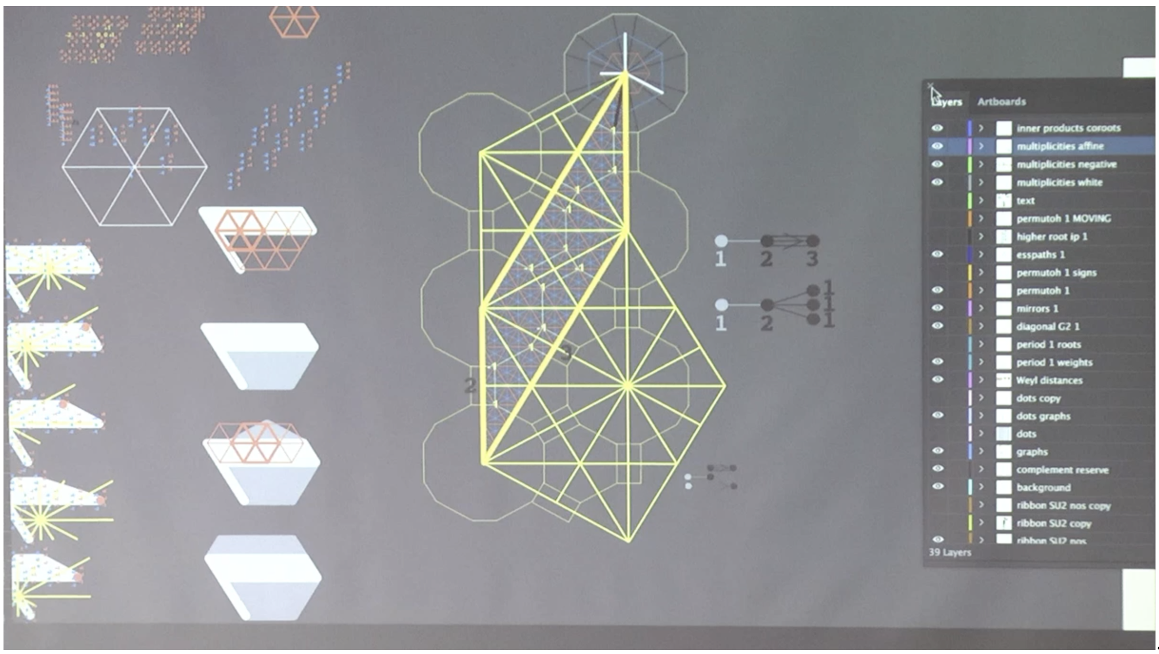
Lecture 29 of Adrian Ocneanu

Notes by the Harvard Group

Lecture notes for 6 November 2017.



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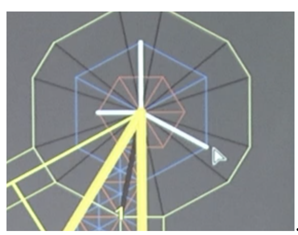
This is the group G_2. These are simple roots and the vertical one is the affine root.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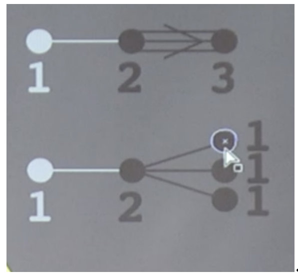
The G_2 roots come from D_4. The three vertices on the right of D_4 are added together. There are two ways to do this: one is to project them on the common axis (the line), which will give you a factor of $1/\sqrt{3}$; or you can add them up (since they form the edges of a cube) and get a factor of $\sqrt{3}$ (the diagonal of the cube).



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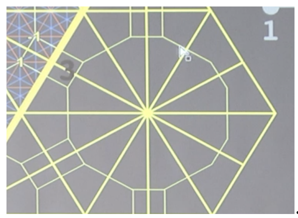
The linear sum of simple roots and the affine root is zero, and the number on the corresponding vertex of the graph is the multiplicity.

We shall work continuously with the action of the Weyl group. The Weyl group action and the permutohedron are here.



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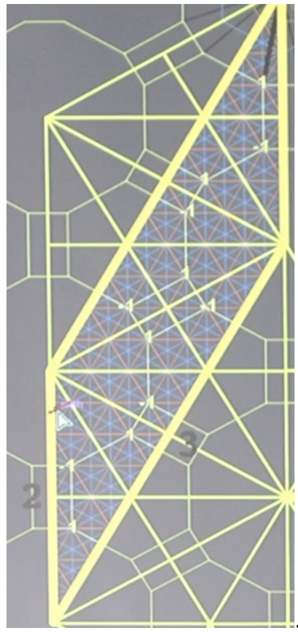
The period is this hexagon and it has 12 alcoves. For us the alcoves are scaled, but for Weyl they had length one, which is why sometimes we take a power of the scaling.

The 12 alcoves also appear in the parallelogram, and $12 = 2 \times 3 \times 2!$. This is the area of the parallelogram measured in number of alcoves. In the case of G_2, there is only one 1 appearing on the graph, but when more than one 1 is present it would give the ratio between the size of the unit in the weight lattice and that in the root lattice (as in D_4).



The black arrow is the Weyl vector. This is the number of essential path with the smallest cutoff (the smallest cutoff for which there is still something left, here the trivial representation). If you move the Weyl vector to the origin, you get the trivial representation. The rest is continued by reflection, and the formula for reflection is the Kac-Walton formula.

There are three kinds of permutohedra, the big thing, the rectangle and the hexagon, corresponding to removing a point in the graph (the first, the second and the third respectively). The structure all comes from the affine root. The product of the exponents plus one gives another formula for the Weyl group (De Concini and Procesi showed a similar formula for the angles). I hope we will be able to give an explanation for this, because we are going to need it to count various things. Once again, this is the graph G_2, a non-type I thing. The period on the graph will be the higher diagonal of G_2.

Question: what are the entries of the higher matrices?

Ocneanu: The entries will be pairs of points on the diagonal, exactly like in the usual case, except that the involution (the star operation) will be replaced by something which is very crystallographic and which depends on everything. So this will be an analogy, you have functions on a real line as a set on which you have the very interesting Fourier transform. Suddenly you will have the group structure, the differential operator and all that. You will have a very intrinsic involution. So we are going there, we are going the diagonal very soon. This, by the way, will be also higher h_{ij}, those numbers 1 and -1. Exactly you have h_{ij}, 1 and -1. This one will have six 1's and -1's.

Now let us concentrate on the series.

$$\frac{1}{1-tu} = 1 + tu + t^2u^2 + \cdots, \ start$$
$$\frac{t^k u^{-k}}{1-tu} = t^{-k}u^{-k} + \cdots + 1 + tu + \cdots, \ shift$$

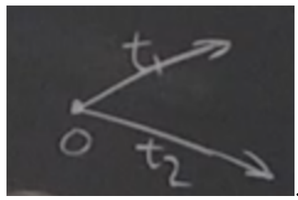
$$\frac{u^{-1}}{1-tu} = u^{-1}t^0 + t^1 + ut^2 + \cdots,\ shift$$

fusion series: g_i matrices of $\sigma \otimes \cdot$ on Vect G, σ_i are fundamental irreps.

$$g_i = \sum_{x \in We\sigma_i} u_x,$$

$$u_{x+y} = u_x u_y, u_{-x} u_x^*,\ unitaries.$$

In the principal Weyl chamber, $t_1, t_2, \cdots$ are shifts corresponding to fundamental weights i.



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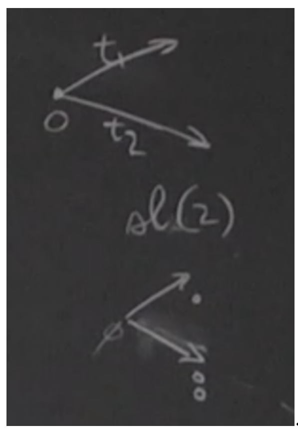
For $sl(2)$, we have



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The series will be sum of fusion or essential paths, (i is fundamental weight):

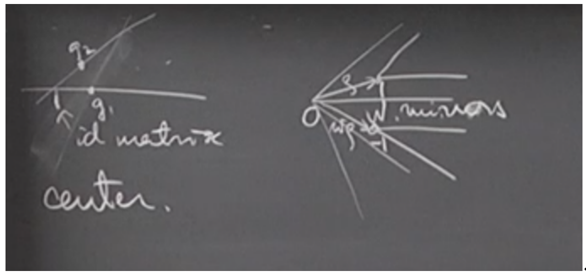
$$\frac{\sum_{w \in W} \varepsilon(w) \Pi_i \frac{w(u_i)}{1-t_i w(u_i)}}{\sum_{w \in W} \varepsilon(w) \Pi_i w(u_i)}.$$

The coefficient of $t_1^{k_1} t_2^{k_2}$ is a polynomial function in matrices g_k. This gives a matrix of tensoring with irreducible with Young diagram (Let us call it Young diagram, although we us it in general.) in $\sigma_1^{k_1} \otimes \sigma_2^{k_2} \otimes \cdots$. The $\sigma' s$ are the generators, highest weight irreducible.

The series subtracts lower order terms from tensor products which are products of generating matrices: $g_1^{k_1} g_2^{k_2}$ corresponds to $\sigma_1^{k_1} \otimes \sigma_2^{k_2}$. These are paths and g_i's are edges of G.

Now we need to prove the above formula. First the formula is written stating the fusion identity

1, g_1, g_2 are generators, ρ is the Weyl vector and w is the Weyl mirror.



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Use shift to start at -1, because the Weyl vector has coordinates $(1, 1)$ and so on. You can see that if we do that, you just get rid of the numerator in the left side. Just compare it with the shift above. Now you get the series

$$\frac{\sum_{w \in W} \varepsilon(w) \Pi_i \frac{1}{1 - t_i w_i}}{\sum_{w \in W} \varepsilon(w) \Pi_i w(u_i)}.$$

Lemma: The series extended from the rational function above is a Weyl antisymmetric, means reflections change signs.

Now we subtract the fusion which should be the result of the series. Which we know this way is that we will get 0 at the $w\rho$. It is still Weyl antisymmetric.

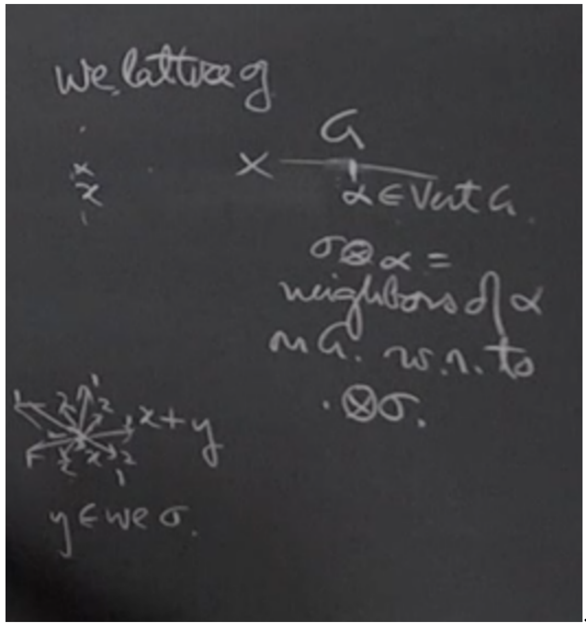
Assume that a function f: weight lattice $\times$ Vect$G \to \mathbb{Z}$ is Weyl antisymmetric, 0 at $W\rho\times$ Vect$G$. Then $f \equiv 0$.

Proof. It is very important that f is biharmonic:

$$f(x, \sigma \otimes \alpha) = \sum_{y \in We\sigma} f(x + y, \alpha),$$

where x is in the weight lattice, α is a vertex in Vect$G$, and $We\sigma$ are weights with multiplicities.

$\sigma \otimes \alpha$ are neighbors of α in G w.r.t. $\cdot \otimes \sigma$. On the other side, in the weight lattice, if you take x, then you shift it by all the y's, $y \in We\sigma$. We know this for generators, then inductively one can prove it in general for any representation. We will use it for irreducible still.



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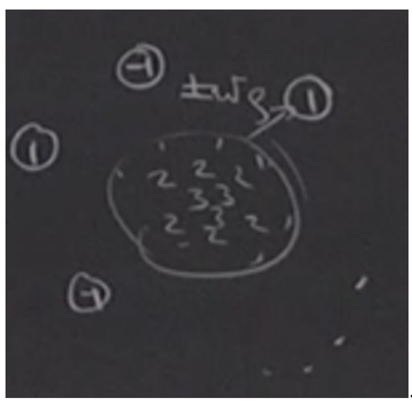
If you take this mountain of multiplicities, you shift it by $\pm W\rho$, you get ± 1 around.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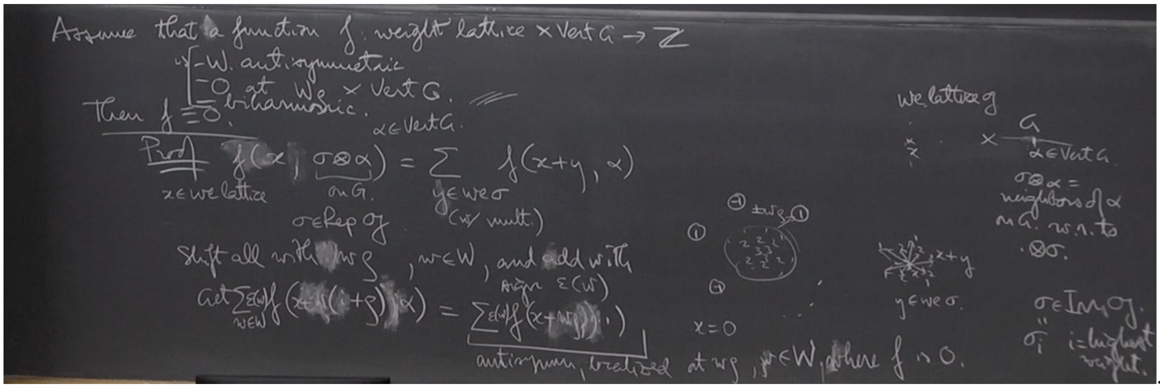
Choose σ in the irreducibles of G, i is the highest weight.
Shift all with $w\rho$, $w \in W$, and add with sign $\varepsilon(w)$. You get

$$\sum_{w \in W} f(x + w(i+\rho)\alpha) = \sum f(x + w\rho\bullet),$$

the $\bullet$ are something we are not interested in. This function is antisymmetric localized at $w\rho$, where f is zero by assumption.



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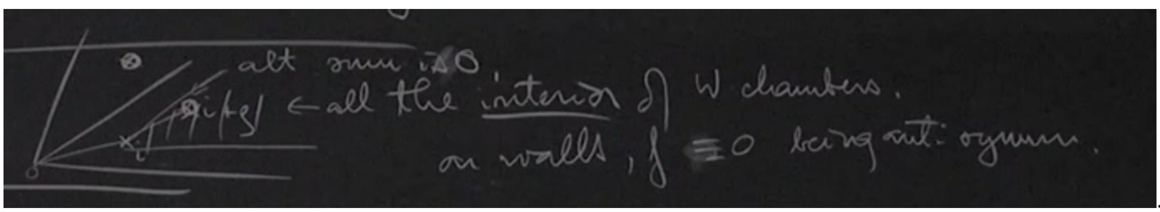
So we have the function now

$$\sum_{w \in W} \varepsilon(w) f(w(i+\rho), \alpha)) = 0,$$

for any $\alpha \in$Vect$G$, any i.

For any i, we have this $i+\rho$. The alternating sum is 0. But our function is assumed to be antisymmetric, this give the interior of the Weyl chambers. On the walls $f \equiv 0$, being antisymmetric.



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The lemma is proved. □

Now all we we have to notice is that our difference function satisfies the lemma.

The proof of the root series from this. The fusion is centered at zero.

$$\frac{\sum_{w \in W} \varepsilon(w) \Pi \frac{1}{1-t_i w(u_i)}}{denominator}.$$

Now we move it back with each $w'\rho$ in turn with sign $\varepsilon(w')$, $w' \in W$. The idea is that you get in this way is

$$\frac{\sum_{w' \in W} \varepsilon(w') \sum_{w \in W} \varepsilon(w) \Pi \frac{1}{1-w'(t_1)w(t_2)}}{denominator} = \sum_{w, w'' \in W} \frac{1}{1-t_i w''(u_i)},$$

where $w'' = ww'$.

So what would happen when you sum these terns, you shift them and you sum back, you apply the Weyl transformation to the t_i as well, so you reflect. So you get a product of two Weyl groups.

This being a group, you get the same. And the coefficient that you get will cancel exactly this denominator. So you get the sum exact as above, which is a much simpler formula for roots. So these are the proofs of the series formulas.

We will start to find the diagonals which was the question at the very beginning. We will start the h_{ij} in the case A_n. From the h_{ij}, the higher version of that, we will build the higher matrices, then the action of these which will be very crystallographic. I am trying to reach as soon as possible that. You will have a lot of very very crystallographic things.

$\square$