Lecture 25 of Adrian Ocneanu

Notes by the Harvard group

Lecture notes for 27 Oct 2017.

Figure 1 shows the graph A_n at level 5 over $sl(3)$. The uppermost point is the trivial representation of degree 0, which is followed by those of degree $1, 2, 3, 4, 5$. The corresponding Coxeter number is the length of the edges of the mirrors; in this case, it is $5 + 3 = 8$. This means that we are at the 8th root of unity. The Young diagrams are not drawn here but can be imagined with 5 columns. The left line gives the symmetric representations, on polynomials in three variables e_1, e_2, e_3. The right line is related to double things like $V \wedge V$, with basis $e_i \wedge e_j$. By Hodge duality, you can map that into the missing e_k, with what in physics is called the totally anti-symmetric tensor on three coordinates. At the 8th root of unity, the ones with degree bigger than 5 would be killed. Remember the Weyl formula for the dimension tells that the dimension is a product of the distances to the mirrors.

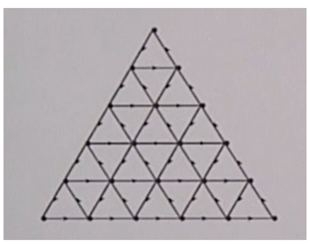

Figure 1: Figure 1

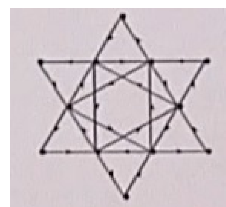

Figure 2: Figure 2

Figure 2 is the first of the exceptionals; these can be classified. The higher Dynkin diagram at the same Coxeter number is the corresponding graph A_n. The octahedron in the middle has wings which are fragments of the A-graph. In representation theory, we have the Schur lemma: if you have the self intertwiner from the representation to itself and if the representation is unitary, then you can show the adjoint is also an intertwiner, then the square is also an intertwiner, and so on, until you find a projection that is an intertwiner. If you have a non-trivial matrix A, then $A + A^*$ and $A - A^*$ are self-adjoint. You can decompose it into sum of self-adjoint elements. Self-adjoint elements can take a polynomial to find a projection, and the projection splits unitary representations into two pieces. That's why irreducible representations have no self-intertwiners.

(Answering several questions from 11:30 to 24:24. Part of the explanation is goes as follows.)

Pictures in Figure 3 both describe the Cartesian products. Bi-harmonicity is where we use the edges here. You tensor with a standard representation on the A_n graph. At the same time you move one edge on the exceptional graph E_5. We start from a corner with a 1 here (see Figure 4 for more details). Then we tensor it with a generator. On the graph E_5, we arrive at a neighbour. There are three edges following the arrows. The product of these edges will be the edges of this ribbon.

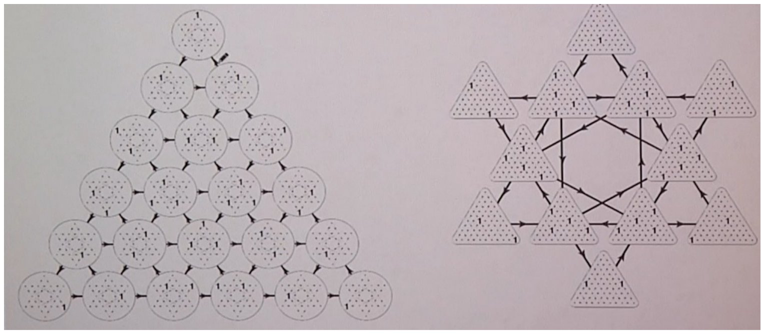

Figure 3: Figure 3

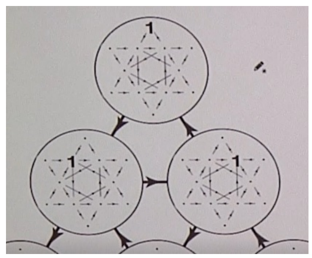

Figure 4: Figure 4

In Figure 5, we have the usual graph D for $sl(2)$. Here the Cartesian product is done in the same way as in the higher case. We take copies of graph D_4 and put it at every point (see the left-hand side). Then you start from here and go to the neighbors, to form roots. The graph has parity, so some points will not appear. We must move one on the A_n graph, and move one on D_4. This is the product of A_n and D_4. This is a root of D_4, the inner product of other roots. The thick red points of half of the ribbon are points will occur. The reason is that if you move one down or up, you must move one down or up in the other graph.

In graph 6, the points that do appear are singled out in thick red. You have one vertex of the graph and the inner product of a particular root. This Cartisian product will be just for $sl(2)$. Remember the formula in the case of $sl(2)$ for finding the inner product of root with other roots. To do that we took the fusion and summed them in two directions, going up and going down. For $sl(3)$, we take the fusion and summing in 6 directions. We can see 6 Weyl chambers (the cones have degree 6), which is exactly the order of Weyl group for $sl(3)$. The period inside the blue lines in Figure 5 separates the ribbon into three.

These modules were found through a technique called conformal inclusion of degree 0 by physicist, and also by Zhengwei Liu. However, the physicists tried without success to find something looking like higher Dynkin diagram, namely, to make angles from the edges. The ribbon worked, and gave the Euclidean root system. The theorems are exactly like the ones for $sl(2)$. Namely, you have the span of the fusion, and the projection of the Dirac mass on the span of fusion gives exactly this geometric root. The inner product is then computed by simply adding the fusions in 6 directions. What you see here is a higher root lattice. Every root has length of $\sqrt{6}$. In the usual A_n case, roots

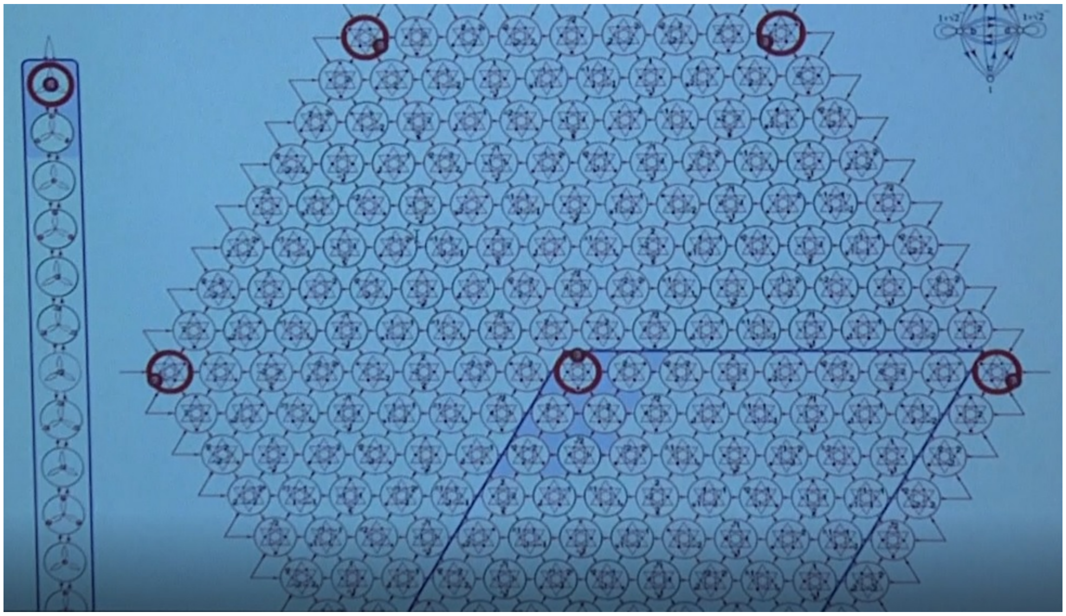

Figure 5: Figure 5

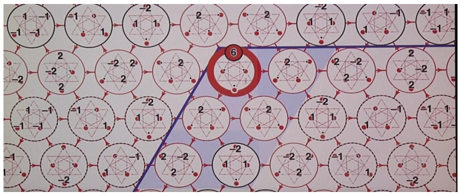

Figure 6: Figure 6

are not abstract but concrete h_{ij}'s with ± 1 and 0 elsewhere. We are going to construct very concrete higher h_{ij}'s for types A, B, C, D, and they will give exactly these inner products. This suggests we should find a vector which has 6 components with ± 1.

Returning to the left-hand side of Figure 5, you can see two adjacent levels. If you take two adjacent levels, the inner product of the root in the center with each of the three, is $+1$. If you take the negatives, then you get the usual simple roots. The basis for the bi-harmonic functions of graph D_4 is four dimensional. This was originally found in the usual representation theory. The same statement here is that the roots in the highlighted 3×2 region are a basis. All the others can be written uniquely as a linear combination of these. If we take the full ribbon, where we get rid of parity problem, three components are of this kind. Three sheets do not communicate with each other. So inner product between the three sheets are 0. The general theorem is that the dimension of the full ribbon is the number of vertices of the graph times the order of underlying (subjacent) Weyl group. We have a Euclidean space with elements roots which are vectors of length $\sqrt{6}$ and have integer product with each other. The whole picture is invariant to translations in three directions. Those translations will appear in the theory: they are the higher analog of the coxeter element.

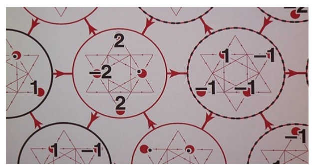

Figure 7: Figure 7

Let us check the bi-harmonicity in Figure 7. See video from 41:05 to 44:00 for more details. We have

$$\mathrm{Hom}[\sigma_i \otimes \alpha, \sigma_j \otimes \sigma_1 \otimes \beta],$$

where σ_1 is a generator.

(Answering questions from 45:00 to 51:00.)

Finally, what we are going to build is actual *representations* of these higher matrices. For these representations what we'll give is the way a matrix element acts and what will happen with the involution is (in the case of sl_3) a permutation of three elements. From 52:00 to 53:00 in the video, self-adjointness is explained with the help of a pad. That will happen in the higher case. You have the representation in the triangular glass pyramid and have an action of the underlying Weyl group on the matrix element, which is the same as permuting the three observers. The matrix units will give us instructions on how to change the vectors exactly like our picture of the Gelfand-Tsetlin representation.

You have something at the base, usually being the actual intertwiner, vectors will grow out of that. Our higher matrices act on these vector spaces and transform them. However, the higher case will be non-associative and the usual definitions will not work.