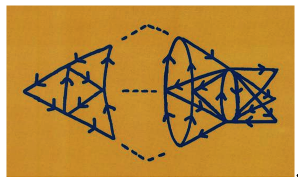
Lecture 28 of Adrian Ocneanu

Notes by the Harvard Group

Lecture notes for 3 November 2017.
The following graph is one of the higher graphs appearing in conformal field theory: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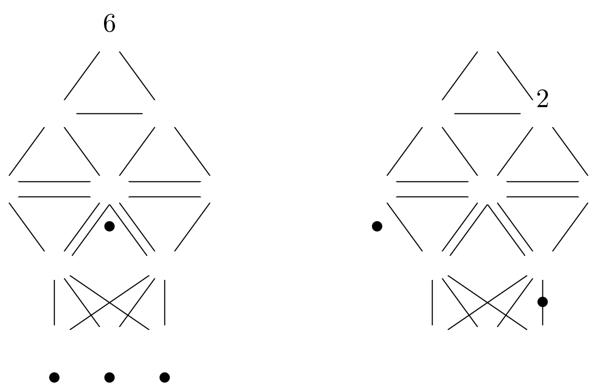
It is a type D graph arising from the $\mathbb{Z}_3$ action of the type A graph, which is a triangle. It appears on the cover of the yellow book "Conformal Field Theory" by Di Francesco, Mathiet and Sénéchal.

The type A graph is a triangle. The type D graph is one-third of the triangle and the central point of the triangle will separate as three points $1, \alpha, \overline{\alpha}$ according to the action of $\mathbb{Z}_3$. That is how a type D higher graph looks.

Today we will prove the formula for the roots. The roots corresponding to this graph have been done by computer. The roots look like this:

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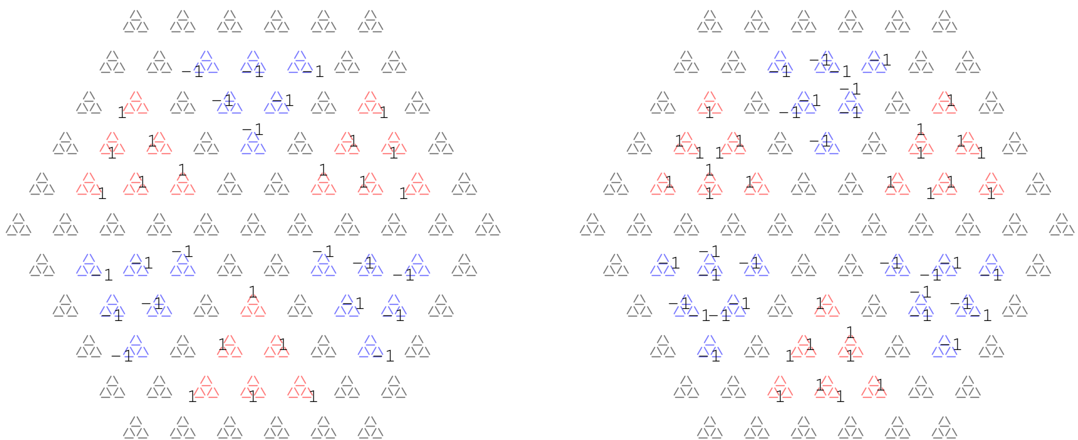
Di Francesco and Zuber found candidates for the higher Dynkin diagrams. The whole yellow book is mostly an attempt to introduce this higher mathematics. Because they had found the candidates for the higher Dynkin diagram. They gave a bottle of champagne for the classification, which I got in Argentina around 2000. But what they could not do was exactly this, so they stopped; everything else was there. They could not do the Euclidian space with higher roots, because you cannot get it by just using graphs for angles as usual.

As I realized while teaching a course on quivers, the quivers could be much better done by using ribbons. The ribbon has a natural construction of roots as we saw in the first part of course. This picture shows the roots, and the scalars in the graph indicate the inner product of the roots.

The diagram is tripartite and oriented, and it is made of representations of sl_3. Remember the usual A, D, E Dynkin diagrams are made of legs which are representations of sl_2 (and some triple points). The higher Dynkin diagrams of sl_3 would be made out of representations of sl_3.

In the graph, up to certain point, it looks like sl_3. You can check by Schur lemma, if there is no intertwiner, then they graph will stay all the time like sl_3 and you will get some triangles. If there is an intertwiner, then the graph starts to break or things start to glue, because the intertwiner may make two representations equivalent. Here in the type D graph the point in the middle breaks. This will be computed using topological quantum field theory (TQFT) and modular theory with argumentation.

We will concentrate on this crystallographic thing. The inner products of neighbors are 2. This is the same as for sl_2, where the inner product was 1 for nearest neighbors.



The black part in the above figure is given by Weyl mirrors. The space between Weyl mirrors is the corresponding graph A_n. (We use a convention where red is plus and blue is minus.) Here we have some paths on the graph; the path on graph are called fusions. The small graph is of type A, and it has, up to symmetry, two vertices: the top and the middle one. In this case, the path is created by tensoring with the irreducible representations. Tensoring by the irreducible representations on the graph is explained at 9:00. By the way, the reflection formula is called the Kac-Walton formula. In the graph on the right, we start from the trivial representation, but in the graph on the left we start from an irreducible representation and so what we get is more interesting (note that we only get two things because the third is annihilated by the mirror on the left).

We have three statement to prove today:

1. If one takes the fusion, it has alternating signs from summing over the Weyl group, as shown in the Kac-Walton formula. This means that the inner product of the inner product of the root with itself is 6, the order of the Weyl group.

The mirrors of the weight lattice are like zeros. The Weyl vector ρ is like 1. Representations are tensorial. The elements in the red Weyl alcove are closed under multiplications. This alcove works like positive real numbers. The center of the weight lattice is the additive center of roots or weights. As for real numbers, the exponential map changes the addition to the multiplication. Here w is additive, but σ_w is multiplicative. That is what appears in the Weyl formula.

The first statement is the following:

$$\mathrm{Proj}_{\mathrm{fusion}}\, \delta_{i,\alpha} = \underbrace{N^{-d}}_{\substack{=(\text{size of}\\ \text{period})^{-1}}} \sum_{w \in W} \varepsilon(w)\, \mathrm{fusion}(i - \rho + w(\rho), \alpha).$$

Here $\delta_{i,\alpha}$ is the Kronecker delta, W is the Weyl group, $\varepsilon(w)$ is the sign. (The statement is explained on the figure from 12:30–14:50.) The proof is similar to the sl_2 case:

Proof the first statement. We want to show that the inner product $\langle \cdot, \mathrm{fusion}\rangle$ with any fusion is the same as the result of formula.

Let i, j be in the weight lattice, and α, β, γ be vertices of the graph G; tensoring takes place on G. In the case of A_n, the graph G is a piece of the weight lattice within the mirrors, namely a Weyl alcove surrounded by mirrors.

The size of the period is N^d. (In the example shown in the figure $N = 5$. The period has 5×5 points.) First of all we have that

$$\begin{aligned} &N^d \langle \delta_{i,\alpha}, fusions(j, \beta)\rangle \\ =&N^d \langle fusion_{j,\beta}(i, \alpha)\rangle \\ =&N^d \dim \hom[\sigma_{j-i} \otimes \alpha, \beta]. \end{aligned}$$

We need to show that this equal to the right hand side of our formula.

$$\begin{aligned} &\sum_{w\in W} \varepsilon(w) \langle fusion(i-\rho+w\rho, \alpha), fusion(j, \beta)\rangle \\ =&\sum_{w\in W} \sum_{k,\gamma} fusion(i-\rho+w\rho, \alpha)(k, \gamma) fusion(j, \beta)(k, \gamma) \\ =&\sum_{w,k,\gamma} \varepsilon(w) \dim \hom[\sigma_{k-i+\rho-w\rho} \otimes \alpha, \gamma] \dim \hom[\sigma_{j-k} \otimes \gamma, \beta] \qquad\qquad (k, \gamma) \in \text{period} \\ =&\sum_{w,k} \varepsilon(w) \hom[\sigma_{j-k} \otimes \sigma_{k-i+\rho-w\rho} \otimes \alpha, \beta] \\ =&\sum_{w,k} \sum_{\substack{x\in \mathrm{we}(\sigma_{j-k}) \\ \text{(with mult.)}}} \varepsilon(w) \dim \hom[\sigma_{x+k-i+\rho-w\rho} \otimes \alpha, \beta]. \end{aligned}$$

Here $k - i + \rho$ is a shift, x is in the weights of σ_{j-k}, and $w\rho$ is the alternating Laplacian. Therefore

$$\begin{aligned} &\sum_{w,k} \sum_{\substack{x\in \mathrm{we}(\sigma_{j-k}) \\ \text{(with mult.)}}} \varepsilon(w) \dim \hom[\sigma_{x+k-i+\rho-w\rho} \otimes \alpha, \beta] \\ =&\sum_{w,k} \varepsilon(w) \dim \hom[\sigma_{k-i+\rho+w(j-k-\rho)} \otimes \alpha, \beta] \\ =&\sum_{k} \dim \hom[\sigma_{j-i} \otimes \alpha, \beta] + \sum_{w\neq 1} \dim \hom[\sigma_{k-wk-i+wj-w\rho} \otimes \alpha, \beta] \\ =&N^d \dim \hom[\sigma_{j-i} \otimes \alpha, \beta]. \end{aligned}$$

□

Here $\mathrm{we}(\sigma_{j-k})$ is the set of weights of σ_{j-k} with multiplicity. The alternating sum $\sum_{x\in We(\sigma_{j-k})}$ of weights σ_{j-k} is an alternating sum. That is Weyl dimension formula: the first alternating sum is the Weyl denominator and the second alternating sum is the Weyl numerator.

The Littlewood-Richardson formula for quantum groups is an open problem: there is no positive formula for the Littlewood-Richardson coefficients in the quantum case.

Tensorial $\equiv$ QFT: quantum field theories have an important tensorial property, whereby

$$\hom[a \otimes b \otimes c, d] = \bigoplus_x \hom[a \otimes b, x] \otimes \hom[x \otimes c, d],$$

which can be written in terms of squares as in figure 1.1.

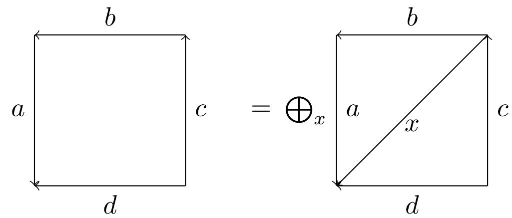


Figure 1: Tensoriality of QFT

This is atensorial property of homomorphisms. Therefore the space of QFT should be made of homomorphisms (homs), namely one needs to fill the space with maps. The objects are on the boundary, and the computations are in the middle and are given by homs.

Preparation for the next time.

Fusion matrices are given in Figure 1.2.

We are going to take an eigenvector v for G, namely

$$\Delta_\alpha^G v = \lambda_\alpha v.$$

We will then write its eigenvalue as a sum of μ_i's, i.e.

$$\lambda_\alpha = \sum \mu_i,$$

and the generator g is a sum of u_i's:

$$g = \sum u_i.$$

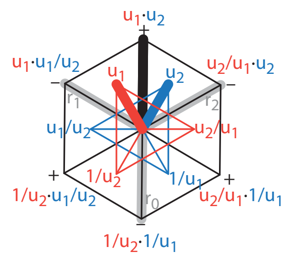
Writing λ_α as a sum of μ_i's will give the eigenvectors for the (commuting) translations on the ribbon (the higher exponents), which are exactly the higher version of the Coxeter element. Why do we need to write λ_α as a sum? That's because the sum of the μ's is the sum of the neighbors on the graph, and therefore the relation

$$\lambda_\alpha = \sum \mu_i$$

is exactly the biharmonicity relation: the sum of the neighbors on the graph is the same as the sum of the neighbors on the ribbon. In this case, the eigenvalues are g_1 and g_2, and you can write them as sums of unitaries: the unitaries and their powers are things on the ribbon. Thus we get the eigenvectors for translation from among the biharmonic functions. After this part we will get to a completely different crystallographic part, where we will find the higher matrices, and after that we will look at their reperesentations.

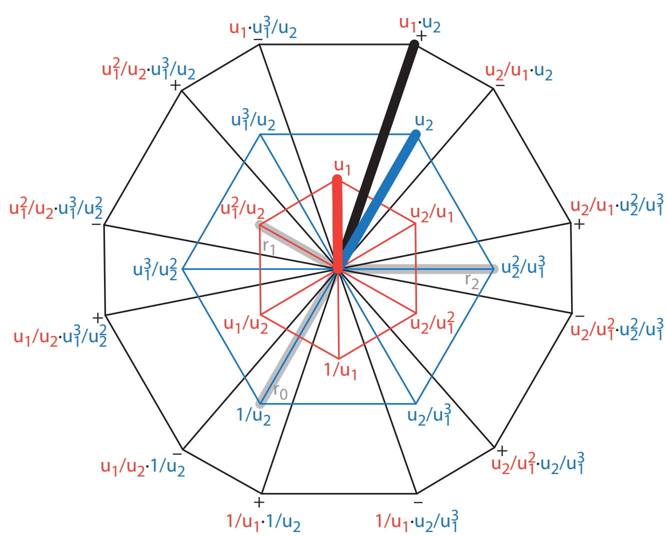
$g_1=u_1+1/u_1$

$$
\begin{aligned}
rt=&+1/((1-t_1\,u_1))\\
&+1/((1-t_1\,1/u_1))=\\
&=(2-t_1\,g_1)/(1-t_1\,g_1+t_1^2)\\
ep=&(+u_1/((1-t_1\,u_1))\\
&-1/u_1/((1-t_1\,1/u_1)))/\\
&(+u_1-1/u_1))=\\
&=1/(1-t_1\,g_1+t_1^2)
\end{aligned}
$$



$g_1=u_1+1/u_2+u_2/u_1$
$g_2=u_2+1/u_1+u_1/u_2$

$$
\begin{aligned}
rt=&+1/((1-t_1\,u_1)\cdot(1-t_2\,u_2))\\
&+1/((1-t_1\,u_1)\cdot(1-t_2\,u_1/u_2))\\
&+1/((1-t_1\,1/u_2)\cdot(1-t_2\,u_1/u_2))\\
&+1/((1-t_1\,1/u_2)\cdot(1-t_2\,1/u_1))\\
&+1/((1-t_1\,u_2/u_1)\cdot(1-t_2\,1/u_1))\\
&+1/((1-t_1\,u_2/u_1)\cdot(1-t_2\,u_2))\\
ep=&(+u_1\cdot u_2/((1-t_1\,u_1)\cdot(1-t_2\,u_2))\\
&-u_1\cdot u_1/u_2/((1-t_1\,u_1)\cdot(1-t_2\,u_1/u_2))\\
&+1/u_2\cdot u_1/u_2/((1-t_1\,1/u_2)\cdot(1-t_2\,u_1/u_2))\\
&-1/u_2\cdot 1/u_1/((1-t_1\,1/u_2)\cdot(1-t_2\,1/u_1))\\
&+u_2/u_1\cdot 1/u_1/((1-t_1\,u_2/u_1)\cdot(1-t_2\,1/u_1))\\
&-u_2/u_1\cdot u_2/((1-t_1\,u_2/u_1)\cdot(1-t_2\,u_2)))/\\
&(+u_1\cdot u_2-u_1\cdot u_1/u_2\\
&+1/u_2\cdot u_1/u_2-1/u_2\cdot 1/u_1\\
&+u_2/u_1\cdot 1/u_1-u_2/u_1\cdot u_2)
\end{aligned}
$$



$g_1=1+u_1+u_2/u_1+u_2/u_1^2+1/u_1+u_1/u_2+u_1^2/u_2$
$g_2=2+u_1+u_2/u_1+u_2/u_1^2+1/u_1+u_1/u_2+u_1^2/u_2+u_2+u_2^2/u_1^3+u_2/u_1^3+1/u_2+u_1^3/u_2^2+u_1^3/u_2$

$$
\begin{aligned}
rt=&+1/((1-t_1\,u_1)\cdot(1-t_2\,u_2))\\
&+1/((1-t_1\,u_1)\cdot(1-t_2\,u_1^3/u_2))\\
&+1/((1-t_1\,u_1^2/u_2)\cdot(1-t_2\,u_1^3/u_2))\\
&+1/((1-t_1\,u_1^2/u_2)\cdot(1-t_2\,u_1^3/u_2^2))\\
&+1/((1-t_1\,u_1/u_2)\cdot(1-t_2\,u_1^3/u_2^2))\\
&+1/((1-t_1\,u_1/u_2)\cdot(1-t_2\,1/u_2))\\
&+1/((1-t_1\,1/u_1)\cdot(1-t_2\,1/u_2))\\
&+1/((1-t_1\,1/u_1)\cdot(1-t_2\,u_2/u_1^3))\\
&+1/((1-t_1\,u_2/u_1^2)\cdot(1-t_2\,u_2/u_1^3))\\
&+1/((1-t_1\,u_2/u_1^2)\cdot(1-t_2\,u_2^2/u_1^3))\\
&+1/((1-t_1\,u_2/u_1)\cdot(1-t_2\,u_2^2/u_1^3))\\
&+1/((1-t_1\,u_2/u_1)\cdot(1-t_2\,u_2))\\
ep=&(+u_1\cdot u_2/((1-t_1\,u_1)\cdot(1-t_2\,u_2))\\
&-u_1\cdot u_1^3/u_2/((1-t_1\,u_1)\cdot(1-t_2\,u_1^3/u_2))\\
&+u_1^2/u_2\cdot u_1^3/u_2/((1-t_1\,u_1^2/u_2)\cdot(1-t_2\,u_1^3/u_2))\\
&-u_1^2/u_2\cdot u_1^3/u_2^2/((1-t_1\,u_1^2/u_2)\cdot(1-t_2\,u_1^3/u_2^2))\\
&+u_1/u_2\cdot u_1^3/u_2^2/((1-t_1\,u_1/u_2)\cdot(1-t_2\,u_1^3/u_2^2))\\
&-u_1/u_2\cdot 1/u_2/((1-t_1\,u_1/u_2)\cdot(1-t_2\,1/u_2))\\
&+1/u_1\cdot 1/u_2/((1-t_1\,1/u_1)\cdot(1-t_2\,1/u_2))\\
&-1/u_1\cdot u_2/u_1^3/((1-t_1\,1/u_1)\cdot(1-t_2\,u_2/u_1^3))\\
&+u_2/u_1^2\cdot u_2/u_1^3/((1-t_1\,u_2/u_1^2)\cdot(1-t_2\,u_2/u_1^3))\\
&-u_2/u_1^2\cdot u_2^2/u_1^3/((1-t_1\,u_2/u_1^2)\cdot(1-t_2\,u_2^2/u_1^3))\\
&+u_2/u_1\cdot u_2^2/u_1^3/((1-t_1\,u_2/u_1)\cdot(1-t_2\,u_2^2/u_1^3))\\
&-u_2/u_1\cdot u_2/((1-t_1\,u_2/u_1)\cdot(1-t_2\,u_2)))/\\
&(+u_1\cdot u_2-u_1\cdot u_1^3/u_2+u_1^3/u_2-u_1^2/u_2\cdot u_1^3/u_2^2\\
&+u_1/u_2\cdot u_1^3/u_2^2-u_1/u_2\cdot 1/u_2+1/u_1\cdot 1/u_2-1/u_1\cdot u_2/u_1^3\\
&+u_2/u_1^2\cdot u_2/u_1^3-u_2/u_1^2\cdot u_2^2/u_1^3+u_2/u_1\cdot u_2^2/u_1^3-u_2/u_1\cdot u_2)
\end{aligned}
$$