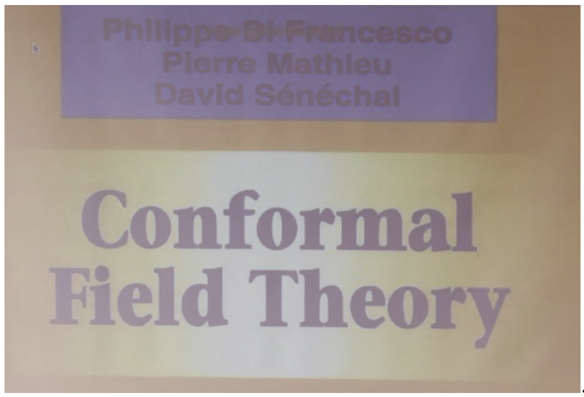
Lecture 30 of Adrian Ocneanu

Notes by the Harvard Group

Lecture notes for 8 November 2017.



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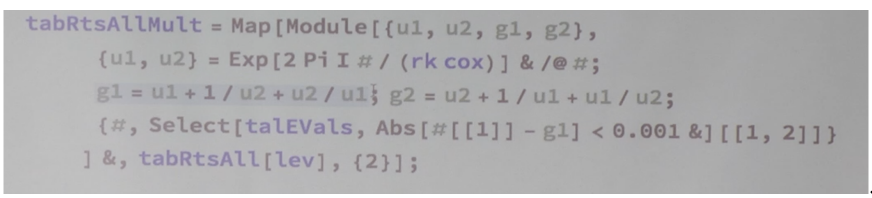
This is a good book for things up to the diagrams. On the cover is the D graph. It contains modular matrices, which are interesting for us. In particular, the matrix S corresponding to a rotation of the torus by $90°$ is the character table for the A_n labels.



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tabRtsAllMult = Map[Module[{u1, u2, g1, g2},
    {u1, u2} = Exp[2 Pi I # / (rk cox)] & /@ #;
    g1 = u1 + 1 / u2 + u2 / u1; g2 = u2 + 1 / u1 + u1 / u2;
    {#, Select[talEVals, Abs[#[[1]] - g1] < 0.001 &][[1, 2]]}
   ] &, tabRtsAll[lev], {2}];
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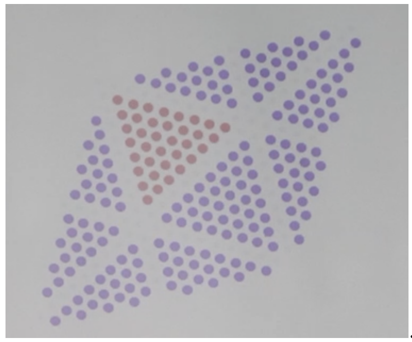
These are the things that we shall use. The idea is that we use everything we know about quantum groups (which was done by and large by the 1990's).

We shall work in the three dimensional representation of sl_3. Here u_1, u_2 corresponds to the highest weights of the fundamental irreducible representations. The graph is A_2 over sl_3.

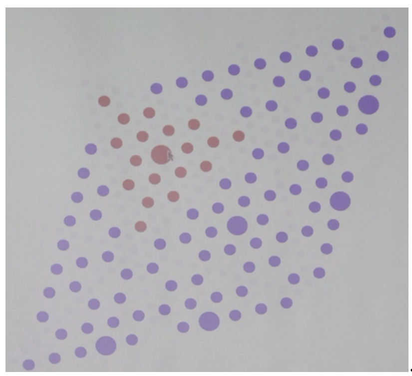
$$g_1 = u_1 + u_2^{-1} + u_2 u_1^{-1},$$

$$g_2 = u_2 + u_1^{-1} + u_1 u_2^{-1}.$$

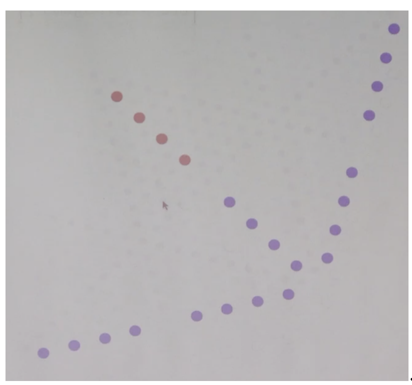
We make g_1, g_2 to be eigenvalues (the above equtions are eigenvalue equations). We find u_1, u_2, which will be some root of unity. The solution are in the graph here, in the weight space of sl_3.

.

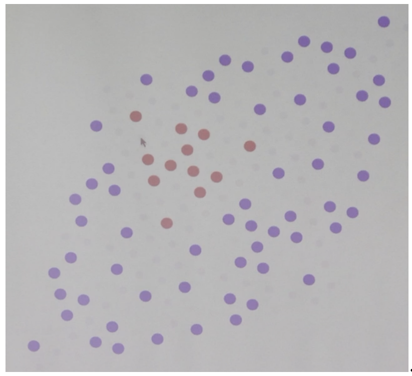
This way you can see the action of the Weyl group. There are other graphs. Below we can see the D graph, the one on the cover of the book, which has a triple root in the middl.

.

This is a small one, a series in way like the D, an orbifold series. It only has things on the diagonal.

.

This is a first exceptional E, that is a part of two series that Zhengwei has found, which are series across all the roots.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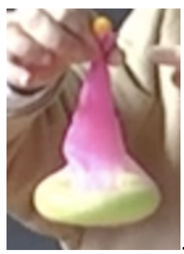
These are the solutions of this equation.

Thsi is what we do with these solutions: we take an eigenvector $v : VectG \to \mathbb{C}$. Here g_1, g_2 are the eigenvalues for fundamental irreps α_1, α_2. Solving

$$g_1 = u_1 + u_2^{-1} + u_2 u_1^{-1},$$

$$g_2 = u_2 + u_1^{-1} + u_1 u_2^{-1}$$

you find the unitaries. How do you know that you find these unitaries? Again by the theory, which uses some modular matrices. I brought some tools: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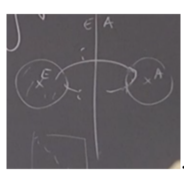
This is a cone over the torus. This is a filling of the torus which is invariant under the modular group. The modular group is the automorphism group on the torus. So whatever you do on the torus goes onto this cone. This filling gives a vector on the boundary which will be modular invariant, that way you get modular invariants

After that, you take the torus with this filling. It is something like a yoyo.

When you contract the torus, you can get this cone. Then the cylinder contracts. When it contracts, it gives the pillow with the center contracted. It is a orthogonal projection onto a disc with a puncture. You can see it is a projection if you put two of them together. This projection give you the Hilbert space of the disc with puncture marked E, where E is an exceptional.



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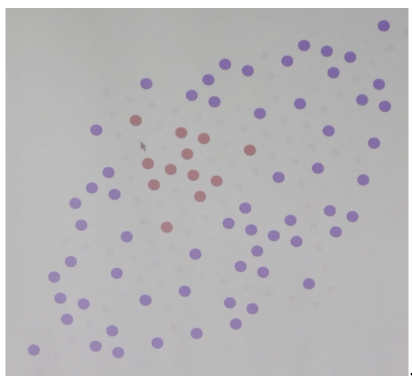
Now you take another one, where you have graph A.Remember that everything we work in representation theory were maps from A into something, for example E. So if you have an object of type $A - E$, then you can surround it by some wires in the A_n (the modular matrix is the identity) you get some diagonals of the modular matrices. These will be the characters of your group: you have a wire surrounded by something, so it goes from an irreducible to the same irreducible. In topological quantum field theories, if you surround the wire, since you can have a map only from an irreducible to the same irreducible, not to other irreducibles, it must be a scalar. This scalar is a character. This was a tiny sketch.

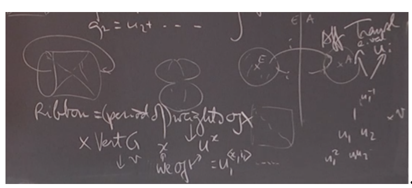
So the formula now is the following: the eigenvalues are u_i, j is a weight in the underling graph $\mathcal{G}$,

$$u_i = e^{\frac{2\pi i}{N} < i,j >},$$

and $< i, j >$ is an inner product between weights of $\mathcal{G}$. These are the eigenvalues, let us call them λ_j. The weights which appear are precisely the diagonal. They are corresponding the eigenvalues of G with multiplicity. They are labelled by the graph A_n. You find these with the periodicity of the affine Weyl group. You can see the mirrors in gray.

.

You can see here all the eigenvalues. These are guaranteed to appear. It is now conjectured that these are the only possible eigenvalues. The number of points is $12 = 3 \times 4$, exactly the number of vertices of the graph. So there are as many eigenvalues as the vertices of the graph. So the number of such eigenvalues is exactly the number of points in the group times the Weyl group. Here you can see the underlying Weyl group, which repeats them. So in this case, it times 6.

.

The effect of this is the following: you take an eigenvector v with g_1, g_2. The ribbon is period of the weights of $\mathcal{G} \times \mathrm{Vect}G$. On $\mathrm{Vect}G$ you take v. On the weights of G, you take, say, u^x, x in weights of $\mathcal{G}$. This means that

$$u^x = u_1^{\langle x, i_1 \rangle} \cdots$$

For $i_1, \ldots$ weights. Consider the affine translations in the Weyl group. They have eigenvalues u_1, u_2. On the other hand if you take the sum of neighbors on G, you get exactly the eigenvalue g_i. If you take the sum of neighbors on weights, then you get

$$\sum_{x \in \mathrm{we}\, \sigma_i} u_x = g_i.$$

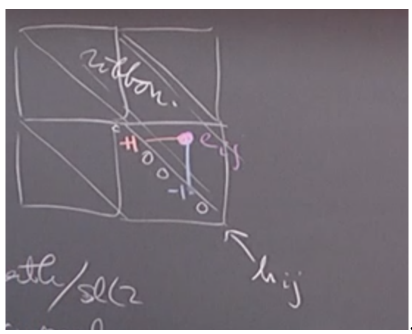
The conclusion is the theorem: The vectors constructed above are

1. biharmornic;
2. eigenvectors of for the translation group in the affine Weyl group. (The group is unitary and abelian.)

The eigenvectors are precisely constructed this way. You can see there you have $72 = 12 \times 6$ eigenvectors on the ribbon. The translations are higher Coxeter elements. The eigenvalues of the u_i's, as embedded in the vertices of the graph A_n, are the higher exponents. These that we have constructed are the exponents of the higher Lie group, which are the fundamental thing.

The part which remains to be shown is that the fusion generates all the biharmonic functions. I will speak about it at some other time. The linear dimension of the biharmonic functions on the full ribbon is exactly the number of vertices of the graph G times the order of the subjacent Weyl group.

We will stop here and move on to other things.



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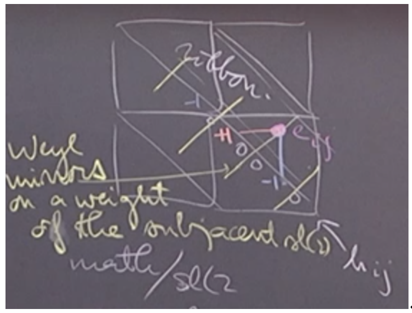
In math over sl_2, the usual math, you repeated the matrix periodically, then you will find the ribbon. You take a point on the ribbon, then you go with red and with blue with ± 1 respectively. (Convention: $+1$ is red, -1 is blue.) This way you get h_{ij}. In the case $G = A_n$, the roots are not just abstract points on the ribbon with given inner products, but they are concrete vectors h_{ij} which realize those inner products $< \cdot, \cdot >$.

Notice the following: there are Weyl mirrors on weights of sl_2 (the subjacent sl_2). They repeat the patterns of ± 1. This is how the h_{ij}'s look like:



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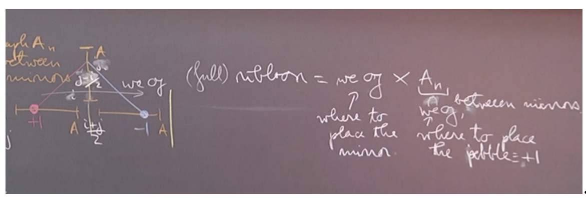
You can see that the part between mirrors is the graph A_n (the orange part). On the graph A_n you put the blue and red pebbles. They are formed by each other by reflection along the mirror (with sign). There is another graph A_n perpendicular to this one.

The horizontal direction are the weights $\mathcal{G}$. The vertical coordinate is $\frac{j-i}{2}$. The other coordinate is $\frac{j+i}{2}$. These are the two coordinates on the ribbon. Recall that the ribbon is a product: we $\mathcal{G} \times A_n$, where $A_n \subset$ we $\mathcal{G}$ is between mirrors. The horizontal coordinate, on the weights of $\mathcal{G}$, tells you where to place the mirror, and the second one tells you where to place the red pebble. The number is $+1$ at i of h_{ij} (the red pebble). You get a -1 at the blue pebble.



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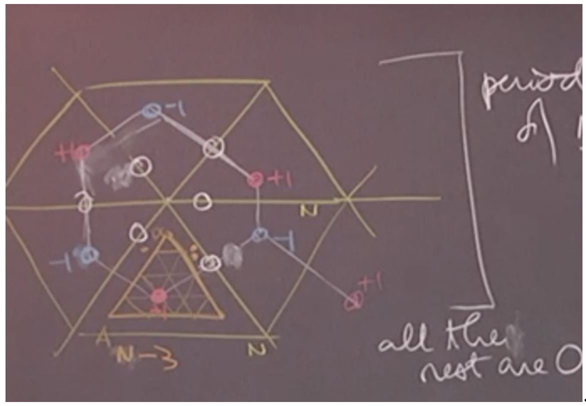
You see what you have to do is to solve an ordinary case for the h_{ij}, but you should try to understand it with methods which work in general.

You try to understand the h_{ij} this way. Once you see that, the thing is clear. We will have the proof next time. We will see the procedure in the case of sl_2 and in general. What we have is the affine mirrors. The length of an interval is N. Remember that the Weyl group is made of reflections in these mirrors. These mirrors are perpendicular to simple roots, and you complete them with the affine mirrors which are perpendicular to the affine root. Remember that the affine root has integers coefficients $1, 2, 3, 4, 5, 6$ for E_8. Moreover, you then scale by the Coxeter number N.

The orange part is the graph A_{N-3}. The number 3 is the distance to the mirrors. You place a red pebble which is $+1$. You reflect it, you get a -1 at the blue pebble, then you reflect it again. This is a highest weight of the representation. You get ± 1 on the hexagon. All the rest are zeros.

You also reflect it also in the affine mirrors.



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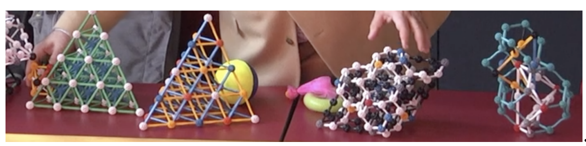
The period of roots of $\mathcal{G}$ will be the diagonal of the higher matrices. The usual diagonal is one-dimensional. This is two dimensional.

This is a higher $h_{i,j}$ and using the formula of Kac-Walton and the formula of Weyl, for the denominator, we will show that the inner product of two such things, which we call hexes, is exactly the inner product that we defined on the roots. The name in the literature for this is *weight permutohedron*. It is not the most general form of the permutohedron which would have arbitrary edges. Here the sum of the length of three edges is a constant. Moreover, you can see here that the Weyl vector ρ plus the highest weight in A_n plus the position of the mirrors should have parity 0, so it should be a root, not a weight. So the graph A_n is partite (for exampleyou know that sl_2 is tripartite, so this full ribbon separates into three).

(In the last five minutes, Ocneanu presented the weight lattice and root lattice of sl_4 and other things on some models.)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We will later continue with higher matrices. We already have the diagonals of the higher matrices; we will later build the off-diagonal elements. After that we will start the crystallography, and then build something in this pyramid in whi thench some vegetables are growing. They grow into things like the Clebsch-Gordan coefficients that the physics students wear on T-shirts sometimes. With crystallography, we will end the course with these representations.