Lecture 26 of Adrian Ocneanu

Notes by the Harvard group

Lecture notes for 30 Oct 2017.

Today we are going to talk about higher roots. Let g_N be (semi) simple Lie group (with adjacent or underlying Lie group sl(2) for usual math) at the N-th root of 1 (also known as quantum group). Let Gb e a module of g_N and the vertices of G, vert G, be the irreducibles of g_N and graphs G^α for every generating irrep α of g (thus, for any representation α G.) Note that the representation ring of g at the N-th root of unity is a quotient.

Basically, our approach is to use,for instance, mirrors. This is the case of $sl(3)$, which we have looked at last time, and are shown in Figure 1.

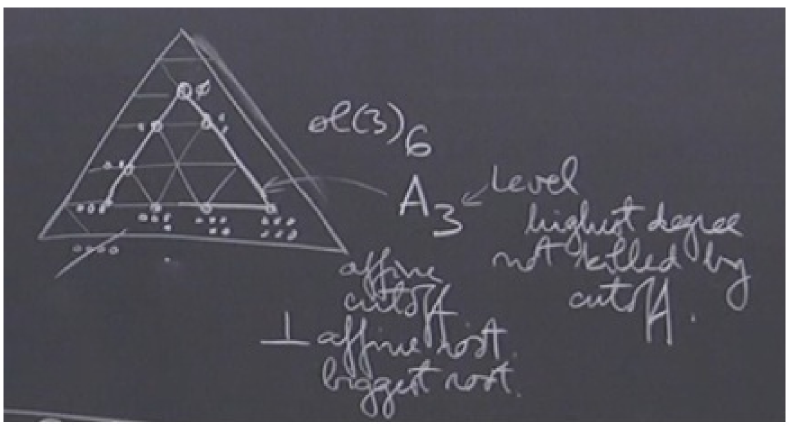

Figure 1: Higher roots for $sl(3)$.

Definition 0.1. *The full ribbon is the cartesian product, Weights of g × Vert G, where Weights of g indicates the weight lattice.*

On this graph we have fusion. We have

$$\text{function}(i, \alpha)(j, \beta) = \dim \text{Hom}[\sigma_{j-i} \otimes \alpha, \beta]$$

where σ_{j-i} is the $j - i$ weight, $\alpha, \beta \in \text{Vert}G$. This extends to tensoring with any representation since we are starting with a module.

The map

$$\text{fusion}(i, \alpha) : \text{ribbon} \to \mathbb{Z}$$

is a biharmonic function.

Let k be the the fundamental representation of g. Then

$$\mathrm{Hom}[\sigma_{j-i}\otimes\alpha,\sigma_k\otimes\beta]\cong\mathrm{Hom}[\bar{\sigma}_k\otimes\sigma_{j-i}\otimes\alpha,\beta]\cong\mathrm{Hom}[\sigma_{\bar{k}+j-i}\otimes\alpha,\beta].$$

by Frobenius reciprocity, where $\sigma_k\otimes\beta$ can be decomposed as

$$\sum_{l\in\text{weight of }\sigma_k}\sigma_l\otimes\beta$$

and $\sigma_{\bar{k}+j-i}$ is the neighbors of j on the weight of g. This is the Laplacian $\Delta_g^{\overline{k}}$ on the weight lattice and Δ_G^k on the veritces (this ensures biharmonicity).

In general, we take a period of weights of g in a multiple of the $N\times\cdots\times N$ torus, so that each higher root appears only once. In the case of $sl(2)$, the period is 2N. For $sl(3)$, the period is N^2.

Theorem 0.2. *The orthogonal projection of the unit vector $\delta_{(i,\alpha)}$ onto the span of the fusion (j,β), multiplied by the size of the period, has entry on (j,β). Said entry is the inner products of the projection of (i,α) and (j,β) and equals the sum*

$$\sum_{w\in\mathrm{Weyl}\,G}\epsilon(w)\,\mathrm{fusion}((i-\rho+w\rho,\alpha),(j,\beta)) \tag{1}$$

where ρ is the Weyl vector of g.

We call $\mathrm{root}(i,\alpha)$, the normalized projection. We have

$$|\,\mathrm{root}(i,\alpha)|^2=|W_g|.$$

We will prove that these power series will generalize Chebyshev polynomials.

Theorem 0.3. *There are unitaries $u_\alpha\in End[\mathbb{C}^{VertG}]$ for α highest weights of the fundamental representation of g, so that extending u : weights $g\to End[\mathbb{C}^{VertG}]$, $\alpha\to u_\alpha$ multiplicatively, we have $G^\alpha=\sum_{\alpha\in wei\ \sigma_\alpha}u_\alpha$.*

In the Weyl chamber (cone), we have

$$\sum\mathrm{fusion}_{0,j}\,t^j=\sum_{w\in W_g}\epsilon(w)\left(\prod_{\alpha\in\mathrm{fund}g}\frac{u_{w\alpha}}{1-t_\alpha u_{w\alpha}}\right)\left(\sum_{w\in W_g}\epsilon(w)\prod_{\alpha\in\mathrm{fund}g}u_{w\alpha}\right),$$

where $\mathrm{fusion}_{0,j}$ is defined as the matrix $(\mathrm{fusion}_{(0,\alpha),(j,\beta)})_{\alpha,\beta}$.

$$\sum\langle\mathrm{root}_{0,j},\cdot\rangle=\sum_{w\in W_g}\prod_{\alpha\in\mathrm{fund}g}1/(1-t_\alpha u_{w\alpha}).$$

In the case of $g=sl(2)$, $G=E_8$

$$\triangle^1_{E_8}=u+u^{-1}. \tag{2}$$

where σ_1 is the fundamental representation of su_2 (spin 1/2). For $\alpha, \beta \in Vert\ E_8$

$$\text{fusion}_{0,j} = \text{fusion}_{(0,\alpha),(j,\beta)_{\alpha,\beta}} = (\frac{u}{1-tu} - \frac{u^{-1}}{1-tu^{-1}})/(u-u^{-1}).$$

$$\langle \text{root}_{0,j}, \cdot \rangle = (\langle \text{root}_{0,\alpha}, root_{j,\beta} \rangle)_{\alpha,\beta} = \frac{1}{1-tu} + \frac{1}{1-tu^{-1}}.$$