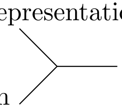
Lecture 36 of Adrian Ocneanu

Notes by the Harvard Group

Lecture notes for 27 November 2017.
Here is a picture of plate as rational linear combination of trees, describing averages and boundaries. See lecture video at 2:15.

Recall from the last lecture that we have expressed the plates as the linear combination of trees. Today, we should extend this. Recall what we did: the plates are permutohedron cones, and what we did was to separate the numbers, here $1, 2, 3$ (the name of coordinates) into lumps. Then we put each of them at the end of all the possible layer trees. Today, we will observe the properties of this map. One is antisymmetry. But let me introduce another thing first.

Consider an intertwiner of a representation, for instance the following is the 3 representation of sl_4, sl_n with $n = 4$ which acts on $V = \mathbb{C}^n$. Here $\cdot := V^{\wedge 1}$, $\cdot\cdot := V^{\wedge 2}$ and $\cdots := V^{\wedge 3}$. Now $V^{\wedge k} \otimes V^{\wedge l} \otimes V^{\wedge l} \supset V^{\wedge k+l+m}$. So if $k + l + m = n$, then this is the identity representation. There is also a similar identity when you take $V^{\wedge(n-k)} = \overline{V^{\wedge k}}$. See Fig. Then diagram  in the Fig.1 is called a *blade*. Zuber and Cocqueraux have written papers in which these are called O-blades.

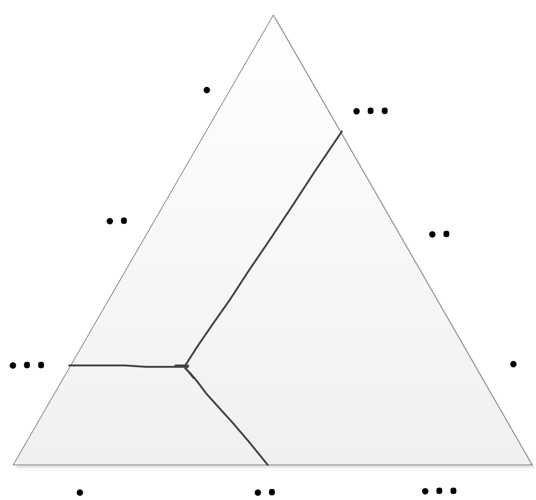
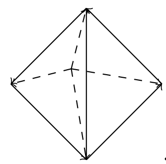


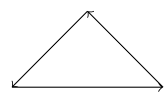
Figure 1: Representation of sl_4.

The diagram of Fig. 1 is affine root of type A_3. If we take some affine permutoherdron, we will obtain this. We can also find the roots in a simplex in general, e.g. Δ^n with $n+1$ vertices.

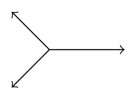


In this case $\tilde{e}_i = Proj_{\sum x_i = k} e_i$, edges are roots. The (affine) simple roots are given by a Hamiltonian path (resp. circuit) on the edges of the simplex.

If you have the simplex like this,



there are two circuits.If you put them afterwards in the same origin, then we have the form of affine roots.

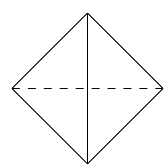


Notice that this is the same picture that appears in the intertwiner.

So this is the system of A_2 affine roots of $sl(3)$. Affine roots = weight of a vertex in the paving by affine permutohedra. See lecture video 14:00. Remember, for plates, we take the neighborhood in one solid permutohedron. Here we take the affine ones, so just on the surface.

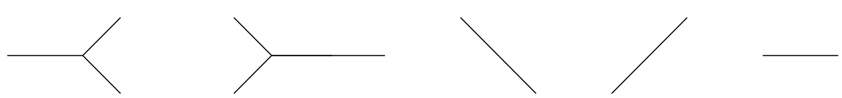
We now define the nondegenerate blades to be the codim 1 surfaces spanned by $n-1$ of the affine roots at the point. See lecture video at 15:30 for a explanation on a model in 3D.

Consider the affine Dynkin diagram

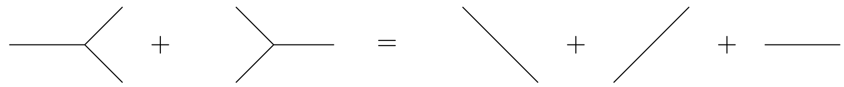


So you have $\binom{d}{d-k}$ faces of codimension k, where d is the number of coordinates. We denote such blade by something by $((S_1)_{s_1}, (S_2)_{s_2}, ..., (S_m)_{s_m})^{(k)}$, where k is the codimension, S_i is the lump coordinates and s_i is the position.

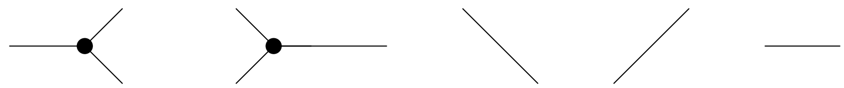
Let us consider case $d=3$, which is in the plane. In this case, around the point, you have the following blades:



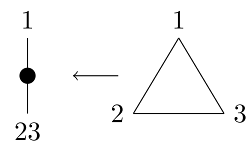
The first two are nondegenerate, and the three on the left are degenerate. These blades satisfy the following relations:



So we actually take characteristic functions modulo lower dimension here. These would be codim 1 blades. These are the ones which appear in all the intertwiner theory of representations of $sl(n)$ (this is known as *breathing*). What I would like to do is to show the importance of high codimenion ones in the plane. For a dot $\cdot$, which as codim 2, we can take full blades. The full blades all have codimensions ≥ 1.

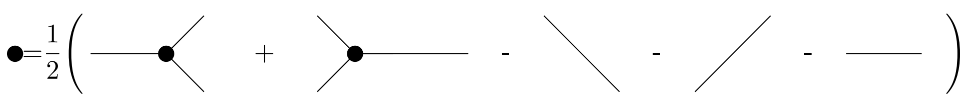


For instance, the following one is a dot on the one dimensional thing, which is traced back.

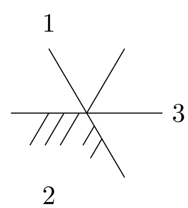


lumping

In this case, the linear dependence relation will give you the definition of the dots.



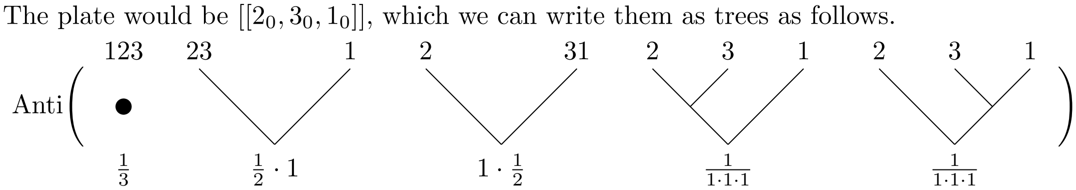
Look at what happened to a plate.



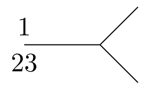
This corresponds to the following equations

$$x_2 \geq 0$$
$$x_2 + x_3 \geq 0$$
$$x_1 + x_2 + x_3 = 0.$$

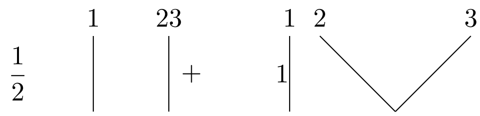
The plate would be $[[2_0, 3_0, 1_0]]$, which we can write them as trees as follows.



If you have $((2, 3, 1))$ as blade, it will be like this:

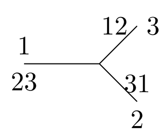


Note that in the previous case we had a distinct normal since we had a solid. Now the two sets are not symmetric, and so we need to use symmetric trees. We call them a *forest*. For this blade we get

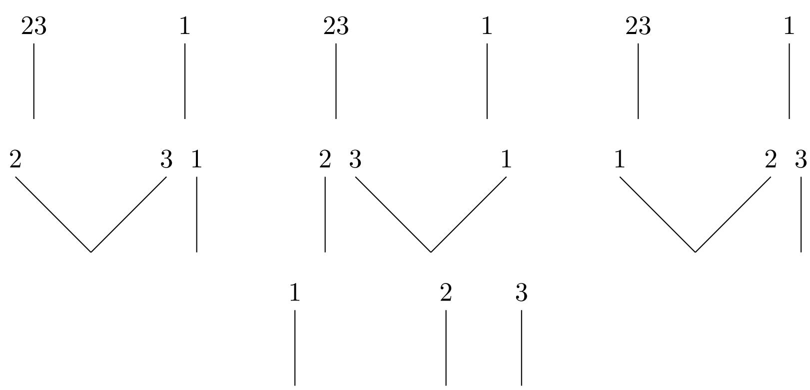


We can now define the full blades. The map is following: if you have $((S_1)_{s_1}, (S_2)_{s_2}, ..., (S_m)_{s_m})^{(k)}$, you take the circular permutation which gives the distinct trees, then lump them. After that, we partition them (in more than 2 parts) into a forest while keeping the order (this is called set composition by combinatorialists). The number of parts minus 1 is equal to the codimension. Then put under each all possible layer trees, with coefficients 1 over the product of the number of S in each lump. Give each lump as coordinate as the sum of the coordinate of the corresponding S. Antisymmetrize each tree.

Let us take $((2_0, 3_0, 1_0))^{all}$.

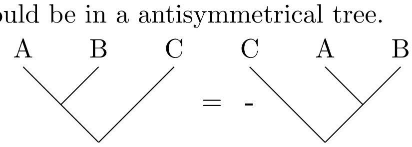


We separate $((2, 3, 1))$ as follows,



See the diagram in the Lecture video at 47:25 for explanation.

In the expansion of the blade, if we have 3 subsets A, B, C, which are lumps, then the lumps should be in a antisymmetrical tree.



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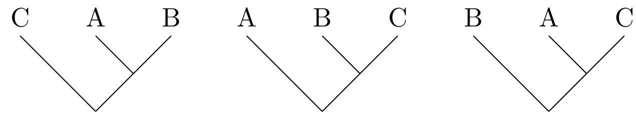
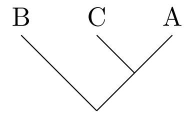
If we look at the following trees,

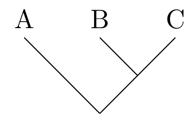


then by anti-symmetry the coefficient of these three trees are +, - and the tree



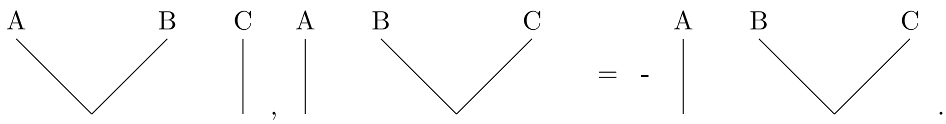
does not appear.

The next result is the following:

$\sum_{\text{circular permutation of ABC}}$coefficients of  =0

This is the Jacobi identity (or the Bianchi identity).

Besides, if you have A, B, C with B is located between A and C, then B can be connected to A or C as follows, which is called grafting (See the lecture video at 53:40)



If we take a branch B and take a forest without some tree B, then you add B in all possible ways. Your B is in some site, and the sum over all possible sites for B will become zero.

$$\sum_{\text{all possible sites for} B} \text{tree } B \text{ tree}' = 0. \tag{1}$$