Lecture 34 of Adrian Ocneanu

Notes by the Harvard Group

Lecture notes for 17 November 2017.
This is the operation we showed in class last time: we took a matrix, we took its coproduct (dual of the product), twist each component with an element w and multiplied back. Then we divided by a number (N^t where t is the number of cycles of the permutation minus 1). This is a random matrix with nonnegative integers, over sl_4. The proof only works for certain things, and I explained it last time. This allows us to axiomatize the higher matrix, which is what we won't do now. Here are all the 120 things tested, and the one that work (for which we get a projection). Some of them only work over sl_2. As you can see in Figure **??**, we color the graph A_n in magenta, we choose a point,

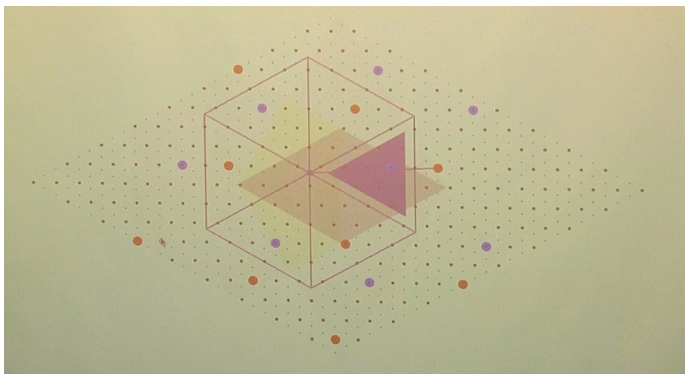

Figure 1: The graph A_n

and reflect it in the mirrors (together with the affine one). The period for the weights can be seen in grey and that for the roots in yellow. We can take as a rectangle or as a permutohedron. Either of them contains exactly 6 points, three $+$'s and three $-$'s. This is the higher h_{ij}'s. The graph D is interesting: in this case we have the mirror in the middle for the h_{ij}'s. Over sl_3 we have a rotation by 180° around the middle point. This is basically the graph D taht you can see on the cover of the book on CFT, which in that case is folded with the point in the middle separated into three This way you have 3 times more things, because you start from A_n. You also get one third of the period. What happens if you have a point in the middle? As I was telling you, in the case D you have an extra slot. So in this case the higher roots they have length $\sqrt{6}$.

This is what we did: we defined roots starting from the ribbon. In the case A_n we have shown that they are concrete vectors, with 6 components in this case; the inner products are exactly those between these mirrors. So in the case D you need 6 more slots (at least over sl_3). This is because when the point is in the middle, you only have 2 points as opposed to 6.

What is a slot? You have a vector, and the space has a basis; the entries are called slots. A few of

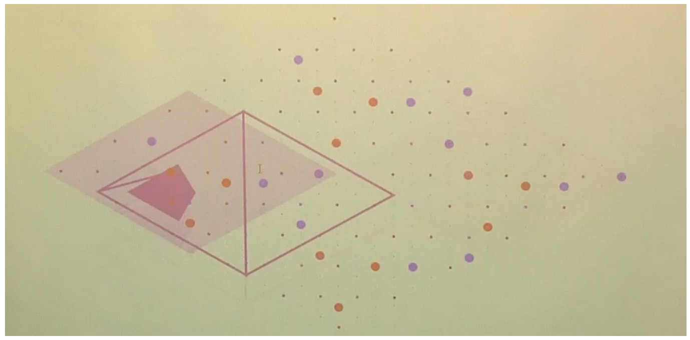

Figure 2: The graph D

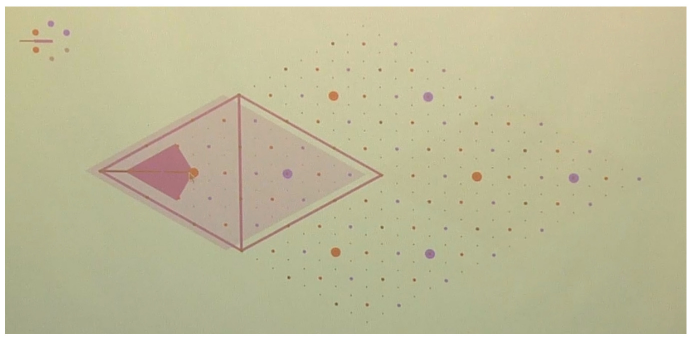

Figure 3: A point in the middle

them (here for example they are the roots in the period), once you reach the middle, instead of 6 we only have 2 points. In general we have 6 more slots, and the entries are determined by the parity of the mirrors, the roots and the weights (roots have a parity too, although it is not often used). So we are talking of the parity of the legs here. Here is the model of another orbifold. For all the orbifolds there are vectors that give inner product of the roots. Roots are *concrete vectors*.

This is what I wanted to say about this. Now we are starting a new chapter, and you can almost forget about the rest. If you are a mathematician or a physicist, this is a completely new type of mathematics that you are used to. What we are going to do is introduce plates. A *plate* is a *permutohedral cone* . We take a permutohedron and a nondegenerate cone around a point. (If you take a point on the edge you get a degenerate cone–we have points with different levels of degeneracy.) We are going to work now in the root space, which is a $d-1$ dimensional root space of $sl(d)$. What we have is a space

$$\{x = (x_i)_{i=1}^d, \sum_i x_i = s\}.$$

Typically $s = 0$, but if you have a simplex you can take it to be the dimension d of the space (e.g. in the triangle they will sum up to 3). The sum does not matter much, and we can change depending on what we are working with. We now define the standard plate:

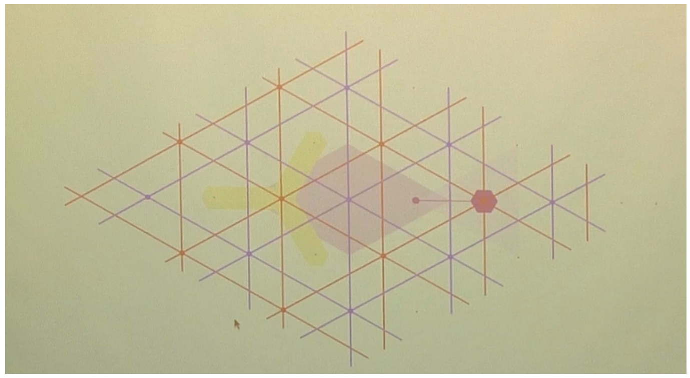

Figure 4: Slots

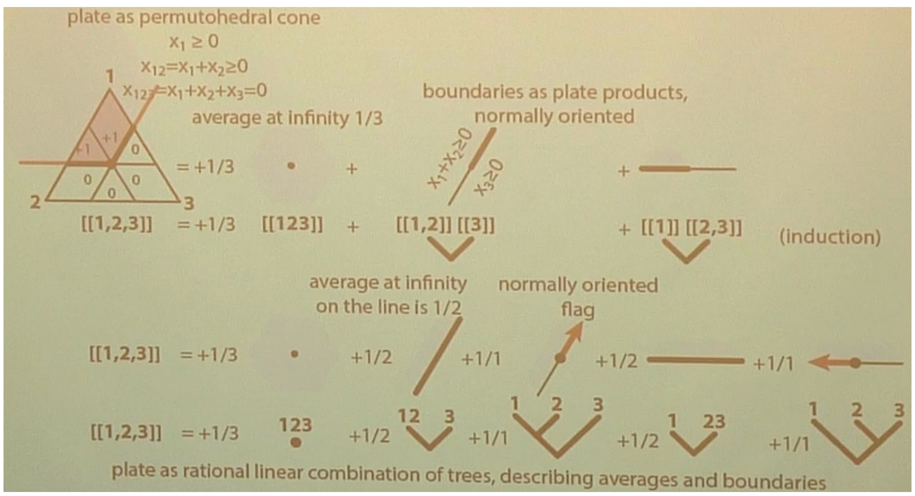

Figure 5: Trees

Definition .1. *The standard plate is defined by*

$$\begin{aligned} x_1 &\geq 0 \\ x_1 + x_2 &\geq 0 \\ &\vdots \\ x_1 + \cdots + x_d &\geq 0 \end{aligned}$$

(the last is actually an equality).

We introduce the notation

$$x_S = \sum_{i \in S} x_i, for\ S \subset \overline{d} = \{1, 2, ..., d\},$$

so that we can introduce the following spaces:

$$\begin{aligned} S_1 \bigsqcup S_2 \bigsqcup ... \bigsqcup S_k =& \bar{d}, s_1 + s_2 + ... + s_k = s, \\ x_{S_1} \geq& s_1, \\ x_{S_1} + x_{S_2} \geq& s_1 + s_2, \\ &\vdots \\ x_{S_1 \bigsqcup S_2 \bigsqcup ... \bigsqcup S_k} \geq& s_1 + s_2 + ... + s_k. \end{aligned}$$

We are going to indicate these plates by double brackets as follows:

$$[[(S_1)_{s_1}, (S_1)_{s_2}, ..., (S_k)_{s_k}]].$$

The S_i's are called lumps. A nondegenerate plate is one such that every lump is a singleton.

Let me give you an overview of what we are doing. You might remember this from Gelfand-Tsetlin.

Definition .2. *We identify a plate with (up to higher codimension) with its characteristic function, and study linear combinations with coefficients in* $\mathbb{Z}, \mathbb{R}, \mathbb{C}$ *of these plates.*

Special hyperplanes and half-planes are of the form

$$x_S = a \in \mathbb{Z} \tag{1}$$
$$x_S \geq a \in \mathbb{Z}. \tag{2}$$

Such hyperplanes appeared in the 50's in the study of retarded potentials.

Special hyperplanes cut space into convex pieces called shards. In the plane,there are only two kinds of shards, one triangle standing up and the other standing down. In dimension 3,there are ten. In dimension 8, even up to permutations, there are about forty thousand kinds of shards. We need to find the relations between these plates, and what I found is that one should go from plates to *trees*.

How do you define a cone in a sphere? One way to define it is to specify its boundary. We think of the boundary as a derivative, a jump. (Demonstration at $37:00$.)

Definition .3. *We map the plate*

$$[[(S_1)_{s_1}, (S_1)_{s_2}, ..., (S_k)_{s_k}]]$$

into a sum of trees as follows:

we call the sequence $(S_1, S_2, ..., S_k)$ *the* canopy.

We partition the canopy into parts which are a set compositions (ordered partition into unordered parts) $(T_1, T_2, ..., T_l)$ *such that*

$$\begin{aligned} T_j &= S_{i_j} \bigsqcup S_{i_j+1} \bigsqcup ... \bigsqcup S_{i_{j+1}-1}, \\ t_j &= s_{i_j} + s_{i_j+1} + ... + s_{i_{j+1}-1}. \end{aligned}$$

Definition .4. *A binary layered tree is a binary tree with one node per layer. (These are in a one-to-one correspondence with permutations.) Given trees* $T = ((T_1), \ldots, (T_l))$ *we consider linear* *combinations*

$$\sum_{T} \sum_{\substack{\textit{layered} \\ \textit{binary} \\ \textit{trees} \\ \textit{underneath}}} \operatorname{coef}((T_1), \ldots, (T_l))$$

where

$$\operatorname{coef} = \frac{1}{\prod_j (\textit{ number of } S_i \textit{ in } T_j)}.$$

This is an expression of a plate in terms of tree. The antisymmetrization of a binary trees is a sum with signs of the permutations of the tree at every node.

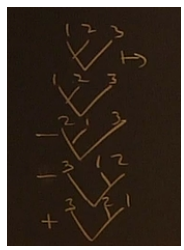

Figure 6:

This way we have expressed every matrix in terms of trees, and now I want to show the opposite part. That is because shards are much more complicated. With trees, on the other hand, you can take linear combinations and form functions on the plane. Next time we are goign to see how to go from trees to blades.