Lecture 35 of Adrian Ocneanu

Notes by the Harvard Group

Lecture notes from 20 November 2017.
Remember that last time we said that we are going to work with plates, which are characteristic functions of permutohedral cones. Later we will sudy blades, their affine version, which will be our higher intertwiner.

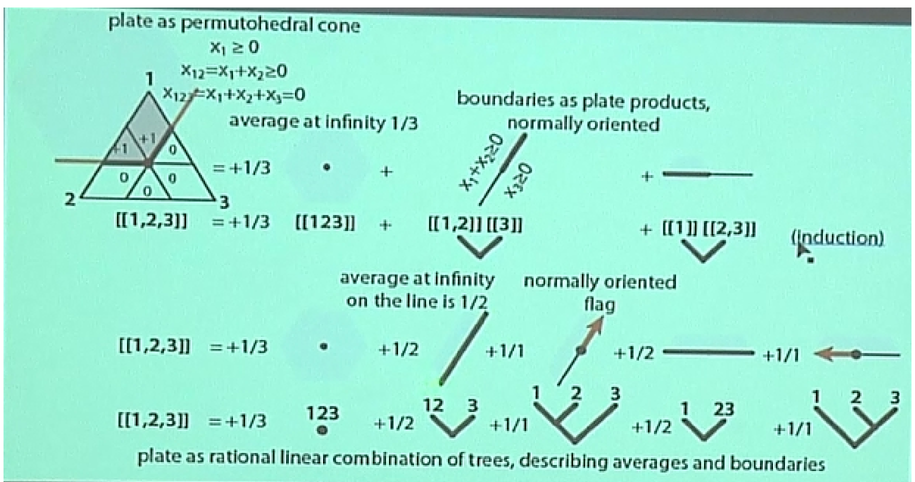

Figure 1: A standard plate

We have here the standard plate $[[1, 2, 3]]$. In this case we put $1, 2, 3$ at the top (we call this a canopy) and we lump them in all possible ways (lumping is also known as *set composition* in combinatorics). This is the same as an ordered partition. This is what we did on the top, and underneath we put all binary layered trees. What is *not* written here is that we need to anti-symmetrize things, and this is in order to glue them (this is a common theme in algebraic topology).

The motivation for this is that we want to take the average at infinity, as is written here. This means that you take a big sphere with a fixed center and take the proporiton as the radius goes to ∞. In This case, we see here that we have regions covering a third, a half and so on. This is a homology map, since the trees give you a boundary. Knowing the average at infinity you can determine a function. What we'll do next is study the inverse map, then study the relation between these plates and move on to blades. These relations appear to be new.

Here is the inverse map. The trees we have seen were counting boundaries. One could actually prove directly that a map characterizes a tree and that a map from linear combination of plates to trees is reversible. What is interesting, however, is that every tree gives rise to a function in our ambient space. Thus trees become concrete objects, and not just labels.

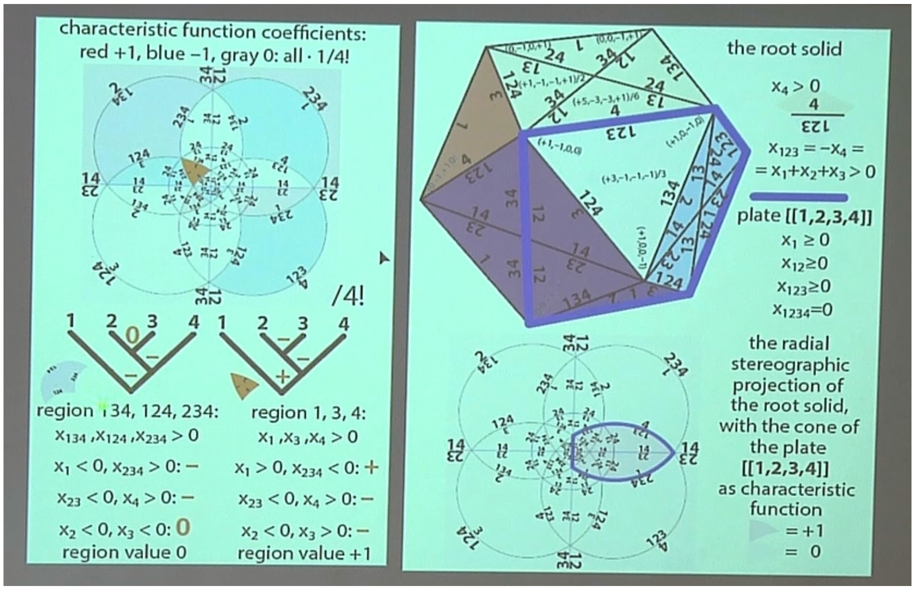

Figure 2: The root solid

What we are doing here is the following: we have the root solid–its 12 vertices are the 12 roots of sl_2, the corresponding h_{ij}. They have four coordinates, but their sum is always 0. We work with our special hyperplanes $x_1 \geq 0, \ldots, x_{1234} = 0$ and so on. The actual coordinates are displayed in black. We are working in a cone around the origin, but in general what we do is study affine plates localized at various points, in which case we have an extra coordinate for the vertex. Here however everything is around 0; this roots solid is a cone around 0. We have all these hyperplanes, and the resulting things are called shards (the number of shards grows as 2^{2^d} where d is the dimension, since they are subsets of subsets). Affine shards are a new concept. Note that if you are in a shard then the sum of any subsets of coordinates has a definite sign.

These objects are not classified, and contain very hard nuber-theoretical problems; for example, points of intersections are solutions to Hadamard matrices. This is why we went to trees: they are much easier.

What I used here is the same projection I used for the octacube: you inflate the object on a sphere and then project it stereographically. You can see the standard plate in blue (12:00). It's a flag of equations. This can be found in the work on retarded potentials.

We now have to map the trees back into functions on the plate. Here is a rule: you take a tree (here the leaves are labeled $1, 2, 3, 4$) and you look at regions (red means $+1$, blue -1, white 0); this is a function, to be divided by its coefficients (15:20) To take such a value, you separate the variables according to the nodes starting from the bottom. If the first is positive and the second is negative the sign is $+$; if the other way around, -1; if they are the same, 0. At the end we take the product of all of this. (Examples until 26:49.)

The proof is goign to involve the derivative and proceed by induction. In this particular case we focus on the verical line, which divides $x_{12} \geq 0$ from $x_{34} \geq 0$. To take the derivative we take the

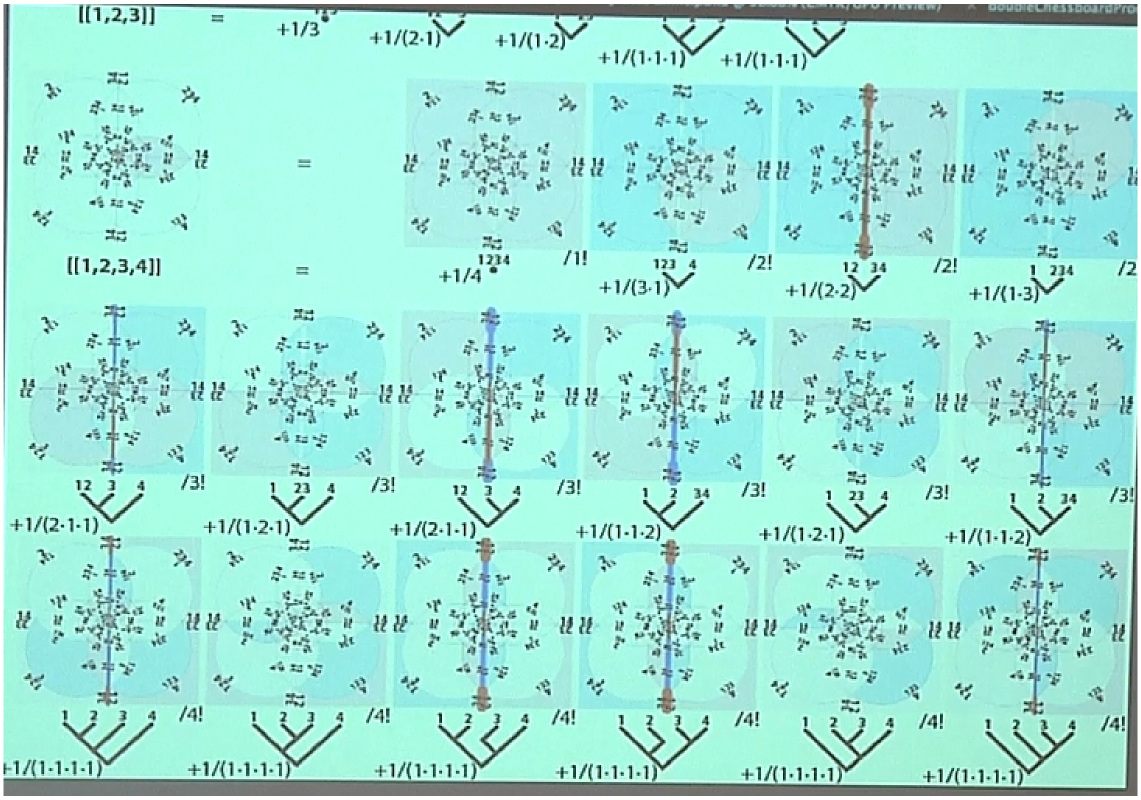

Figure 3: Functions and derivatives

difference between the region which contains 1 and the region which doesn't. Red means $+1, +2$ (2 is thicker) and similarly $-1, -2$ for blue.

After this we summ all of these differences after dividing by the number of leaves factorial and then dividing by the lumping number.

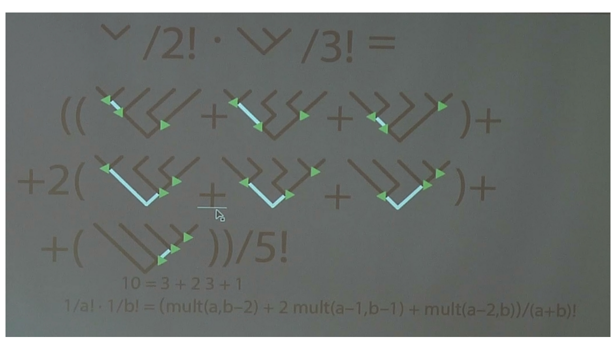

Figure 4: Product of trees

The statement now is the following: if you take these differences and add them up, you will have the derivative on this hyperplane (vertical in this case), and if the two regions are contiguous then what you get is exactly the product of things computed by induction, as we will see later. If they are not ordered, then the sum is going to be 0. The proof can be seen in Figure 4. It works by induction.

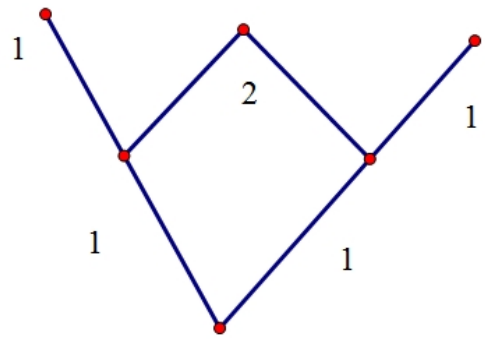

Figure 5: Pascal's triangle

The multinomial coefficients arise for the same reason as they do for the Pascal triangle, shown in Figure **??**. The result is that the sum is the multinomial of a, b, namely

$$\frac{(a+b)!}{a!b!}.$$

This gives equality of coefficients. In Figure **??** we can see how induction works.

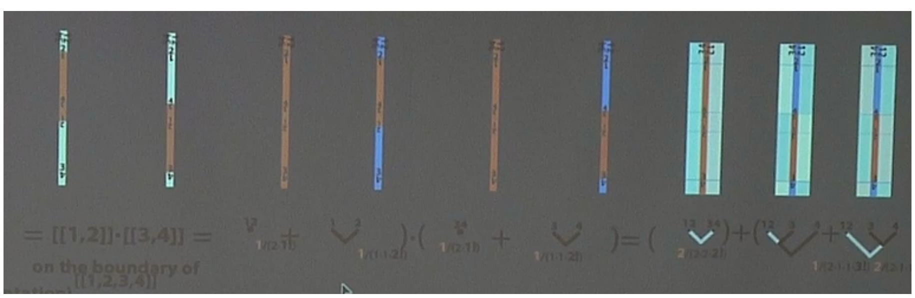

Figure 6: Sum of differences

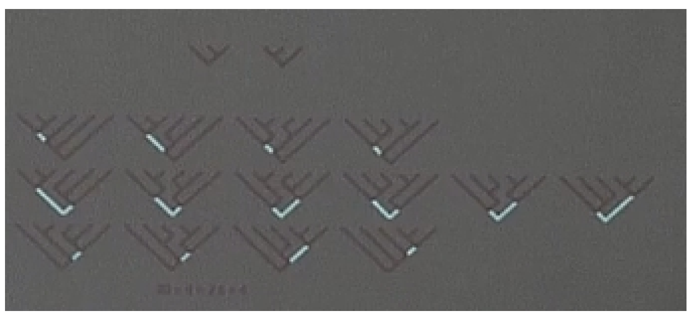

Figure 7: Induction step