Lecture 33 of Adrian Ocneanu

Notes by the Harvard Group

Lecture notes for 15 November 2017.
What we did last time generalizes naturally to the higher case. In usual case with $N = 4$ for $sl(2)$, the matrix element is given by an arrow on the diagonal, and then we can get corresponding e_{ij} (see Figure 1).

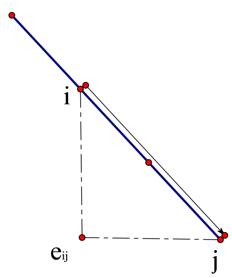

Figure 1: Figure 1

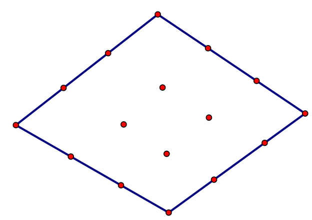

Figure 2: Figure 2

Similarly, we take N^r, where r is the rank underlying the subjacent, $r = n - 1$ for $sl(n)$. Take i, j in a period of subjacent roots (see Figure 2). Here i, j are in the diagonal, and you have an e_{ij} has the higher matrix unit which has the usual behavior $e_{ij}e_{kl} = \delta_{ik}e_{il}$. There is an action. We are going to define our e_{ij}^w, with $w \in W$. The motivation for this is: equalities between the e_{ij}'s are very sparse, they are very special to lines and involve a very large number of terms. So they are not suitable for a direct definition of a law as the usual ones. However, for $sl(2)$, we may put everything interesting in this involution.

We choose a sequence of simple roots and use the associated reflections. Let i correspond to the identity in the Weyl group and j be indexed by the corresponding Coxeter element in W. Two reflections are shown in Figure 3(a). They are the left actions, one substituting digits 1 and 2, the other substituting 1 and 3. The perpendiculars to these reflections form a basis, so this picture is unique, it is completely determined by i and j. Now we start with an element w and applying the same reflecting with the same order. Then we get $w(i, j)$ from (i, j). It is a generalisation of the case in Figure 3(b).

We can now define the higher commutator as

$$[A, B]_w = \sum_{w \in W} \epsilon(w)(A^w B^w)^{w^{-1}}.$$

The Coxeter element w here is chosen once and for all. Now we show an example. Let w be one of the main reflections in W. Then

$$\text{mult}\,(w \otimes w(\text{comult}(A))) = N\text{Proj}_{\delta_w} A.$$

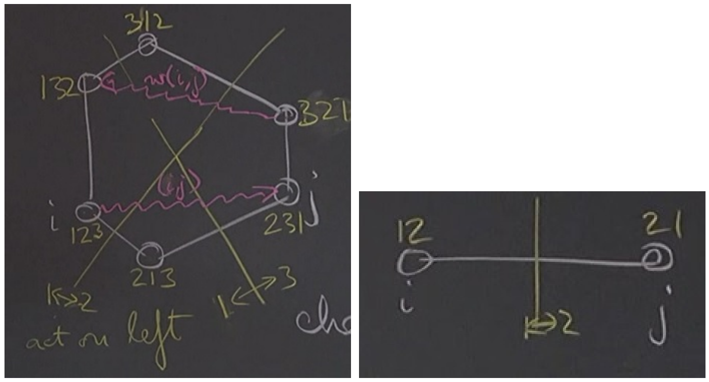

Figure 3: Figure 3(a), 3(b)

Here $\delta_w = \{A : wA = A\}$ where w acts on the left and

$$\text{comult}(e_{ij}) = \sum_k e_{ik} \otimes e_{kj}.$$

These tensors are all products which give e_{ij}. See figure 4 for the i, j, k, the corresponding hexes and main reflections. If we start from $i' = 213$, we arrive at 123 by the first reflection and then arrive at k' by the second. From $k'' = 213$, similarly we arrive at j''. For the product, the two match. Notice that one of them is in the direction of the axis from k to k''. The product is nonzero would imply $k' = k''$. But k' is in another direction in the picture. So the only possible way is that $k' = k'' = k$. Then $i = i'$. It continues in the same direction. The conclusion is that (i, j) is in the plane of reflection 12 acting on the right. The hexes degenerate to triangles as in Figure 5. In particular, i and j need to be exactly in this direction. Before we get that, we need to divide it by N and get the projection onto the pairs (i, j) which has this direction.

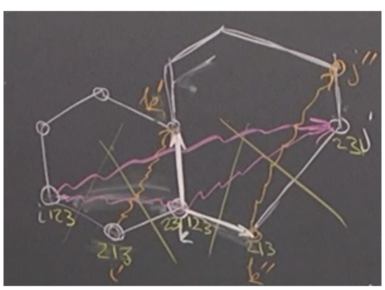

Figure 4: Figure 4

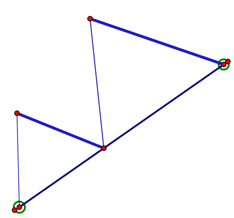

Figure 5: Figure 5

If we know intersection of w entirely, then we can determined the diagonal of the matrices up to permutation. This means we know the basis of the matrices. Therefore, we are going to define the inner product as following. Let $A, B, C \ldots$ be N elements, with N being the subjacent Coxeter number. We define

$$\langle A, B, C, D, \ldots \rangle = \text{tr}(AB^c C^{c^2} D^{c^3} \ldots)$$

where c is the Coxeter element. See Figure 6 for $\langle e_{ij}, e_{ij}, e_{ij} \rangle \neq 0$.

See the video from 32:54 to 42:20 for related materials in the case with the diagonal of 4×4.

One important thing for action of the Weyl group is the following:

Theorem .1. *The map $A \mapsto A^w$, with $w \in W$, is an action of W.*

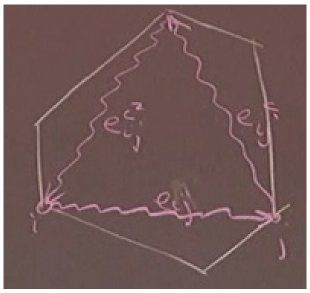

Figure 6: Figure 6

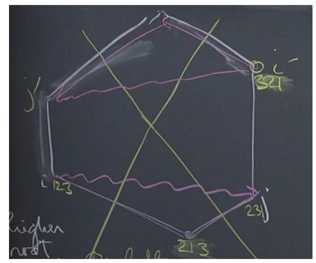

Figure 7: Figure 7

The higher root in Figure 7 appears in the construction of the map. If we are in the non-degenerate situation, exactly like the usual case, when you have a root e_{ij} with $i \neq j$, each higher root with respect to A_2 appears 6 times. Our higher root only starts from the bottom point. These main reflections (in the numbers) act on the left. If you start from somewhere else, say, from i' to j', they will give us the same refelctions in the same order, so we will get the same hex. All the permutations on the new permutahedron are multiplied by the starting point. If we continue to act, compose such operations, then we get the operation corresponding to the product of the w's.

See the video from 49:20 to 49:35 for some models of the diagonals and roots of $sl(4)$.