}$	(0,0,1,0)
12	$\begin{smallmatrix}0&0&1\\&0&\end{smallmatrix}$	(0,0,0,1)

$\square$

4 ribbon

Ribbon of Type ADE graph G over $\mathfrak{sl}(2)$ which is $\mathbb{Z}_{/2N} \times_{\mathbb{Z}_{/2}} G$. For D_4, $N = 2 \cdot 4 - 2 = 6$,



$\#$ Ess path$(a, \cdot)$ is biharmonic, which is $(\Delta_G - \Delta_{vert})f = 0$. The functions are from vertices of ribbon into $\mathbb{R}$ or $\mathbb{Z}$.

σ_1 is irreducible representation of $\mathfrak{sl}(2)$, then $\sigma_1 \otimes \sigma_k = \sigma_{k-1} \oplus \sigma_{k+1}$. The biharmonic functions are determined by two adjacent levels.

dim biharmonic function on the ribbon of G is $\#$ vertices of G. Idea: What coordinate should we use in the linear span of roots (or weights).

Instead of $\langle \cdot,$ simple root, we use $\langle \cdot, anyroot \rangle$ which is overdetermined. The coordinates could be biharmonic.

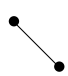
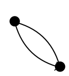
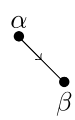
Move 2 down on the ribbon is the Coxeter element. Each Coxeter element determines a ribbon.

Geometric structure of roots on the ribbon. Dots $\equiv$ orthonormal basis.

Theorem 4.1. *The projection of any point on the biharmonic subspace is the geometric root up to a scalar.*

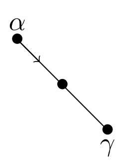
$\langle root(a), root(b) \rangle = const \langle \text{proj } a, \text{proj } b \rangle$ the inner product of any $root(a)$ with other roots ($\equiv$ of coordinates of $root(a)$) = fusion down from $a+$ fusion up from a. Fusion $=\#$ essential paths.

G is an A, D, E graph,

 $=$  $=$ 

$\in$ Hom $[\sigma_1 \otimes \alpha, \beta]$,

where σ_1 is the generator of $\mathfrak{sl}(2)_N$, $N = CoxG$



$\in$ Hom $[\sigma_1 \otimes \sigma_1 \otimes \alpha, \gamma]$, Hom $[\sigma_2 \otimes \alpha, \gamma]$,

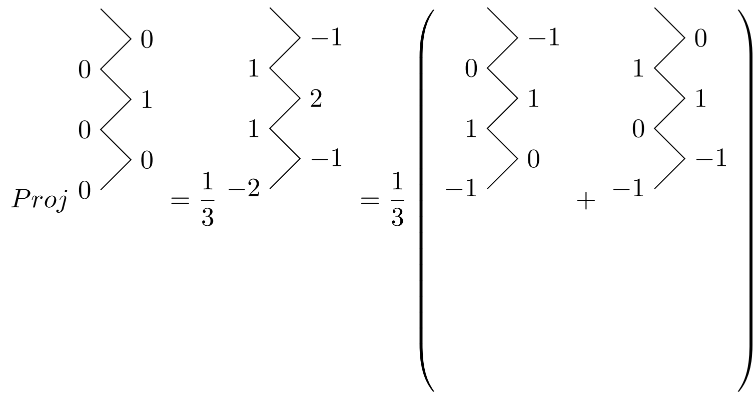
is the essential paths having dimension fusion from (i, α) to $(i + 2, \gamma)$

We will show that

$$Proj_{\text{span fusion}} \delta_{(i,\alpha)} = \frac{1}{N}(\text{fusion}_{i,\alpha} - \text{fusion}_{i+2,\alpha})$$

which is the span of fusion.

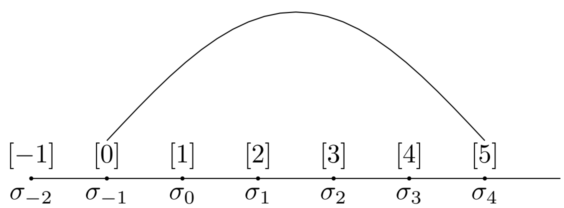
$$\langle \delta_{i,\alpha}, fusion(j,\beta)\rangle = \frac{1}{N}\langle \text{fusion}(i,\alpha) - \text{fusion}(j,\beta), fusion(j,\beta)\rangle.$$



where the second is the fusion upward.

Recall that

$$Hom[u,v] = \oplus_{\text{basis } w} Hom[u,w] \otimes Hom[w,v]$$



$$\sigma_{-2} = -\sigma_0, \quad \sigma_0 \otimes \sigma_3 = \sigma_{0-3} + \sigma_{0-1} + \sigma_{0+3} = \sigma_3.$$

$$\sigma_{-1-k} = \sigma_{-1+k}, \quad \sigma_{N-1-k} = -\sigma_{N-1+k}.$$

$$Hom[\sigma \otimes \alpha, \beta] \simeq Hom[\alpha, \overline{\sigma} \otimes \beta] = Hom[\alpha, \sigma \otimes \beta]$$

$$
\begin{aligned}
&N\langle \delta_{(i,\alpha)}, fusion(j,\beta)\rangle\\
=&N\#Hom[]\sigma_{i-j}\otimes\beta,\alpha]\\
&\langle fusion(i,\alpha)-fusion(i+2,\alpha), fusion(j,\beta)\rangle\\
=&\sum_{(k,\gamma)}\#Hom[\sigma_{k-i}\otimes\alpha,\gamma]\#Hom[\sigma_{k-j}\otimes\alpha,\gamma]\\
&-\#Hom[\sigma_{k-i-2}\otimes\alpha,\gamma]\#Hom[\sigma_{k-j}\otimes\alpha,\gamma]\\
=&\sum_k \#Hom[\sigma_{k-i}\otimes\alpha,\sigma_{k-j}\otimes\beta]-\#Hom[\sigma_{k-i-2}\otimes\alpha,\sigma_{k-j}\otimes\beta]\\
=&\sum_k \#Hom[\sigma_{k-j}\otimes\sigma_{k-i}\otimes\alpha,\beta]-\#Hom[\sigma_{k-j}\otimes\sigma_{k-i-2}\otimes\alpha,\beta]\\
=&\sum_k \#Hom[(\sigma_{k-j-k+i}+\sigma_{k-j-k+i+2}+\cdots+\sigma_{k-j+k-i}-\sigma_{k-j-k-i+2}-\cdots-\sigma_{k-j+k-i-2})\otimes\alpha,\beta]\\
=&\sum_k \#Hom[\sigma_{i-j}\otimes\alpha,\beta]-\#Hom[\sigma_{2k-i-j}\otimes\alpha,\beta]\\
=&N\#Hom[\sigma_{i-j}\otimes\alpha,\beta]+0
\end{aligned}
$$

The root is defined as $root(i,\alpha)$=2N $proj_{biharm}\delta_{(i,\alpha)}$.

Theorem 4.2. *$root\{(i,\alpha), i=0,1\}$ forms a basis. The span of fusion $\equiv$ biharmonic.*

$root(0,\alpha)$ and $root(N-1,\alpha)$ are simple roots.

a simple root basis consists of points a 2 levels $N-1$ apart, $\langle a,b\rangle = 2\delta_{a,b}-\Delta_{a,b}$, where $\Delta_{a,b}$ is # of edges. $2-\Delta$ positive definite $\|\Delta\|=[2]=2\cos\frac{\pi}{N}<2$.

Weights are eigenvalues for the diagonal. Roots are eigenvalues of the adjoint representation and $\langle weight, root\rangle\in\mathbb{Z}$. Weight= integer valued biharmonic function on the ribbon. Fusion is biharmonic (associativity).

For any point on the 2 consecutive levels, there is a biharmonic function which is 1 at the point and 0 elsewhere.

If you know 2 consecutive levels of biharmonic functions, we can compute all others inductively by biharmonicity.

Theorem 4.3. *biharmonic functions are the span of the fusion and have dimension $|G|$.*

Root lattices consists of linear combination of simple roots with integer coefficients.

Theorem 4.4. *Let $a\neq 0$ in the root lattice. Then $|a|^2\geq 2$ and if $|a|^2=2$, then a can be mapped into a simple root by a succession of simple root reflections (=reflection in the hyperplane orthogonal to the roots).*

We work with roots as expressed in the basis of the simple roots. Assume that $|a|^2=2$. Then the coefficients of a in the simple basis have al the same sign and this support is connected on the graph G.

Proof. If the support is not connected, then G splits into 2 subgraphs and $|a|=2$ on each component. Split a into the part with sign + and sign $-$, applying inductively them on each. $\langle x,y\rangle=-1$, the norm $|\cdot|^2$ together $> \sum|\cdot|^2$. x simple roots, $y_1, y_2, \ldots$ are simple roots $\langle x, y_i\rangle = -1$. $\mathrm{Refl}_x(y_i) = y_i + x$, $\mathrm{Refl}_x(x) = -x$. Coefficient for roots at lattices point a. $\mathrm{Refl}_x(a) = -a+$sum of neighbors coefficients. ☐

Lemma 4.5. *$\exists$ reflection which lowers a coefficient strictly.*

Proof. If not, for every coefficient of a, the sum of neighbours is $\geq 2a_i$. Then the norm $\|\Delta_G\| \geq 2$. This ends the proof as we keep lowering coefficient by simple reflection until a single nonzero remain, that one must be 1. ☐

Theorem 4.6. *The $root(i,a)$ and the $fusion(i,a)$ are biharmonic in (i,a).*

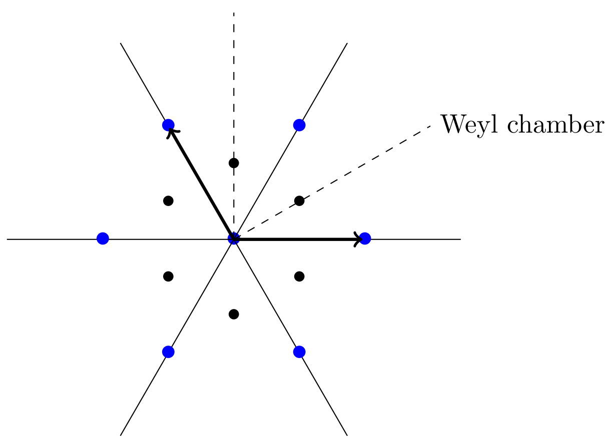
Proof. $\langle root(i,a),(j,b)\rangle$ is biharmonic in (j,b) as sum of fusion $= \langle (i,a), fusion(j,b)\rangle$. ☐

Coxeter element = product of reflections in the hyperplanes perpendicular to a set of simple roots in some order. The Coxeter element is unique up to conjugacy by reflections. The Weyl group is generated by the reflections.

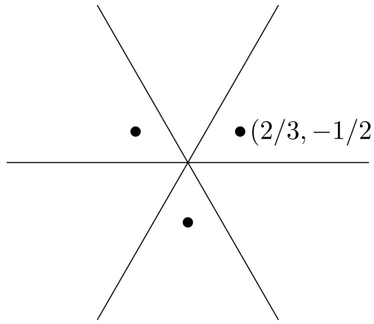
Theorem 4.7. *If we choose simple roots on 2 rows $N-1$ apart on the ribbon, and define the Coxeter element by taking the product of reflection on each row, and then multiply them. Then the Coxeter element moves the ribbon by 2 vertically. $Cox^N = id$.*

Theorem 4.8. *The choice of a Coxeter element organizes the roots into a ribbon on which the Coxeter element is translation by 2. Uniquely up to translations.*

5 SU(3)



Weight lattices are eigenvalue of representations and root lattices are eigenvalue of adjoint representations



Theorem 5.1 (Weyl)**.** *Every irreducible representation has a highest weight ω in the Weyl chamber. The multiplicity of the weights are*

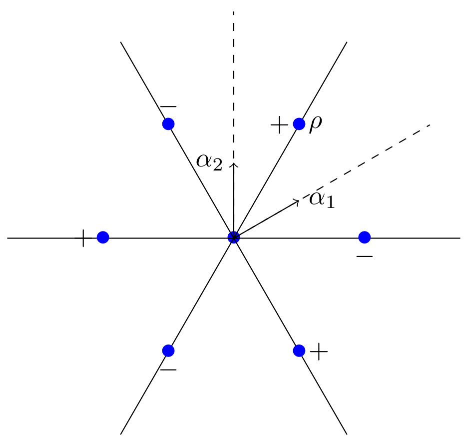
$$\frac{\sum_{w\in W} \varepsilon(w) e^{w(\omega+\rho)}}{\sum_{w\in W} \varepsilon(w) e^{w\rho}},$$

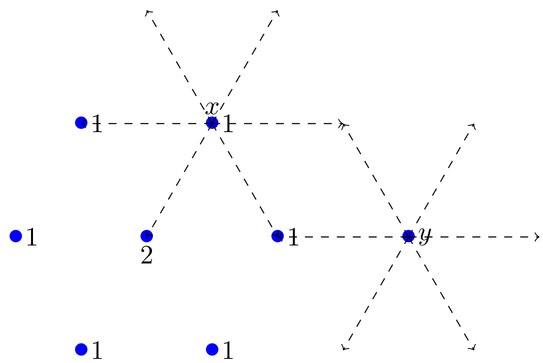
where $\varepsilon(w)$ is the sign of $w \in W$.

The denominator gives the alternating Laplacian Δ^{alt} defined by: f : weight lattice $\to \mathbb{C}$, $\Delta^{alt} f$ is the alternating sum of $w\rho$-weightors

$$(\Delta^{alt} f)(x) = \sum_{w\in W} \varepsilon(w) f(x + w\rho),$$

where ρ is the sum of fundamental weights.





$$\Delta^{alt} mult(x) = 0, \quad \Delta^{alt} mult(y) = 1.$$

The vertices of the polygon are W (highest ω)

$$\Delta^{alt} w(\alpha + \rho) = 1$$

Kostant expansion: $K(\alpha) = \#$ of ways in which α can be written as a sum of positive root. $r_{12} = r_{12} = r_1 + r_2$ in which there are 2 ways.

$$mult = \sum_{w \in W} \varepsilon(w) transl_{-w\alpha}.$$

A, D, E graph G with n vertices The order of Weyl group

$$|W| = |G|! \prod_{i \in \text{Vertice of affine} G} \mu(i) |\{i \in \text{Vert affine } G | \mu(i) = 1\}|,$$

where $\mu(i)$ is the Frobenius-Perron eigenvalue. $|\{i \in \text{Vert affine } G|\mu(i) = 1\}|$ is the center of Lie groups $\equiv$ weight/roots $\equiv$ characters of subgroup of $SU(2)$ in Mackay correspondence.

E_6, E_6^{aff}

$$|W| = 6! \cdot (1112223) \cdot 3,$$

where the last 3 is the number of 1s, which is also $\mathbb{Z}_{/3}$.

Chevalley

$$|W| = \prod_{i \in \text{exponents } G} \frac{i+1}{i}.$$

$\mathfrak{sl}(n+1)$

$$|W| = n!(n+1) = (n+1)!$$

Fundamental period of weights parallelepiped in the direction of fundamental weight α $\mu(\alpha) \to$ volume $()\mu(\alpha)$ in unit parallelepiped $\equiv n!$ simplexes (Weyl alcoves). Period roots = period of weights $\frac{\text{weight lattices}}{\text{root lattices}}$.