Lecture 38 of Adrian Ocneanu's Course "Higher Representation Theory"

Notes by the Harvard Group

Lecture notes for 1 December 2017.

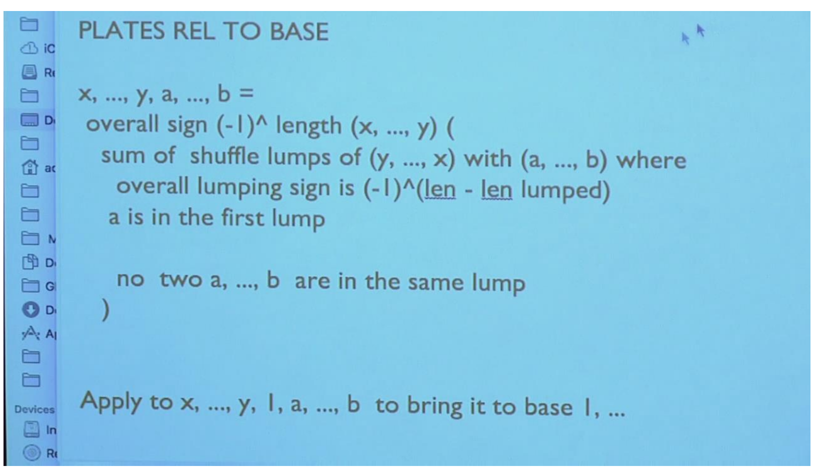
Relations for plates and (codim 1) blades

When we had a plate $[[1, 2, 3, ..., n]]$, it is the plate that $x_1 \geq 0, x_{12} \geq 0, ...$ The basis of the plates are the plates which have 1 in first lump. By the way, the permutohedron are generated by plates.

The first type of relation is **bring to basis**. In this case, if you have a plate like $[[5, 4, 3, 1, 2]]$, which could be replaced by lumps, the lumps by unions. What I mean by this is that you can have 1 be replaced by (234), this is a lump; and then 2 replaced by (57), and then the lump (12) replaced by (23457). All relations are invariant to such things.

$$[[5, 4, 3, 1, 2]] = \sum (-1)^{|A|+\text{lumping sign}} \text{lump}'(A^{\text{rev}} shuffle' B)$$

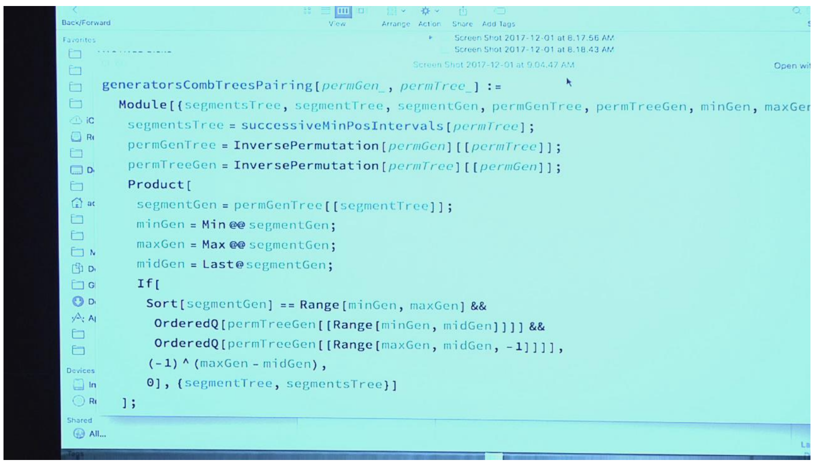
Where $A = (5, 4, 3)$ and $B = (1, 2)$. $shuffle'$ means that 1 is in the first lump. And lump' is



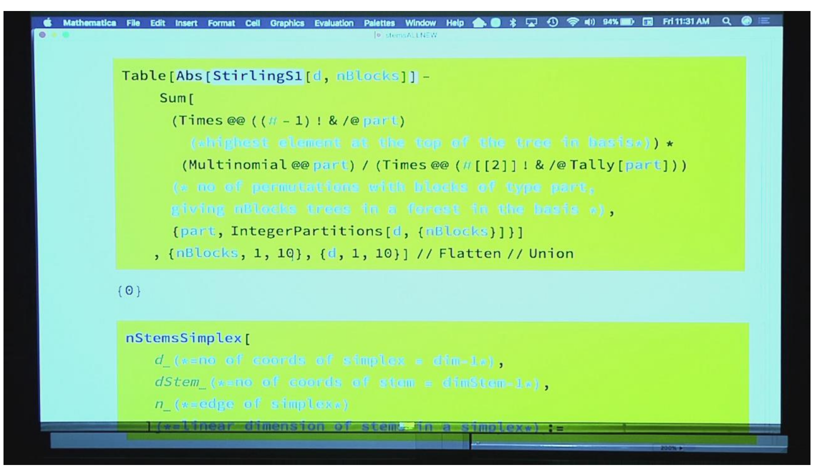
(see video 8:05). What is a shuffle in mathematics, you have a few sets, and you take their union, such that when you restrict the order to each of them then you get it in this order. So if you shuffle 1,2,3, with 4,5,6, you need to have always 1,2,3 in the order 4,5,6 in order, and the rest is the same. So this is exactly shuffling cards.

This a apply to x,..,y,1,a,...,b to bring it to the base, which starts with 1. This is the relation of the plates to the base. The proof of this which will discuss will involve writing them in terms of trees that's I found it that was a reason or going to those trees, because if you want the values of the plate at the point, that point depends on the, I mean if you want to a shards depends on the knowledge to shards on the shards are unclassifibly. No many mathematics.

Apply to $1, x, ..., y, 2, a, ..., b = x, ..., y, 2, a, ..., b, 1$ to bring it to base $2, ..., 1 = 1, 2, ...$ This gives you an arbitrary blade or plate in terms of things which are in the base. Once you have that you can bring anything to the base. Now the last one will all the a,b's are in the lump, this lump may not contain 1, and in that case you continue inductive. So this is the blade relations to a base. But the blades have a simplify, I mean, the understanding of that this prog things is. There is a big matrix which has a blades in all the codimensions. And this big matrix is invertible. So relation between blades like this one, corresponds to constructing a higher codimension blade. if you do this to the full blades that we described last time, then you get the higher codimension blades.



Now this is a big matrix, and I am going to describe it.



These are the functions in mathematica. If you have a permutation with 5 cycles, that must be the identity. These are stirling S1. The second kind are very interesting. They give you the following, if you have a, in our case, if you have a set of coordinates, so in this case the coordinates are the vertexes of the simplex. So you see how many coordinates do you have. Here we have 4 coordinates. We are in 3-d space but we have 4 coordinates because the sum of the coordinates is 0. So this is the rot lattice. When you have degenerated a blades, for instance a degenerated blade can be a horizontal one. Now that one, if you have all instances horizontal plane like this, means that you take the bottom 3 points you grow up them into a single one. So you take the simplex like this, you group the bottom ones into one, the top into one, so then you have a simplex like this, which has only 2 coordinates. Here you take a blade, which is just a point in this case. And this blade gives you a blade here. Now the number of ways to degenerate the lump together, k valuables into l subsets, that is the stirling number of the second kind. So this will be the ones which group 4 into 2. So this will be a part of the stirling 4,2, how many do we have, we have 4 of them of this kind, will we have 1 and the other 3, then we have 3 which we have 2 on the other 2. If you have 2 on the other 2, here you have 2 of them on the other 2. You see again you group them, exactly like before. Now you going to group 2, 1, 2. And the point here if going to give you a triangle. Here which is just another kind of a higher intertwiner. So this is a degenerate intertwiner. And you have 7 of them, of the stirling these 2 kind. Now the number of points in a simplex is a binomial. These are exactly binomial numbers. So the elements on the line is n+1 if you have an segments in the plane you have something like (n+1)(n+2)/2. Because for a triangle, you take a triangle, and you a second one like that, and now you have (n+1)(n+2) points. And it's over 2 because you have 2 triangles. So the number of elements in a simplex of integer coordinate points are binomial numbers. When you put together the stirling guess 1, the stirling guess 1 gives you the number of degrees of freedom, for every point. It's known that the sum of the stirling guess 1 is a factorial. If you take that factorial, and you take this binomials, so this is here, (n+2)!/(2!n!). For instance here of independent blades at the point. If you take the inner points, then you have in this case, $\frac{(n-1)!}{(n_3!2!)}$. There are 4! blades of higher codimensions that every point that using a stirling of the first kind. And so what you get is $\frac{(n-1)!}{(n_3!2!)}$. Now when you take this, so this is for the inside. You have a similar formula, for the inside of an edge, and for a point. So when you sum these, against the stirling 2, of 2 and 1, you sum these also for all those quotients. The amazing thing that you get is that the total number is exactly n^{dim-1}, the total blades in all dimensions. This is exactly the number of points on the higher matrix diagonal. In 2-d case, the number of diagonals of a 2-d matrix is exactly n, which you can see. Now the importance of this is the fact that when you have a usual representation theory, you have sl(n), GL(n), GL(n) has n elements of the diagonal, one of them is separated which is the determiner the sum, if you look at the Lie algebra, the sum of the elements. That because the determinant has its only representation, it's separated on the last.

Local (conservation) relations

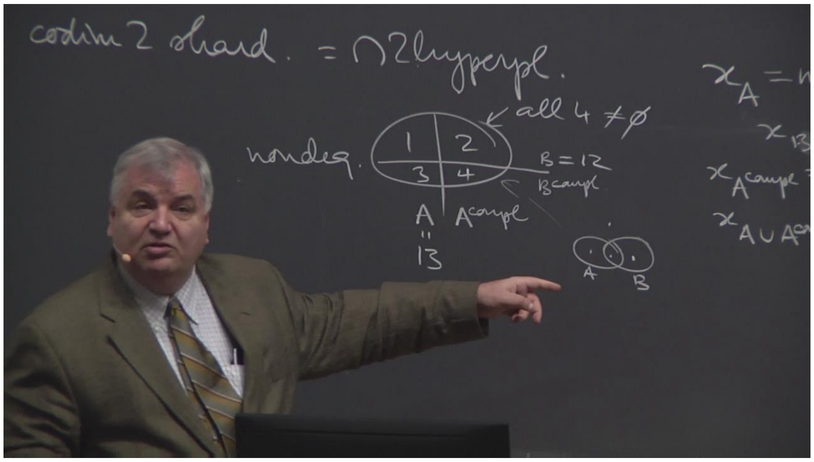
Now for the conservation relations. When you have a codim-2 shard,

$$\text{codim 2 shard} = \bigcap_2 \text{hyperplanes}$$

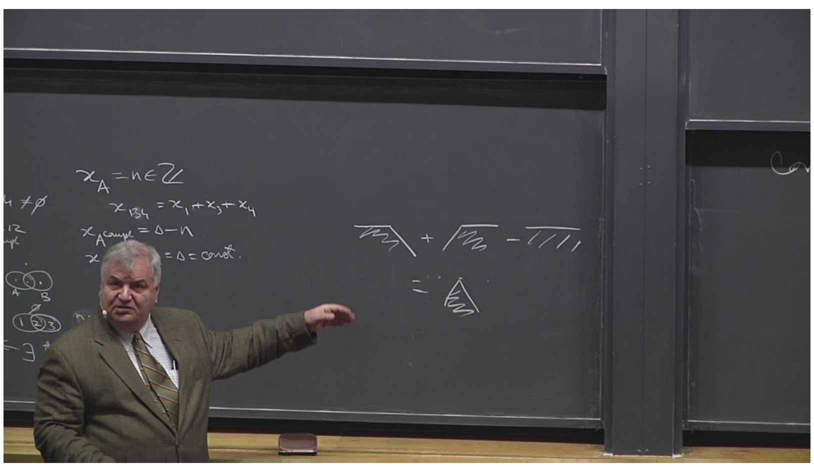
where a hyperplane is $x_A = n \in \mathbb{Z}$, A is a subset. In this case, $x_{A^c} = s - n$, because

$$x_{A \cup A^c} = s = const.$$

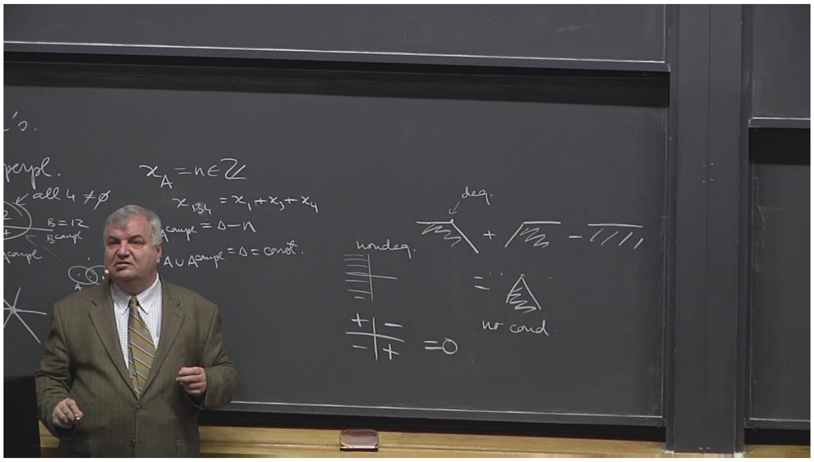
Now if you have A, A^c and B, B^c, these are 2 hyperplanes. There are 2 situations here which in the nontrivial case, namely when all 4 subsets are nonempty. So for instance 1,2,3,4, A=13,B=12. We call such a intersection nondegenerate. This is the Venn diagram of 2 subsets.



Now when you look at the plates, then you can get



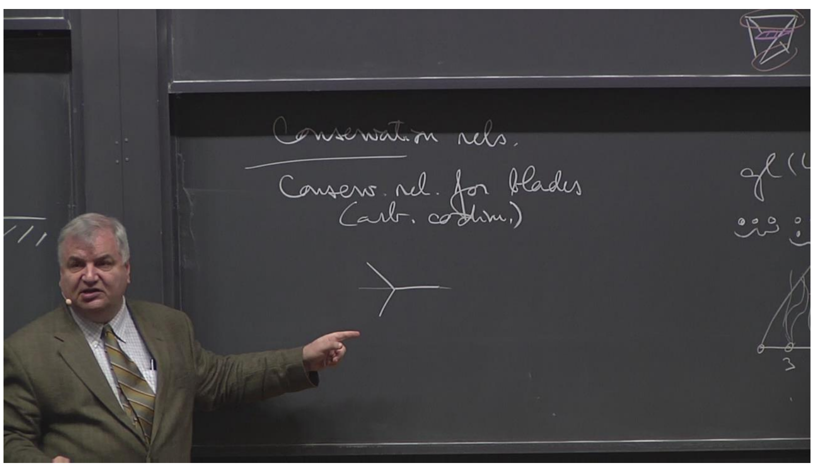
It means that you can generate around the degenerated plate you can generate everything. So no condition. Around the degenerated plates there are no condition, eh, degenerated intersection there is no condition for plates. You can generate everything. However, when you look at the nondegenerated plate, it always have empty or everything.



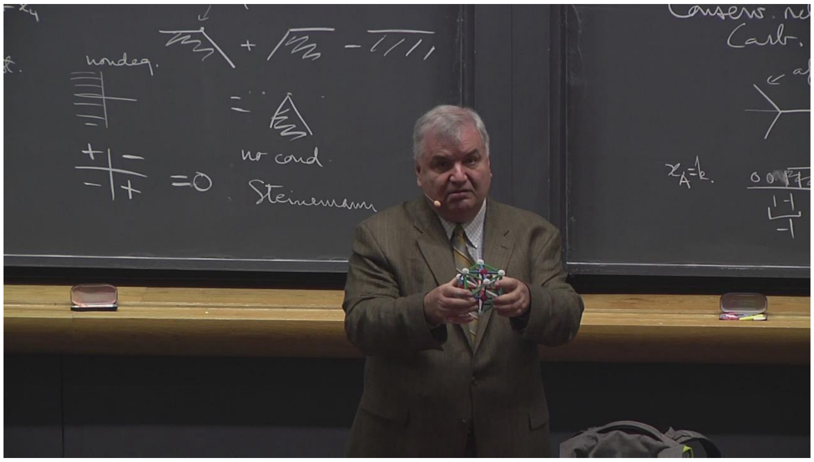
The sum of this sides, +,-,+,-, you take the multiplicity, this sum is 0. Whenever you take sums with multiplicity of half planes, if you take around a point, if you take half plane and the other half plane, for each half plane it satisfy this condition.

The main theorem is that, in the multiplicity on shards of codimension 0, satisfying the relations on the nondegenerated points, and it's necessary sufficient for them to be a sum of plates.

What's equivalent condition for the. I am going to give you the **conservation relation for blades**, this is in arbitrary dimension. If you have for instance this blade,



The conservation relation here is for any special hyperplane, which means hyperplane for some form, the number of affine roots on one side of the hyperplane is equal to the other side. The proof is the following: relabel: choose the affine roots to be 12,23,...,(n-1)n,n1. For instance, if $A = \{2, 3, 4, 6\}$, then the inner product of A and 12 is -1, and the inner product of A and 45 is 1. So there are equally many ascents and descents in this, and you get the conservation relation. The big theorem is that this conservation relation is necessary and sufficient for codimension 1 blades. I haven't check it but it's likely to be true also in the high codimension. This is a local condition which is necessary and sufficient. Once again if you have hyperplane you look for instance at, eh, you look at it then you linear convolution of blades the number of, the multiplicity of the things above, see clear above it is the same as the multiplicity of the things below, no matter what hyperplane you take. For instance here you have 4 such segments and 4 under knee,



then you can have in any with any answer. So on is as longer as the conservation are satisfied. So this shows you can speak about as things growing and so on. I have to stop here. And next week it will be an overview of the representation and I will give the explicity the representation for a generator. So the way the matrix act on this vegetable. (see video 59;20, Adrian play his toys.)