Lecture 37 of Adrian Ocneanu

Notes by the Harvard Group

Lecture notes for 29 November 2017.
What I would like to mention here is the definition of plate. In codim 0, we can have plates, and they are defined modulo higher codimensions. The above is a collection of sets of the form

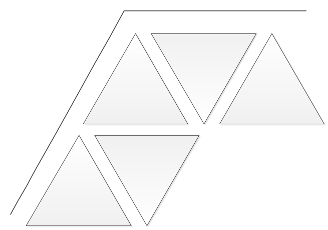


Figure 1: Open shards.

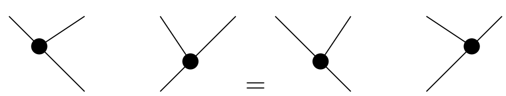
$x_S = \sum_{i \in S} x_i = n \in \mathbb{Z}$. If we cut the space with subspace $\sum x_i = N$, hese will give you the shards. Similarly, when you have a blade, it is defined with open shards. (See Fig.1)

If you take a full blade, you also add a dot to this. But this dot does lives in this space, where only dots live. Each dimension live in a separate space, otherwise you will get lots of problem.

One more observation is the following properties of plates and blades. Plates and blades are mapped into some trees, and we have the properties of the images. For instance, the images are not all the linear combination of the trees.

Also, if we write tree 1+ tree 2+tree 3, this means that the coefficients in the image satisfy that relation, like the Jacobi identity. Let us discuss the properties of the image.

(1) Delayering. If you have two nodes of a tree which have different heights, then they can be put into other oder as well. See the following example.

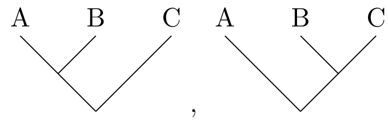


This is the trees for associahedra (not permutohedra), which means you have the trees with the nodes as high as possible. The proof of this property is that

$$\text{plate\&blade} \to \text{lumped} \to \text{all trees underneath } \to \text{antisymmetrization} \tag{1}$$

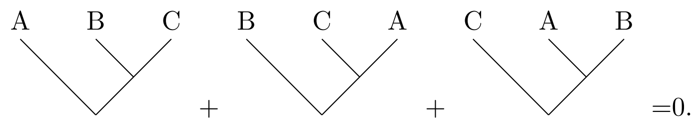
(2) Antisymmetry (by definiiton, introduced so that we can glue things).

(3)Jacobi identity. If you have three trees A, B, C,they can appear in the following different ways,

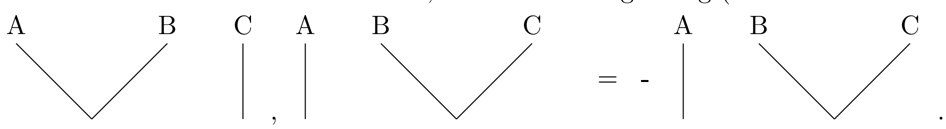


+antisymmetry.

In the blade case, we take the cyclic permutation ≥ 2 trees, which means we get another D, i.e., A, B, C, D. This implies that C does not appear, otherwise D will appear between C and A.

Coefficients of 

(4) For blades in all codimensions, if you have A, B, C with B is located between A and C, then B can be connected to A or C as follows, which is called grafting (See the lecture video 44:52)

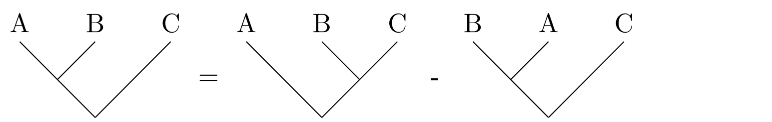


If you take a forest without some tree B, then you add B in all possible ways. Your B is in some site, and the sum oover all possible sites for B will become zero.

$$\sum_{\text{the possible sites for} B} \text{tree } B \text{ tree}' = 0. \tag{2}$$

The first three properties are known in the theory of free Lie algebra, and the last one has never been seen before.

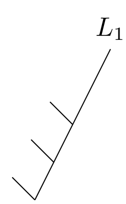
Using these properties, let us first try to find which trees are linearly independent. So for plates, using the Jacobi identity as the following,



If you do the differential geometry, you might be reminded of something like

$$\bigtriangledown_{[A,B]} = [\bigtriangledown_A, \bigtriangledown_B]. \tag{3}$$

where $\bigtriangledown_A$ means putting a branch A. This implies that all nodes can be pushed right, so that you can get the trees as follows:

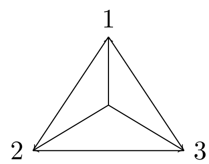


which is called right combed tree.

Using the Jacobi and antisymmetry, you can put 1 at top to get the lump containing 1 at the top L_1. These form a basis.

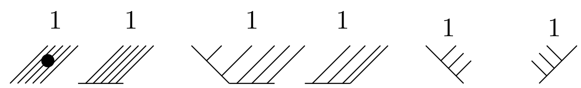
The corresponding plates $[[S_1, ..., S_n]]$ such $1 \in S_1$ form a basis.

If we are in plate like this,



coordinate $\sum_i x_i = 0$

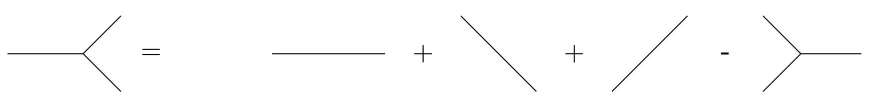
Remember what we do is that we take the standard basis and project onto the subspace where the sum of the coordinates is constant. This is the picture of the basis. In this case, the plates which form a basis are the following,



Any plate is a linear combination of these plates.

For cordim 1 blades, this would be something like $((S_1, ..., S_n)) = ((S_2, ..., S_n, S_1))$ by definition. For the basis, you have $S_1 \ni 1$, $S_2 \ni$ smallest corodinate not in S_1. If we we take 1,2,3, then these plates would be the following:

The others one would be expressed in a linear combination of the above of plates, for example,



The full plate are linear independent.

We are now going to see the structure of shards (see the picture at 25:00 or Fig. 2).

In dimension 0, we have one dot. This is the affine simplicial complex obtained by cutting with a special hyperplane. We are going to study the representation next week; however, I want to say in advance that the special hyperplanes here, which give us the whole structure, have been studied extremely little. There may have been 4 papers in the last 15 years about them. The main hyperplanes are those such that the sum of coordinates is an integer. These appear in sl_2 over anything. If you work with a tetrahedron, this is exactly the hyperplane which appears there. Then you will see, in the representation part, they appears, for example, in the Clebsch-Gordan symbols, 6-j symbols and all the things which are considered extremely complicated in physics and chemistry. In general, we are building now is a higher version of Gelfand-Tsetlin representations.

Here we are on the line. We have dimension 0, the center. This is a integer point. We have the segment, which has dimension 1. The segment has coordinates $01, 10$.

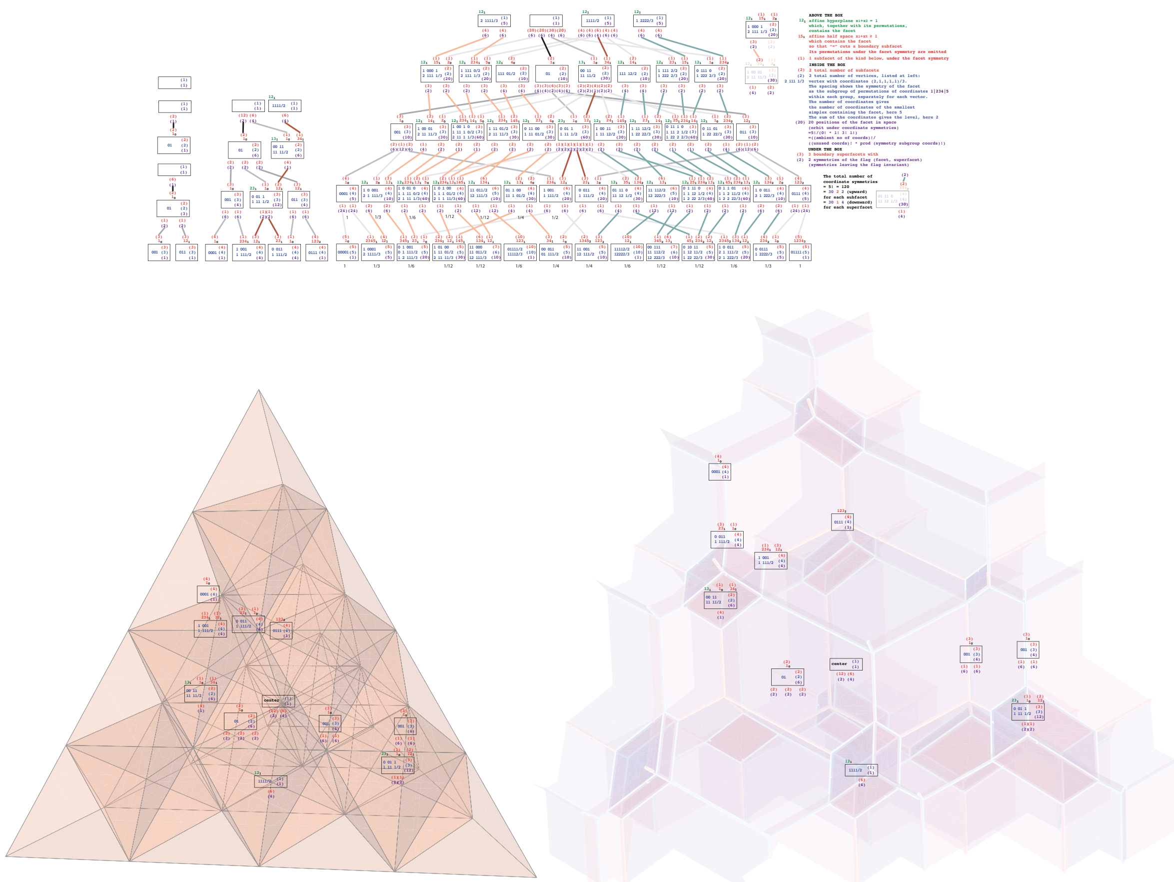
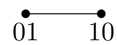
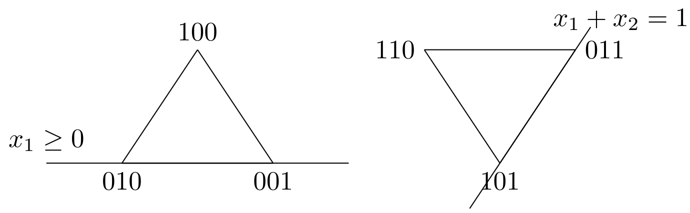


Figure 2: The structure of shards.



The next one are the shards on 2 dimensions, with 3 coordinates.



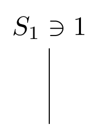
Here is a structure of shard in 3D. See Fig.2.

$(1,1,1,1)/2$ is called gem. (See 33:26 in the lecture video.) These have the property that all equations of the form $x_S \in \mathbb{Z}$ determine it. These are the vertices of the shards. There is no such shard in the literature. Denominators of shards are determinants of Hadamard matrices, which have entries $0, 1$ or ± 1. The 3D dimension case is presented in a model see 35:54 of the lecture video. This is the blue point in the middle of the octahedron. You can see here there is a white point, which is an integer point. Then you have the green edges which comes from the lower codimension. Then you have triangles standing up and down, which also come from lower dimensions. For all the triangle given here, you can find if it is a $+$ or a -1.

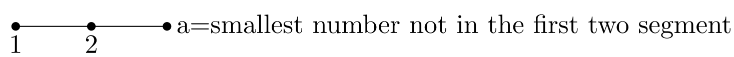
The explanation of Fig.2 begins from 37:26 to 43:00 in lecture video.

The most important thing here is the Riemann curvature, if you work in 4 coordinates (it also appears in higher cases), and in codimension 1 blades.

We have a forest basis for all codim blades. So if you have the forest basis, with grafting, then you can make the first tree without branches, because you can graft them to the others. The first tree would have a set $S_1 \ni 1$. The others would be combed trees. And $S_2 \ni$ smallest corodinate not in S_1,



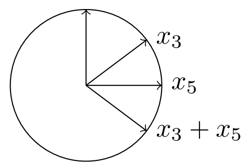
Similarly $S_2 \ni$ smallest corodinate not in $S_1 \cup S_2$. So this forms a basis. In this basis, if you have a permutation π starting with 1, then you have a subsequence of successive minima. Namely, if you have 1, the next one is such that the number betweem them is larger than a. See the following example.



You can make these exactly into trees, where each of these segments will become a combed tree.

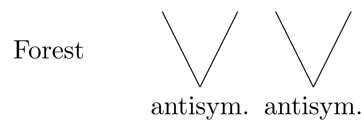
Answering a question from a student. In general, the Hasse diagram gives you the distance to faces of permutohedron. If you have a function on Hasse diagram, that is the same as a generalized permutohedron. You can take a permutohedron like this, where the face are indexed by subsets of $\{1,2,3,4\}$. Then you can move every face as wish, and this is a generalized permutohedron. There are some conservation relation which are not satisfied by shards. Shards, typically, can not be realized by linear combinations of plates and blades. They are more general.

If you are in a simplex, then the coordinates have sum N. So you take a clock and divide it into 1/N portions. If you have a plate here, such as $x_3 \geq$, $x_3 + x_5 \geq \ldots$. Then you can draw the clock as follows:

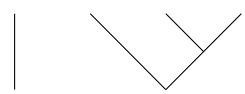


So you can put hands for the clock in this way. Inside the plate, these are bigger. This means that hte hands move forward without passing each other. So that describe all the inside points of the blades; they are parametrized by the clock. If you do the analogous thing in the line, then you get the parametrization of the infinite blades.

We can see a bit of the Riemann curvature. If you take the blades of codimension 1, then on the one hand, you have the tree with the following form.



Each tree is anti-symmetric. You have something symmetric in the two trees, because they are part of the forest. Remember, these are exactly the symmetries of R_{ijkl} of the Riemann curvature tensor. Now by grafting, you can move it into



where the latter satisfies the Jacobi (or Bianchi) identity.