Lecture 27 of Adrian Ocneanu

Notes by the Harvard Group

Lecture notes for 1 November 2017.

Let me bring back what we did last time in a picture (Fig. 1). Here is the statement for $\mathfrak{su}(2)$: we have a matrix g_1, which stands for the matrix of the graph. In the case of A_n this matrix has 0's on the diagonal and 1's underneath and so forth:

$$\begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Since the eigenvalues of this matrix are the sum of unitaries and their inverses, this matrix is written as $g_1 = u_1 + u_1^{-1}$. We are going to look at two formulas: one for the number of essential paths and one for the inner product of roots. If a and b are two points on the graph, if you look at the ribbon and travel k levels down (written t_1^k), you can count the number of essential paths on the ribbon from a to b. As we said, we write

$$g_1 = u_1 + u_1^{-1}.$$

Then, at a distance k the power series for the distance is given by the kth coefficient of the power series

$$\left(\frac{u_1}{1 - t_1 u_1} - \frac{u_1^{-1}}{1 - t_1 u_1^{-1}}\right) / (u_1 - u_1^{-1}) = \frac{1}{1 - t_1 g_1 + t_1^2}$$

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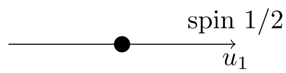
In general, for $\mathfrak{su}(N)$ we sum over the elements of the Weyl group, which in this case has order 2. This fraction defines a power series. The denominator tells us that if you take a value on the ribbon and add it to the corresponding value two floors dow, you get the value in the middle multiplied by the generator g_1. This is exactly the biharmonicity condition.

The root inner product is

$$\mathrm{rt} = \frac{1}{1 - t_1 u_1} + \frac{1}{1 - t_1 u_1^{-1}}.$$

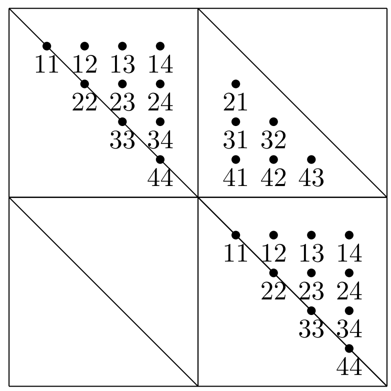
First of all, we look at the Weyl group.



Here u_1 is the vector for spin 1/2. We are using here exponential notation (so u_1^{-1} corresponds to spin $-1/2$). In the work of Weyl there is an interplay between vectors in exponential and additive notations, corresponding to the structure of representations on the one hand and that of roots and weights on the other hand. We can write the above as follows:

$$\mathrm{rt} = \frac{1}{1-t_1u_1} + \frac{1}{1-t_1u_1^{-1}} = \frac{2-t_1g_1}{1-t_1g+t_1^2}.$$

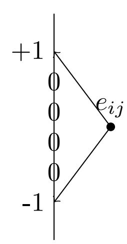
This is the generating function for the inner product of a root with other roots. For the case A_n, the corresponding ribbon is represented as follows for $n=4$.



Note that this is the product of A_3 and a period. The diagonal is the roots of $\mathfrak{sl}(2)$, and the antidiagonal is the rows of the ribbon cut the diagonal at the weights of $\mathfrak{sl}(2)$.

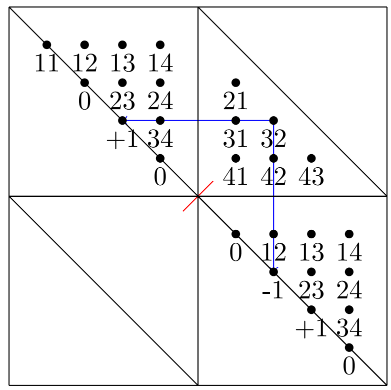
The ribbon coordinate of e_{ij} (viewed as roots h_{ij} on the ribbon) is given by the following: the horizontal (antidiagonal) coordinate is $\frac{j-i}{2}$, and the vertical (diagonal) coordinate is $\frac{i+j}{2}$.

Given a point e_{ij} on the ribbon, in the case of A_n there is an important property (which holds in the case of orbifolds too, but is trickier). The corresponding element h_{ij} is shown below:

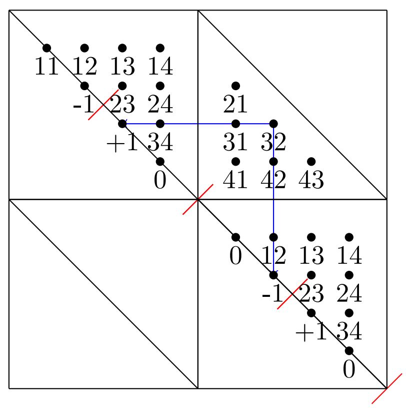


Here, $h_{ij} = e_{ii} - e_{jj}$ or a diagonal $e_i - e_j$.

In the matrix itself, you just go horizontally and vertically. According the periodicity, we get the following matrix. The idea is that you have some mirrors:

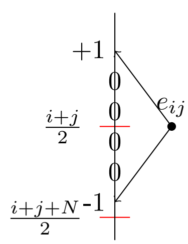


For the case A_n (and B, C, D's), the roots are explicit (concrete) vectors h_{ij}. The formula for the length of the roots for inner product between roots and so on, is given by the concrete vectors.



where denotes the affine Weyl group scaled by N. The affine Weyl mirrors are normally at a distance of 1 weight. Notice they come on *weights*, not on roots.

The mirrors are located on $\frac{i+j}{2}$, and the next one is located on $\frac{i+j+N}{2}$. The distance between these two mirrors is of N weights.



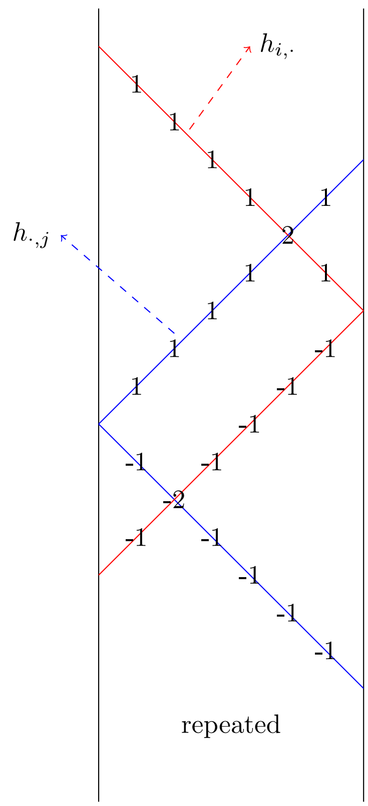
The period is therefore N roots of $\mathfrak{sl}(2)$, starting at an arbitrary points (namely it will be twice the distance between the mirrors).

This is like a kaleidoscope, especially in the case of $\mathfrak{sl}(3)$.

In our particular case, we have a graph A_3 in the space between the mirrors. In a way, that's what A_n is: a cutoff of the weights of $\mathfrak{sl}(2)$. In fact, in our case we would start from spin 1/2 and cut off the weight at spin 2 and end up with the graph A_3.

If we place a red pebble in between two mirrors, it will be reflected by a mirror and become a blue pebble. We will have the same mechanism in higher representations, where the higher graphs A_n will occupy the space between mirrors and these mirrors will be used to reflect pebbles.

Now, for the case A_n, let us see what the inner product of roots is. Given a root h_{ij} (located at e_{ij} on the matrix), the following picture shows the innner product of h_{ij} with the others.



In the picture above, the red segment going through h_{ij} has constant coordinate i (i.e. roots of the form $h_{i,\cdot}$), and the blue segment has constant j (roots of the form $h_{\cdot,j}$). The inner product with roots lying on these lines will be 1, since they share one coordinate. When the line reflects across the mirror, the result will be the same multipied by a factor of -1.

Going back to our formula

$$\frac{2 - t_1 g_1}{1 - t_1 g_1 + t_1^2},$$

recall that the denominator expresses the biharmonicity condition. The 2 in the numerator means that a root times itself is equal to 2. The $-t_1 g_1$ in the numerator means that a root times its neighbor on graph is equal to 1. The reason of the minus sign is that we need to expand the fraction as a power series. With Mathematica (36:10) we see that the first few terms are

$$2+g_1t_1+(g_1^2-1)t_1^2+(g^3-3g)t^3+O(t^4).$$

By going down two rows we see a factor of g_1^2, which means that we take the neighbor twice (on paths of length two) and subtract 2. We can do the same for the other series, the one for the essential paths:

$$\frac{1}{1-t_1g_1+t^2} = 1+gt+(g^2-1)t^2+(g^3-2g)t^3+O(t^4)$$

The coefficients of this power series are the Chebyshev polynomials, defined by

$$\begin{aligned} P_n(\cos x) &= \cos nx \\ Q_n(\cos x) &= \frac{\sin nx}{\sin x}. \end{aligned}$$

By the definition we see that $|P_n(x)| \leq 1$ for $|x| \leq 1$. In the higher case we will have generalizations of the Chebyshev polynomials, which appear to be new. The Q_n polynomials are more or less quantum numbers. Note tha in Figure 1 (Mathematica slides at $43:16$), the second example introduces a second unitary, and the two unitaries correspond to the fundamental weights of $\mathfrak{su}(3)$. Recall that we are using exponential notation. The rest follows analogously to the previous case. Since the Weyl group permutes the weights, the previous formulae are generalized b summing over the Weyl group. As a result, we are able to write the formulae in terms of the two generators g_1 and g_2. In the case of the essential path number formula, we have numerators and signs corresponding to the vertices of the hexagon, which were not present in the previous case. In general, for the higher cases we will interpret the denominators as conditions on permutohedra (in the case of $\mathfrak{sl}(2)$ we had the biharmonicity condition on perumtohedra for S_2, namely segments).

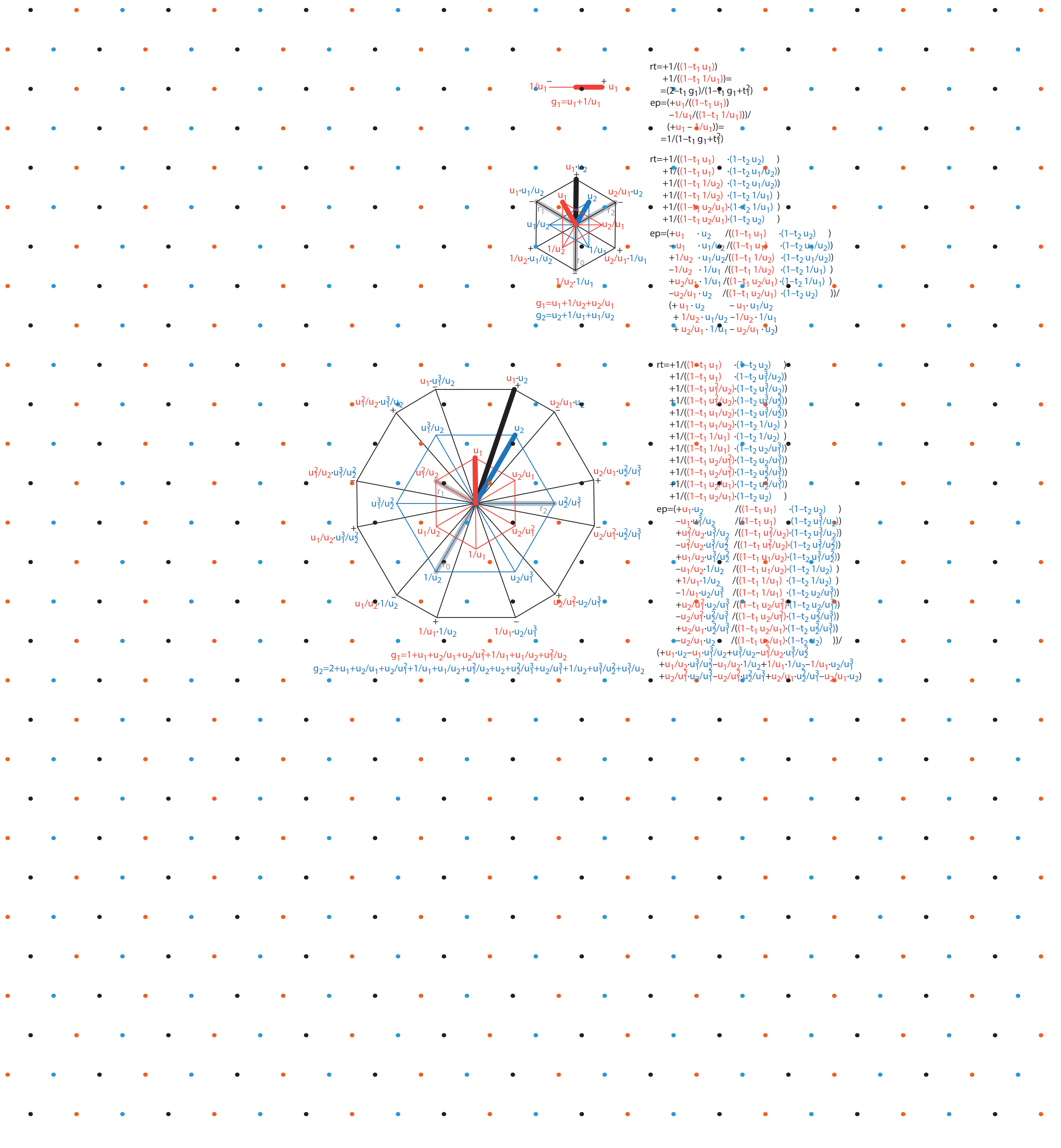


Figure 1: Higher roots and weights series.