Lecture 32 of Adrian Ocneanu

Notes by the Harvard Group

Lecture notes for 13 November 2017.
See the picture for bi-harmonicity in Figure 1. In the case A_n, remember that the ribbon is a Cartisian product between a weight and an element of the graph, the higher Dynkin diagram. First, the weight is where you put the center of mirrors (the center is right in the middle in Figure 1). Then we take an element of the A_n graph which is a point in the principal Weyl chamber, which is not on the mirrors, and we reflect it. The points which are mirrored are given by blue one, red one and purple one. What appear on the ribbon are $+1$'s and -1's, with $+1$ down and then alternating signs. Here we keep in the same point of the weights, but we take the sum of neighbors on the graph of A_n (the graph of A_n is not pictured in Figure 1). We take three neighbors of the arrow, i.e. three hexes which have the same center. They give you exactly identical hexes, the orange, the light blue and the dark blue one, which are translated. The left side of the ribbon gives the position of the mirror. And you have three neighbors on the graph G which is A_n here.

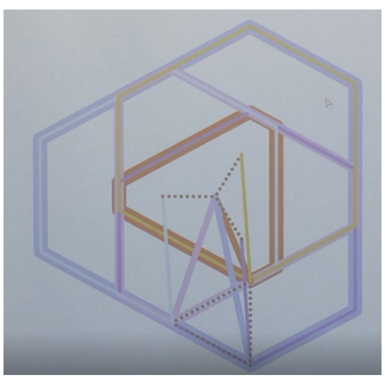

Figure 1: Figure 1

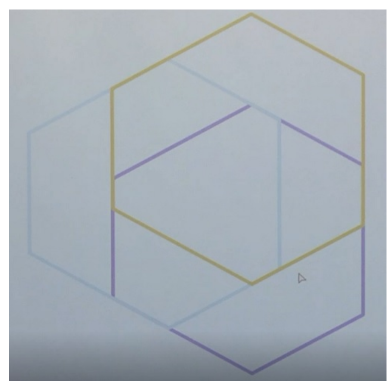

Figure 2: Figure 2

In picture 2 there are three hexes which are identical in shape. This means that they come from the same point of the graph A_n, but are centered in three different ways. They give you exactly the same thing (this is just translation). If you do the denominators of the power series (which we did), you find the sum of four hexes, which is trivial, with the same max. The sum of the blue and the yellow, is the same with the sum of the other two (see Figure 3). There are some identities satisfied by these hexes.

Let us see how matrices are done in the usual case. Here we have some matrices (see Figure 4). We will see the matrix elements as arrows on diagonal. The red arrow on this element, say e_{14}, is h_{ij}. The product of the matrices then is matching: here you have an element of the red matrix

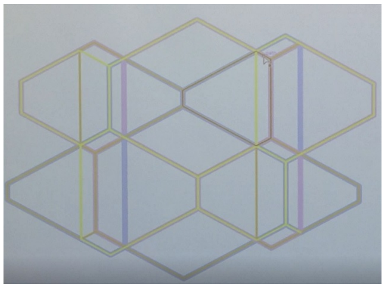

Figure 3: Figure 3

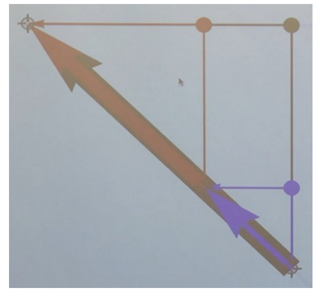

Figure 4: Figure 4

and an element of the blue matrix, i.e., e_{14} and e_{46}, respectively. So e_{14} times e_{46} equals e_{16}. So the product of matrices is done by putting arrows. Now, arrows may have integer multiplicity. For example, if we have 2 red arrows and 3 blue arrows here; in this case then the product would have each arrow with every other arrow (every red arrow with every blue arrow). So the coefficient is $2 \times 3 = 6$ for the black arrow. This is how you make diagonal into matrices. The product of matrices is the compositions of the arrows on the diagonal.

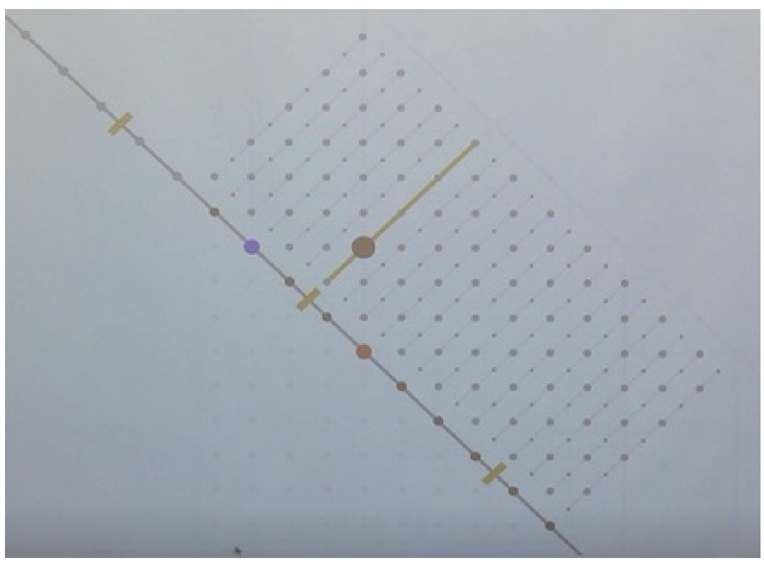

Figure 5: Figure 5

In Figure 5, one may see H_{ij} on the ribbon, some mirrors and the graph A_n. The matrix itself is in light green. It is the matrix on which we work. From the graph A_n, we use every other vertex for the matrix. The diagonal is using the roots of $sl(2)$, while the mirrors are on weights of $sl(2)$ that are half integers. In this case, the length here is the correct one for the length for root, i.e. $\sqrt{2}$, and the weight length is $\sqrt{2}/2$. If we move the largest point in Figure 5 on the ribbon, we gain all the matrix elements.

How about B, C, and D? This part is also new. The usual basis for the B, C, D's are different from this one. The matrix here is twice as big, but you have another mirror, in the middle of graph A_n. Remember that the B, C, D's are obtained by folding the graph A_n in various ways: you can fold the graph A_n after you build the roots, and then projecting on the plane of symmetry. The roots that lie on the plane of the symmetry remain long. Others are perpendicular and are projected, becoming shorter. There is also another way, in which you think of mathematical objects such as vector spaces on the vertices of graph A_n. In that case, when you have a usual orbit, you get just one of them, and when you have something of $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z}$ acting on the middle one, it breaks into eigenspaces and D_n appears. This should be something like either of the B, C, D's up to this point. What appears in the middle of the room all of a sudden is a new mirror, the magenta mirror. It has the property that it preserves the sign. The blue here is reflected into blue, and the red to red. This mirror also appears in the middle of the graph A_n. It is the symmetry with respect to $\mathbb{Z}/2\mathbb{Z}$. Because of this new mirror, we get twice as many points on the diagonal, and we half the period. This is exactly how you learn about orthogonal symplectic matrices. They are not build out of Dynkin diagram, but are represented as usual matrices, which may be twice bigger and subject to some symmetry. Now the period is half the previous period. In the symplectic case, for instant, the two red points are ${}_iq$'s, and the blue are p_i's. There is a matrix element made of the product of p_i and q_j. Please see the video from 13:50 to 15:04 to find how they moves and enter the sign-preserving mirror. There are two possibilities. One is that the p_i and p_j which enter give you p_i^2. Before, you have something like $(-1, 1, \ldots)$, where the first two digits are on position p_i, p_j. The length of this is $\sqrt{2}$. Now when they come together we get -2 in the middle, i.e., $(0, 0, 2, 0, \ldots)$. The length is $2 = \sqrt{2} \cdot \sqrt{2}$, which is bigger.

This is the symplectic case, and its graph C_n is a quotient of A_n, obtained by projecting the roots on the plane of symmetry as in Figure 6(a). Notice the direction, the two points on the left of the graph A_n are orthogonal to each other. The roots in the plane of symmetry keep the same length.

The insteresting question is how to get all the properties of the symplectic group form this graph. On answer is based on the following. There is one long root, so this a the theory which has squares and is commutative. You have polynomials in p_i and q_j. The non-commutativity and the anti-symmetry would appear from the inner product, in this case the Poisson bracket. In the orthogonal case, if you enter the middle mirror, the middle mirror has a $1 = \sqrt{2}/\sqrt{2}$. In the case B_n you will have a root which is shorter. The B_n comes out of the D by projecting the roots onto the plane of symmetry. These two ones which are orthogonal to each other would be shorter (see Figure 6(b)).

In the D_n case, we have sign reflecting mirrors, between which is a sign preserving mirror. (See Figure 6(c)) Consider the two red points that come to the mirror. Before this, the length is $\sqrt{2}$. What happens when them enter the sign preserving mirror? Because D_n is unimodular, all the roots have length $\sqrt{2}$. What appears is that we have a 1 in the mirror, and the mirror itself has a slot which is 0 above and is ± 1 when a point enters the sign-preserving mirror. The Dynkin graph D_n has two legs, written by $+$ and $-$, since they comes out of the action of $\mathbb{Z}/2\mathbb{Z}$ (the point here is not just to do the B, C, D here, but to do the higher B, C, D as well). The sign of the mirror is equal to the product between the sign of the position of the mirror times the sign of the leg of the D-diagram. The statement is that if you take the sign for the mirror, then the inner products will be exactly the inner products on the ribbon or inner products of D_n. Remember that D_n is done in a completely different way. The slot means that you have $n+1$ entries for the vector, in this case the higher H_{ij}. You have one extra position. That position is 0 unless you are on one of the legs of the D_n. Our ribbon was the $\mathbb{Z}/2N\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with the graph. You have a parity from $\mathbb{Z}/2N\mathbb{Z}$ and from the leg. That gives ± 1, and you can check that it gives the correct inner products between root.

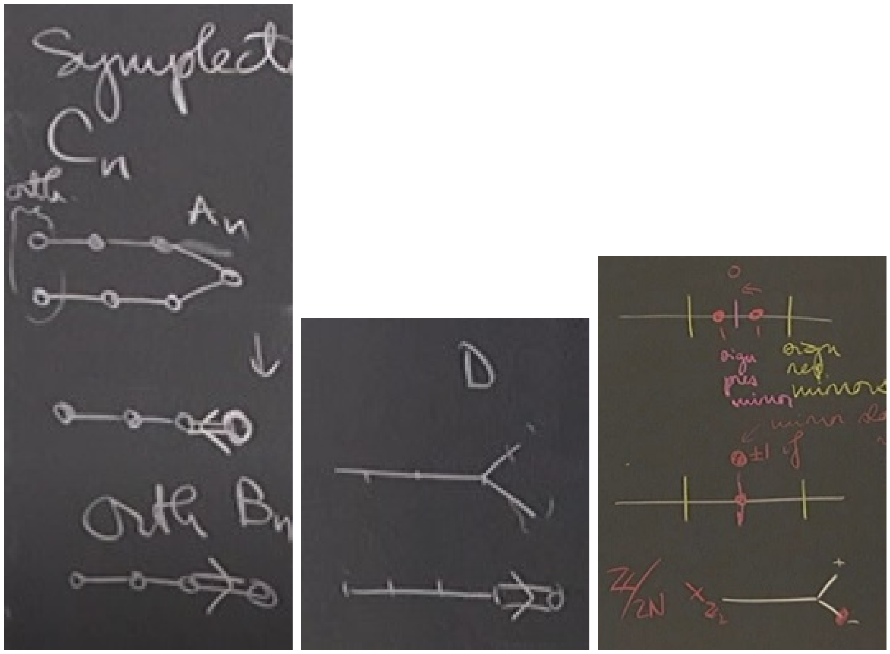

Figure 6: Figure 6(a), 6(b), 6(c)

(Answering several questions from 31:30 to 35:20.)

Now we turn to higher matrices in the case of higher A over any semisimple Lie algebra $\mathfrak{g}$ at the N-th root of unity. We take a period of roots to be the higher diagonal. With i, j, k, l in the diagonal, let $e_{ij}e_{kl} = \delta_{jk}e_{il}$ and $e_{ij}^* = e_{ji}$. If we work on $sl(3)$, Figure 7 shows a diagonal in the period 4×4. The action of the Weyl group is important. In addition to the usual involution, we have an action of the subjacent Weyl group W.

For A_2 or sl_3, take the permutahedron as in Figure 8. Edges correspond to switching labels by position, so switching adjacent labels. The position of a permutation π is π^{-1} in homogeneous coordinates. So, you take the inverse of the permutation, and it gives you three coordinates here, with sum 6. Position switching will be the right action of W on itself. The left action would be given by geometric reflections (see red line in Figure 8), and it switches labels wherever they are. Both are used to define involution.

Choose once and for all a product of simple reflections, acting on the left (having as product a Coxeter element). Given i, j in the diagonal, translate these simple reflections (defined as the main sequence of reflections) so that they move i onto j. Label i with $1 \in W$. Complete the permutohedron with the main sequence of reflections. To act with $w \in W$, take i' as the point labeled by w and j' to be the image through the main sequence of reflections. Then $w(i, j) = \langle i', j' \rangle$. This is actually an action of the Weyl group and is a fundamental one. If you want to do some unusual multiplication, you can turn your elements with an element in Weyl group, then multiply and turn back, and you can have commutators by multiply in various directions. The video, from 54:00 to 56:00, shows how the higher matrix looks like when we turn it. The following are two examples.

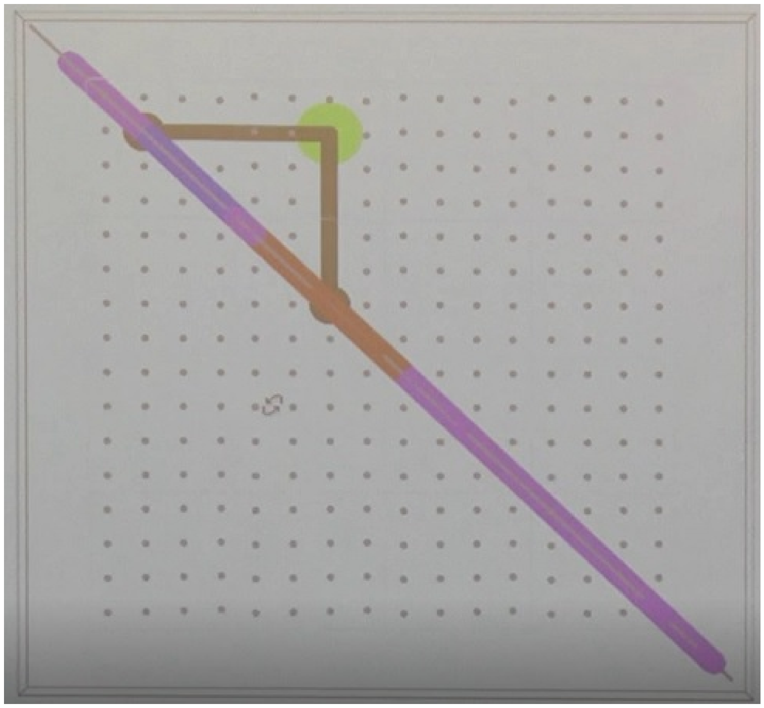

Figure 7: Figure 7

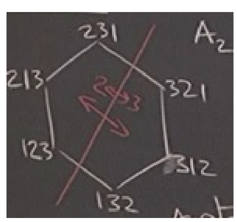

Figure 8: Figure 8

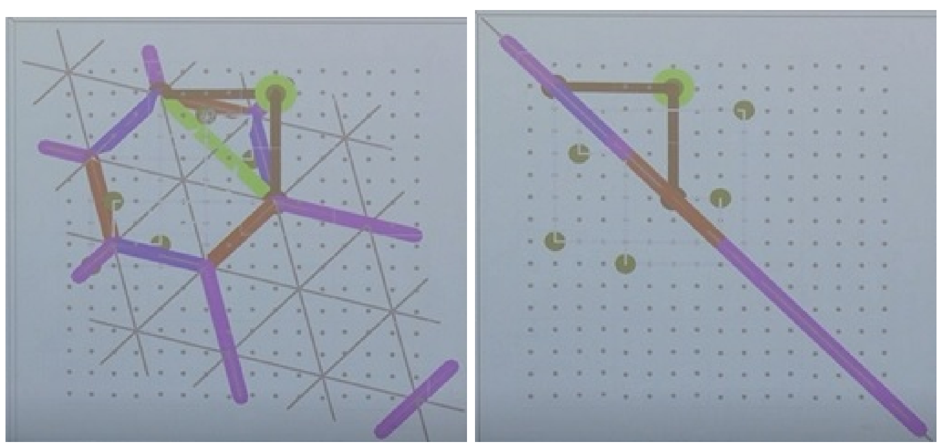

Figure 9: Figure 9(a), 9(b)